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BROWNIAN REPRESENTATION OF A CLASS OF LÉVY PROCESSES AND ITS APPLICATION TO OCCUPATION TIMES OF DIFFUSION PROCESSES

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Dedicated to the memory of Professor J. L. Doob

Abstract. It is well known that a class of subordinators can be represented using the local time of Brownian motions. An extension of such a representation is given for a class of Lévy processes which are not necessarily of bounded variation. This class can be characterized by the complete monotonicity of the Lévy measures. The asymptotic behavior of such processes is also discussed and the results are applied to the generalized arc-sine law, an occupation time problem on the positive side for one-dimensional diffusion processes.

1. Introduction

Let $\{B(t)\}_{t\geq 0}$ be a standard Brownian motion and $\ell(t,x)$ be its local time. Then, for a Radon measure $m(dx)$ on $\mathbb{R}$, the processes

$$X^+(t) = \int_{(0,\infty)} \ell(\ell^{-1}(t,0),x)\, m(dx), \quad t \geq 0, \tag{1.1}$$

$$X^-(t) = \int_{(-\infty,0)} \ell(\ell^{-1}(t,0),x)\, m(dx), \quad t \geq 0 \tag{1.2}$$

are independent subordinators. For example, if $m(dx) = |x|^{1/\alpha-2}\,dx$, then $c_+X^+(t) - c_-X^-(t)$ $(c_+, c_- \geq 0)$ is a stable Lévy processes with index α provided that $0 < \alpha < 1$. On the other hand, for $1 \leq \alpha < 2$, the above integrals diverge *a.s.* and do not make sense. Indeed, the processes that can be expressed as above are necessarily of bounded variation. Nonetheless, some Lévy processes of unbounded variation can be given a similar Brownian representation by modifying the integral as something similar to the Cauchy principal value or the Hadamard finite part of a divergent integral. For example,

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using the Hölder continuity in x of $\ell(t,x)$, we see that

$$Z(t) = \lim_{\epsilon\to+0}\int_{x>\epsilon}\{\ell(\ell^{-1}(t,0),x) - \ell(\ell^{-1}(t,0),-x)\}\,x^{1/\alpha-2}\,dx$$

converges $a.s.$ even if $1 \le \alpha < 2$, and is symmetric α-stable by self-similarity. Further, if one needs completely asymmetric α-stable Lévy processes with $1 < \alpha < 2$, then

$$Z(t) = \lim_{\epsilon\to+0}\int_{x>\epsilon}\{\ell(\ell^{-1}(t,0),x) - t\}\,x^{1/\alpha-2}\,dx$$

will do. (When $\alpha = 1$ we need a slight modification.) We note that a similar representation of stable processes has been already discussed by M. Yor [13, p. 4].

In the present paper, we generalize these examples to obtain a similar representation for a class of Lévy processes. In Section 2, we introduce a class $\mathbb{M}$ of Radon measures m on an interval $(0,l)$, $0 < l \le \infty$, and define Lévy processes $T^{\pm}(m;t)$ associated with m by modifying the integrals (1.1) and (1.2) similarly as in the above examples of stable processes. In Section 3, we first treat these examples of stable processes and study the long time asymptotics of the processes defined in Section 2. In Section 4, we apply the results in the preceding sections to a limit theorem for occupation times of one-dimensional diffusion processes. If the diffusion is positively recurrent, or, more generally, if the tails of the speed measure are slowly varying with the same order, then it is known that the ratio of the time spent on the positive side converges to a constant. Our results here are concerned with limit theorems for the fluctuation. Section 5 is devoted to a remark on Lamperti's class of distributions in connection with the result in Section 4. In Section 6, we complement Section 2 to describe the class of Lévy processes having the Brownian representation of Section 2. It is a class of Lévy processes with the Lévy measure supported on $(0,\infty]$ and with a completely monotone density, and, furthermore, without the Gaussian part. By a recent result of S. Kotani, we can see conversely that any such Lévy process has a Brownian representation.

2. Lévy processes represented by Brownian local time

We denote by $\mathbb{M}$ the set of all functions $m : (-\infty,\infty) \to (-\infty,\infty]$ such that

(i) $m(x) \equiv 0$ on $(-\infty, 0]$,
(ii) $m(x)$ is nondecreasing, right-continuous on $(0,\infty)$, and
(iii) we have

$$\int_{0<x<\delta} m(x)^2\,dx < \infty, \quad \exists\delta > 0.$$

For $m \in \mathbb{M}$, we define

$$l(m) = \sup\{x; m(x) < \infty\},$$

and for $m_\lambda, m \in \mathbb{M}$, we say $m_\lambda \to m$ if and only if
(m.1) $m_\lambda(x) \to m(x)$ at every continuity point x of m, and
(m.2) $\lim_{\delta \to 0+} \limsup_{\lambda \to \infty} \int_{0<x<\delta} m_\lambda(x)^2 \, dx = 0$.

Typical examples, which will often appear later, are the following:

EXAMPLE 2.1. Let $0 < \alpha < 2$ and put $m^{(\alpha)}(x) = 0$ for $x \le 0$ and

$$m^{(\alpha)}(x) = \begin{cases} x^{\frac{1}{\alpha}-1}, & 0 < \alpha < 1, \\ \log x, & \alpha = 1, \\ -x^{\frac{1}{\alpha}-1}, & 1 < \alpha < 2, \end{cases}$$

for $x > 0$. In these cases, $l(m^{(\alpha)}) = \infty$.

REMARK 2.2. The class $\mathbb{M}$ may look strange at first but the authors were informed by S. Kotani the following fact (cf. [8]): For $m \in \mathbb{M}$, define its *dual* $m^*(x)$ $(-\infty < x < \infty)$ by

$$m^*(x) = \inf\{u > 0; m(u) > x\}.$$

Then $m^* : (-\infty, \infty) \to [0, \infty]$ is a nondecreasing, right-continuous function such that $m(-\infty) = 0$, and the condition (iii) can be rewritten as

$$\int_{-\infty} x^2 \, dm^*(x) < \infty.$$

This condition is equivalent to the condition that the boundary $-\infty$ of the operator $\frac{d}{dm^*}\frac{d}{dx}$ is of the *type limit circle*.

Throughout the paper $\{B(t)\}_{t \ge 0}$ denotes a standard Brownian motion starting at 0 and $\ell(t, x)$ the local time with respect to $2dx$; $\ell(t, x)$ is continuous in (t, x) and

$$\int_0^t f(B(s)) \, ds = 2 \int_{-\infty}^{\infty} f(x) \, \ell(t, x) \, dx, \quad a.s.$$

for every bounded continuous $f(x)$.

DEFINITION 2.3. For $m \in \mathbb{M}$ we put

$$G(x) = \int_0^x m(u) \, du, \ -\infty \le x < l(m), \quad \text{and} \quad \zeta_\pm = \inf\{t \mid B(t) = \pm l(m)\},$$

and define the following stochastic processes with state space $\mathbb{R} \cup \{\infty\}$:

$$S^+(m;t) = \begin{cases} -\int_0^t m(B(s))\,dB(s) + G(B(t)), & 0 \le t < \zeta_+, \\ \infty, & t \ge \zeta_+, \end{cases}$$

$$S^-(m;t) = \begin{cases} \int_0^t m(-B(s))\,dB(s) + G(-B(t)), & 0 \le t < \zeta_-, \\ \infty, & t \ge \zeta_-, \end{cases}$$

$$T^+(m;t) = S^+(m;\ell^{-1}(t,0)), \quad t \ge 0,$$

$$T^-(m;t) = S^-(m;\ell^{-1}(t,0)), \quad t \ge 0.$$

Here, $\ell^{-1}(t,0) = \inf\{s > 0 \,|\, \ell(s,0) > t\}$.

These definitions are rather artificial but the meaning will be clear later in Corollary 2.6. Here, we mention that the stochastic integrals exist, i.e., that

$$\int_0^t m(\pm B(s))^2\,ds < \infty, \qquad 0 \le \forall t < \zeta_\pm \quad a.s. \tag{2.1}$$

Indeed, we have

$$\int_0^t m(\pm B(s))^2\,ds = 2\int_0^\infty m(x)^2\,\ell(t,\pm x)\,dx,$$

and, if $0 \le t < \zeta_\pm$, then $x \mapsto \ell(t,\pm x)$ is continuous with compact support in $(-\infty, l(m))$. Since $m \in L^2_{loc}((-\infty, l(m)), dx)$, we obtain (2.1).

Lemma 2.4. *Let $m \in \mathbb{M}$. If $m(0+) > -\infty$, then*

$$S^\pm(m;t) = \int_{x>0} \ell(t,\pm x)\,dm(x) + m(0+)\,\ell(t,0), \quad 0 \le t < \zeta_\pm,$$

$$T^\pm(m;t) = \int_{x>0} \ell(\ell^{-1}(t,0),\pm x)\,dm(x) + m(0+)\,t, \quad 0 \le t < \ell(\zeta_\pm,0).$$

Proof. We first note that $G(\pm x)$ is a difference of two convex functions such that

$$D^+G(x) = \begin{cases} 0, & x < 0, \\ m(x), & 0 \le x < l(m), \end{cases}$$

where D^+ denotes the right-hand derivative. Therefore, it holds that

$$d(D^+G)(x) = 1_{(0,l(m))}(x)\,dm(x) + m(0+)\,\delta(dx)$$

in the sense of signed measures, where $\delta(dx)$ denotes the unit mass at 0. Hence, applying the Itô-Tanaka formula, we derive, for $0 \le t < \zeta_\pm$,

$$G(\pm B(t)) = \pm\int_0^t m(\pm B(s))\,dB(s) + \int_{x>0} \ell(t,\pm x)\,dm(x) + m(0+)\ell(t,0)$$

because $G(0) = 0$. Substituting this into the definition of $S^{\pm}(m;t)$ we have the first half of our assertion. The latter half follows from the first. $\square$

We note that the class of Brownian additive functionals $S^{\pm}$ contains those introduced and studied by Yamada [12] and Yor [13].

THEOREM 2.5 (Continuity theorem). *Suppose $m_\lambda, m \in \mathbb{M}$ and $m_\lambda \to m$. Then,*

$$\liminf_{\lambda\to\infty} (\zeta_\lambda)_\pm \geq \zeta_\pm, \qquad a.s.,$$

and

$$\sup_{t\in[0,(\zeta_\pm-\epsilon)\wedge T]} |S^{\pm}(m_\lambda;t) - S^{\pm}(m;t)| \xrightarrow{P} 0,$$

$$\sup_{t\in[0,\ell((\zeta_\pm-\epsilon)\wedge T,0)]} |T^{\pm}(m_\lambda;t) - T^{\pm}(m;t)| \xrightarrow{P} 0,$$

for every $T > 0$ and $\epsilon > 0$.

Proof. Let $l(m) = \sup\{x; m(x) < \infty\}$ as before. Since $m_\lambda \to m$, it holds that $m_\lambda(x) \to m(x)$ for every $x \in (0, l(m) - \epsilon)$ such that $m(x-) = m(x)$ ($\forall\epsilon > 0$). Therefore, $l(m_\lambda) \geq l(m) - \epsilon$ for all sufficiently large λ. Hence we obtain the first part of the theorem. Also (m.2) implies that $\{m_\lambda\}_\lambda$ is bounded in $L^2_{loc}((-\infty, l(m)), dx)$. Hence $m_\lambda(x) \to m(x)$ in $L^1_{loc}((-\infty, l(m)), dx)$. Therefore, we have

$$G_\lambda(x) := \int_0^x m_\lambda(u)\,du \to G(x) := \int_0^x m(u)\,du, \quad \forall x \in (-\infty, l(m)),$$

which combined with the definitions of $S^{\pm}$ and $T^{\pm}$ proves the assertion. $\square$

COROLLARY 2.6. *If $m \in \mathbb{M}$, then for every sequence $\varepsilon_1 > \varepsilon_2 > \cdots \to 0$,*

$$S^{\pm}(m;t) = \lim_{n\to\infty} \int_{x>\varepsilon_n} \ell(t, \pm x)\,dm(x) + m(\varepsilon_n)\,\ell(t,0),$$

$$T^{\pm}(m;t) = \lim_{n\to\infty} \int_{x>\varepsilon_n} \ell(\ell^{-1}(t,0), \pm x)\,dm(x) + m(\varepsilon_n)\,t,$$

where the convergence holds, uniformly for $0 \leq t \leq \zeta_\pm - \epsilon$, in probability ($\forall\epsilon > 0$).

Proof. The assertion follows immediately from Lemma 2.4 and Theorem 2.5 by putting $m_n(x) = m(x \vee \epsilon_n)$. $\square$

Now, for $m \in \mathbb{M}$ and $c \in \mathbb{R}$, we define $m + c \in \mathbb{M}$ by

$$(m+c)(x) = \begin{cases} 0, & x < 0, \\ m(x) + c, & x \geq 0. \end{cases}$$

Corollary 2.7 (Linearity). *Let $m_1, m_2 \in \mathbb{M}$, $a, b > 0$, and $-\infty < c < \infty$. Then,*

$$S^{\pm}(am_1 + bm_2 + c; t) = aS^{\pm}(m_1; t) + bS^{\pm}(m_2; t) + c\,\ell(t, 0),$$
$$T^{\pm}(am_1 + bm_2 + c; t) = aT^{\pm}(m_1; t) + bT^{\pm}(m_2; t) + c\,t.$$

Proof. The assertion can easily be seen from the definition, but also follows immediately from Corollary 2.6. □

Theorem 2.8. *Let $m^+, m^- \in \mathbb{M}$. Then $T^+(m^+; t)$ and $T^-(m^-; t)$ are independent Lévy processes without discontinuities of negative jumps.*

Proof. In view of Corollary 2.6 it suffices to prove that

$$\int_{x>\varepsilon_n} \ell(\ell^{-1}(t, 0), \pm x)\, dm(x) + m(\varepsilon_n)\, t$$

have the desired properties. However, the independence is well known and it is clear that they have no negative jumps. □

As we have seen above, the Lévy measures of $T^{\pm}(m; t)$ vanish on $(-\infty, 0)$. This fact allows us to consider Laplace transforms instead of Fourier transforms. We will review the Lévy-Khintchin form of the Laplace exponent in Section 6. In what follows we study the limit process of $T^{\pm}(m_\lambda; t)$, $m_\lambda \in \mathbb{M}$, in the case where (m.1) holds with $m \in \mathbb{M}$ but where (m.2) does not hold (and hence $m_\lambda \to m$ fails). We first consider the simplest, but most important, case where $m(x) \equiv 0$:

Theorem 2.9 (Central limit theorem). *Let $m_\lambda^+, m_\lambda^- \in \mathbb{M}$ and suppose that $m_\lambda^+(x), m_\lambda^-(x) \to 0$ for every $x > 0$. We further assume that*

$$\lim_{\lambda\to\infty} \int_{0<x<\delta} m_\lambda^{\pm}(x)^2\, dx = \sigma_\pm^2 (\geq 0).$$

(This condition does not depend on the choice of $\delta > 0$.) Then,

$$\begin{aligned}&\big(S^+(m_\lambda^+; t), S^-(m_\lambda^-; t), B(t)\big)\\ &\qquad \xrightarrow{\mathcal{L}} \big(\sqrt{2}\sigma_+ B^{(+)}(\ell(t, 0)), \sqrt{2}\sigma_- B^{(-)}(\ell(t, 0)), B(t)\big)\end{aligned}$$

and

$$\big(T^+(m_\lambda^+; t), T^-(m_\lambda^-; t), B(t)\big) \xrightarrow{\mathcal{L}} \big(\sqrt{2}\sigma_+ B^{(+)}(t), \sqrt{2}\sigma_- B^{(-)}(t), B(t)\big)$$

in the sense of convergence in law on the Skorohod space $D([0, \infty) : (-\infty, \infty]^3)$ of cádlág functions. Here, B and ℓ are the same as before and $B^{(+)}$, $B^{(-)}$ are copies of B such that $B^{(+)}$, $B^{(-)}$ and B are independent.

Proof. Recall the definition of $S^{\pm}$:

$$S^{+}(m_\lambda^{+};t) = -\int_0^t m_\lambda^{+}(B(s))\,dB(s) + G_\lambda(B(t)),$$
$$S^{-}(m_\lambda^{-};t) = \int_0^t m_\lambda^{-}(-B(s))\,dB(s) + G_\lambda(-B(t))$$

where

$$G_\lambda(x) = \int_0^x m_\lambda(u)\,du.$$

Let us see that, under our conditions,

$$G_\lambda(x) \to 0, \quad \lambda \to \infty \quad (x \in \mathbb{R}), \tag{2.2}$$

the convergence being uniform on every finite interval $[0, A]$. Indeed, for $x \in [0, A]$, we have

$$\begin{aligned}|G_\lambda(x)| &\le \int_0^\epsilon |m_\lambda^{\pm}(u)|\,du + \int_\epsilon^x |m_\lambda^{\pm}(u)|\,du \\ &\le \left(\epsilon \int_0^\epsilon m_\lambda^{\pm}(u)^2\,du\right)^{1/2} + \big(|m_\lambda^{\pm}(\epsilon)| + |m_\lambda^{\pm}(A)|\big)A,\end{aligned}$$

and hence

$$\limsup_{\lambda\to\infty} \sup_{0\le x\le A} |G_\lambda(x)| \le \sqrt{\epsilon}\,\sigma_{\pm},$$

for any $\epsilon > 0$. Thus we have (2.2). Hence the proof of the theorem is reduced to determining the limiting processes of

$$\begin{aligned}&(M_1^\lambda(t), M_2^\lambda(t), M_3^\lambda(t)) \\ &\qquad := \Big(-\int_0^t m_\lambda(B(s))\,dB(s), \int_0^t m_\lambda(-B(s))\,dB(s), B(t)\Big).\end{aligned}$$

Since our assumptions imply that

$$m_\lambda^{\pm}(x)^2 \to \sigma_{\pm}^2 \delta(x),$$

where $\delta(x)$ is the delta function, it is easy to see that

$$\langle M_1^\lambda \rangle_t = \int_0^t m_\lambda(B(s))^2 ds = 2\int_0^\infty \ell(t,x) m_\lambda(x)^2 dx \longrightarrow 2\sigma_+^2 \ell(t,0),$$

the convergence being uniform for $t \in [0, T]$ $(\forall T > 0)$ *a.s.* Analogously, we have

$$\langle M_2^\lambda \rangle_t = \int_0^t m_\lambda(B(s))^2 ds \longrightarrow 2\sigma_-^2 \ell(t,0), \quad a.s.$$

Similarly, by (2.2) we have

$$\langle M_1^\lambda, M_3^\lambda \rangle_t = 2\int_0^\infty \ell(t,x) m_\lambda(x)\,dx \to 0, \quad \lambda \to \infty,\ a.s.,$$

as well as $\langle M_2^\lambda, M_3^\lambda\rangle_t \to 0$ while it is obvious that $\langle M_1^\lambda, M_2^\lambda\rangle_t = 0$. These facts imply the tightness of the processes $\{M_i^\lambda(t)\}_\lambda$ $(i = 1, 2, 3)$ over the function space $C([0,\infty))$. Indeed, this follows from the following observation. As is well known, $M_1^\lambda(t)$ can be expressed as $B^\lambda(\langle M_1^\lambda\rangle_t)$ for suitable Brownian motions B^λ by extending the probability space if necessary. Thus tightness of $\{M_1^\lambda(t)\}_\lambda$ is reduced to that of $\{(B^\lambda(t), \langle M_1^\lambda\rangle_t)\}_\lambda$, which obviously holds because both components converge in law. Similarly, $\{M_2^\lambda(t)\}_\lambda$ is tight. Therefore, the tightness is shown and it remains only to identify the joint limiting processes. To this end recall that we have

$$\lim_{\lambda\to\infty} \langle M_i^\lambda, M_j^\lambda\rangle_t = \begin{cases} 2\sigma_+^2 \ell(t,0), & i = j = 1, \\ 2\sigma_-^2 \ell(t,0), & i = j = 2, \\ t, & i = j = 3, \\ 0, & i \neq j. \end{cases}$$

Therefore, any limiting process (M_1, M_2, M_3) can be expressed as follows in terms of the 3-dimensional Brownian motion $(B^{(+)}, B^{(-)}, B)$:

$$\big(B^{(+)}(2\sigma_+^2\ell(t,0)), B^{(-)}(2\sigma_-^2\ell(t,0)), B(t)\big)$$

as desired (see, e.g., [5] for the representation theorem of continuous martingales). Thus we have the convergence of $S^\pm(m_\lambda; t)$. The convergence of $T^\pm(m_\lambda; t)$ also follows because

$$T^\pm(m_\lambda; t) = S^\pm(m_\lambda; \ell^{-1}(t, 0)). \qquad \square$$

The following theorem is a mixture of Theorems 2.5 and 2.9.

THEOREM 2.10. *Let $m_\lambda^+, m_\lambda^-, m^+, m^- \in \mathbb{M}$. If*

(i) *$m_\lambda^+(x) \to m^+(x), m_\lambda^-(x) \to m^-(x)$ at all continuity points $x (> 0)$ of $m^\pm$, and*

(ii) *there exist σ_+^2 and σ_-^2 such that*

$$\lim_{\delta\to 0} \limsup_{\lambda\to\infty} \Big| \int_{0<x<\delta} m_\lambda^\pm(x)^2\, dx - \sigma_\pm^2 \Big| = 0,$$

then,

$$\big(S^\pm(m_\lambda^\pm; t), B(t)\big) \xrightarrow{\mathcal{L}} \big(S^\pm(m^\pm; t) + \sqrt{2}\sigma_\pm B^{(\pm)}(\ell(t,0)), B(t)\big),$$

$$\big(T^\pm(m_\lambda^\pm; t), B(t)\big) \xrightarrow{\mathcal{L}} \big(T^\pm(m^\pm; t) + \sqrt{2}\sigma_\pm B^{(\pm)}(t), B(t)\big),$$

where $(B^{(+)}, B^{(-)}, B)$ and ℓ are defined as in Theorem 2.9.

Proof. Let $\{A_\lambda\}_\lambda$ be real numbers tending to $-\infty$ as $\lambda \to \infty$ and let

$$m_{\lambda,1}^\pm(x) = m_\lambda^\pm(x) \vee A_\lambda, \quad m_{\lambda,2}^\pm(x) = m_\lambda^\pm(x) - m_{\lambda,1}^\pm(x).$$

Notice that

$$m_{\lambda,1}^\pm(x) \to m^\pm(x), \quad m_{\lambda,2}^\pm(x) \to 0, \quad \lambda \to \infty,$$

at every continuity point x of the limiting functions. By Corollary 2.7 it also holds that

$$S^{\pm}(m^{\pm}_{\lambda};t)=S^{\pm}(m^{\pm}_{\lambda,1};t)+S^{\pm}(m^{\pm}_{\lambda,2};t).$$

Now let us see that Theorems 2.5 and 2.9 are applicable to the first and second terms, respectively. Since

$$\lim_{\delta\to 0}\limsup_{\lambda\to\infty}\int_{0<x<\delta}(m^{\pm}_{\lambda}(x)\vee A)^2\,dx=0,$$

for every fixed A, we have

$$\lim_{\delta\to 0}\limsup_{\lambda\to\infty}\int_{0<x<\delta}m^{\pm}_{\lambda,1}(x)^2\,dx=0, \tag{2.3}$$

provided that $\{A_\lambda\}_\lambda$ tends to $-\infty$ slowly enough. This proves that $m^{\pm}_{\lambda,1}\to m^{\pm}$ in $\mathbb{M}$. Furthermore, it holds that

$$\lim_{\delta\to 0}\limsup_{\lambda\to\infty}\Big|\int_{0<x<\delta}m^{\pm}_{\lambda,2}(x)^2\,dx-\sigma^2_{\pm}\Big|=0. \tag{2.4}$$

Indeed, since

$$\begin{aligned}\Big|\Big(\int_{0<x<\delta}m^{\pm}_{\lambda,2}(x)^2\,dx\Big)^{1/2}&-\Big(\int_{0<x<\delta}m^{\pm}_{\lambda}(x)^2\,dx\Big)^{1/2}\Big|\\&\le\Big(\int_{0<x<\delta}m^{\pm}_{\lambda,1}(x)^2\,dx\Big)^{1/2}\longrightarrow 0,\quad \lambda\to\infty,\end{aligned}$$

by (2.3), our assumption (ii) implies (2.4). Thus, Theorems 2.5 and 2.9 are applicable to $S^{\pm}(m^{\pm}_{\lambda,1};t)$ and $S^{\pm}(m^{\pm}_{\lambda,2};t)$, respectively, and it remains to show the joint convergence of these processes. As in the proof of Theorem 2.9, the problem is reduced to determining the limiting processes of

$$\begin{aligned}&\big(M^{\lambda}_1(t),M^{\lambda}_2(t),M^{\lambda}_3(t),M^{\lambda}_4(t),M^{\lambda}_5(t)\big)\\&\quad=\Big(-\int_0^t m^{+}_{\lambda,1}(B(s))\,dB(s),-\int_0^t m^{-}_{\lambda,1}(B(s))\,dB(s),\\&\qquad\int_0^t m^{+}_{\lambda,2}(-B(s))\,dB(s),\int_0^t m^{-}_{\lambda,2}(-B(s))\,dB(s),B(t)\Big).\end{aligned}$$

To this end it suffices to determine the limiting processes of $\langle M^{\lambda}_i,M^{\lambda}_j\rangle$, $i,j=1,\dots,5$. Indeed, as in the proof of Theorem 2.9, we see that

$$\langle M^{\lambda}_2\rangle_t\to 2\sigma^2_{+}\ell(t,0),\quad \langle M^{\lambda}_4\rangle_t\to 2\sigma^2_{-}\ell(t,0),\quad \langle M^{\lambda}_5\rangle_t=t$$

and also it is easy to see that other $\langle M^{\lambda}_i,M^{\lambda}_j\rangle_t$ converge to 0. This completes the proof of the theorem. □

3. The case of stable Lévy processes and limit theorems

Let $m^{(\alpha)} \in \mathbb{M}$ $(0 < \alpha < 2)$ be as in Example 2.1. We shall first show that $T^+(m^{(\alpha)};t)$ and $T^-(m^{(\alpha)};t)$ are independent, completely asymmetric α-stable Lévy processes. As we have mentioned in Section 2, $T^+(m^{(\alpha)};t)$ and $T^-(m^{(\alpha)};t)$ are independent Lévy processes without negative jumps, and therefore their common Laplace transform makes sense. So let $\psi^{(\alpha)}(s)$ be its exponent, i.e.,

$$E[e^{-sT^+(m^{(\alpha)};t)}] = E[e^{-sT^-(m^{(\alpha)};t)}] = e^{-t\psi^{(\alpha)}(s)}, \quad t \geq 0,\, s > 0.$$

Theorem 3.1. *We have*

$$\psi^{(\alpha)}(s) = \begin{cases} \frac{\Gamma(2-\alpha)}{\Gamma(\alpha)}\{\alpha(1-\alpha)\}^{\alpha-1}s^\alpha, & 0 < \alpha < 1, \\ -s\,(\log s + 2\gamma), & \alpha = 1, \\ -\frac{\Gamma(2-\alpha)}{\Gamma(\alpha)}\{\alpha(\alpha-1)\}^{\alpha-1}s^\alpha, & 1 < \alpha < 2, \end{cases} \tag{3.1}$$

where $\Gamma(x)$ *is the usual gamma function and* $\gamma = 0.577\ldots$ *is Euler's constant.*

Proof. For $0 < \alpha < 1$, the result is already known (see, e.g., Kotani-Watanabe [9]). In order to consider its analytic continuation, let

$$\widetilde{m}^{(\alpha)}(x) = \begin{cases} \frac{\alpha}{1-\alpha}(x^{1/\alpha-1} - 1), & 0 < \alpha < 2,\, \alpha \neq 1, \\ \log x, & \alpha = 1, \end{cases}$$

and define $\widetilde{\psi}^{(\alpha)}(s)$ by

$$E[e^{-sT^\pm(\widetilde{m}^{(\alpha)};t)}] = e^{-t\widetilde{\psi}^{(\alpha)}(s)}.$$

Then, since

$$T^+(\widetilde{m}^{(\alpha)};t) = \begin{cases} \frac{\alpha}{1-\alpha}\left\{T^+(m^{(\alpha)};t) - t\right\}, & 0 < \alpha < 1, \\ T^+(m^{(\alpha)};t), & \alpha = 1, \\ \frac{\alpha}{\alpha-1}\left\{T^+(m^{(\alpha)};t) + t\right\}, & 1 < \alpha < 2, \end{cases}$$

it holds that

$$\widetilde{\psi}^{(\alpha)}(s) = \begin{cases} \psi^{(\alpha)}(\frac{\alpha}{1-\alpha}s) - \frac{\alpha}{1-\alpha}s, & 0 < \alpha < 1, \\ \psi^{(\alpha)}(s), & \alpha = 1, \\ \psi^{(\alpha)}(\frac{\alpha}{\alpha-1}s) + \frac{\alpha}{\alpha-1}s, & 1 < \alpha < 2. \end{cases}$$

Therefore, (3.1) is equivalent to

$$\widetilde{\psi}^{(\alpha)}(s) = \begin{cases} -\alpha^{2\alpha}\frac{\Gamma(2-\alpha)}{\Gamma(1+\alpha)}\frac{s^\alpha}{\alpha-1} - \frac{\alpha}{1-\alpha}s, & 0 < \alpha < 2, \alpha \neq 1, \\ -s\,(\log s + 2\gamma), & \alpha = 1. \end{cases}$$

By a standard argument we see that both sides are analytic in α on $\{\alpha \in \mathbb{C}\,; 0 < \mathfrak{Re}\,\alpha < 2\}$. Hence the proof of the theorem is complete. $\square$

This proof also gives the following result, which we need in Section 6.

THEOREM 3.2. *As* $\alpha \uparrow 1$,

$$\frac{1}{1-\alpha}\{S^{\pm}(m^{(\alpha)};t)-\ell(t,0)\} \xrightarrow{P} S^{\pm}(m^{(1)};t),$$

and

$$\frac{1}{1-\alpha}\{T^{\pm}(m^{(\alpha)};t)-t\} \xrightarrow{P} T^{\pm}(m^{(1)};t),$$

in $C[0,\infty)$ *and* $D[0,\infty)$, *respectively* .

Proof. Let $\widetilde{m}^{(\alpha)}$ be as above. Then the assertions can be rewritten as

$$S^{\pm}(\widetilde{m}^{(\alpha)};t) \to S^{\pm}(\widetilde{m}^{(1)};t), \quad S^{\pm}(\widetilde{m}^{(\alpha)};t) \to S^{\pm}(\widetilde{m}^{(1)};t), \quad \alpha \to 1,$$

respectively. Thus the problem can be reduced to the continuity of $\widetilde{m}^{(\alpha)}$ in α. □

THEOREM 3.3 (The case $1<\alpha<2$). *Let* $m(dx)$ *be a finite Borel measure on* $[0,\infty)$ *and suppose that*

$$m(x,\infty) \sim x^{-\beta}L(x), \quad x \to \infty \tag{3.2}$$

for some $0<\beta<1/2$ *and slowly varying* $L(x)$. *(*$f(x) \sim g(x)$ *means that* $f(x)/g(x) \to 1$ *throughout the paper.) Then, for* $m \in \mathbb{M}$ *defined by* $m(x) = -m(x,\infty)$, *we have*

$$\left(\frac{1}{\lambda^{1/\alpha}L(\lambda)}S^{\pm}(m;\lambda^2 t), \frac{1}{\lambda}B(\lambda^2 t)\right) \xrightarrow{\mathcal{L}} \left(S^{\pm}(m^{(\alpha)};t), B(t)\right), \tag{3.3}$$

$$\left(\frac{1}{\lambda^{1/\alpha}L(\lambda)}T^{\pm}(m;\lambda t), \frac{1}{\lambda}B(\lambda^2 t)\right) \xrightarrow{\mathcal{L}} \left(T^{\pm}(m^{(\alpha)};t), B(t)\right), \tag{3.4}$$

where $\alpha = 1/(1-\beta)$, *so that* $1<\alpha<2$.

Note that, by Lemma 2.4, the assertion may be rewritten in a more familiar way. For example, (3.4) implies

$$\frac{\lambda^{1-1/\alpha}}{L(\lambda)}\left(\frac{1}{\lambda}\int_{(0,\infty)} \ell\big(\ell^{-1}(\lambda t,0),\pm x\big)\, m(dx) - m(0,\infty)\, t\right) \xrightarrow{\mathcal{L}} T^{\pm}(m^{(\alpha)};t).$$

Proof of Theorem 3.3. For $\lambda>0$, define $m_\lambda \in \mathbb{M}$ by

$$m_\lambda(x) = \frac{1}{\lambda^{1/\alpha-1}L(\lambda)} m(\lambda x).$$

Then the condition (3.2) implies that $m_\lambda \to m^{(\alpha)}$ in $\mathbb{M}$. Indeed, (m.1) is an immediate consequence and (m.2) can be checked by the following well-known fact.

$$\lim_{\lambda\to\infty}\int_0^\delta m_\lambda(x)^2\,dx = \int_0^\delta x^{-2\beta}\,dx = \frac{1}{1-2\beta}\delta^{1-2\beta}, \quad \delta>0$$

(see Seneta [10, pp. 66–67, Thm. 2.7] or Bingham *et. al.* [3, p. 26, Prop. 1.5.8]).

Since

$$\left(\frac{1}{\lambda}B(\lambda^2 t), \frac{1}{\lambda}\ell(\lambda^2 t, x)\right) \stackrel{(d)}{=} \left(B(t), \ell(t, \frac{x}{\lambda})\right),$$

where "$X \stackrel{(d)}{=} Y$ "means that X and Y are equally distributed, we have

$$\begin{aligned}
&\left(\frac{1}{\lambda^{1/\alpha}L(\lambda)}S^{\pm}(m;\lambda^2 t), \frac{1}{\lambda}B(\lambda^2 t)\right) \\
&\quad = \left(\frac{1}{\lambda^{1/\alpha}L(\lambda)}(\int_{x>0}\ell(\lambda^2 t, \pm x)\,m(dx) - m(0,\infty)\ell(\lambda^2 t, 0)), \frac{1}{\lambda}B(\lambda^2 t)\right) \\
&\quad \stackrel{(d)}{=} \left(\int_{x>0}\ell(t,\pm x)\,dm_\lambda(x) - m_\lambda(0)\ell(t,0), B(t)\right) \\
&\quad = \left(S^{\pm}(m_\lambda; t), B(t)\right).
\end{aligned}$$

Hence (3.3) follows from Theorem 2.5. As for (3.4), recall that $T^{\pm}$ are defined through $S^{\pm}$ and that the inverse process of $\frac{1}{\lambda}\ell(\lambda^2 t, 0)$ is the process $\frac{1}{\lambda^2}\ell^{-1}(\lambda t, 0)$. Then we can derive that

$$\left(\frac{1}{\lambda^{1/\alpha}L(\lambda)}T^{\pm}(m;\lambda t), \frac{1}{\lambda}B(\lambda^2 t)\right) \stackrel{(d)}{=} \left(T^{\pm}(m_\lambda; t), B(t)\right).$$

Therefore, (3.4) follows from Theorem 2.5 as well. □

Example 3.4. A typical example of (3.2) is

$$m(dx) = (1+x)^{-\beta-1}dx, \quad x \geq 0, \quad (0 < \beta < 1/2).$$

We next study the extreme case $\beta = 0$ in Theorem 3.3 or, equivalently, $\alpha = 1$. In this case, $m(dx)$ can be both a finite and an infinite measure.

Theorem 3.5 (The case $\alpha = 1$). *Let $m(dx)$ be a Radon measure on $[0,\infty)$ and let $L(x)\,(x \geq 0)$ be a slowly varying function at infinity such that both $L(x)$ and $1/L(x)$ are locally bounded on $[0,\infty)$. If*

$$\frac{m(0,\lambda x] - m(0,\lambda]}{L(\lambda)} \longrightarrow \log x, \quad \forall x > 0, \quad \lambda \to \infty, \tag{3.5}$$

then, for $m \in \mathbb{M}$ defined by $m(x) = m(0,x]$, we have

$$\begin{aligned}
&\left(\frac{1}{\lambda L(\lambda)}\left(S^{\pm}(m;\lambda^2 t) - m(\lambda)\,\ell(\lambda^2 t, 0)\right), \frac{1}{\lambda}B(\lambda^2 t)\right) \\
&\qquad\qquad \xrightarrow{\mathcal{L}} \left(S^{\pm}(m^{(1)}; t), B(t)\right)
\end{aligned} \tag{3.6}$$

and

$$\left(\frac{1}{\lambda L(\lambda)}\left(T^{\pm}(m;\lambda t) - m(\lambda)\lambda\, t\right), \frac{1}{\lambda}B(\lambda^2 t)\right) \xrightarrow{\mathcal{L}} \left(T^{\pm}(m^{(1)}; t), B(t)\right). \tag{3.7}$$

EXAMPLE 3.6. If $m(dx) = dx/(1+x)$, then the assumptions are satisfied with $L(\lambda) = 1$ and hence we have, for example,

$$\frac{1}{\lambda}\left\{\int_0^\infty \ell(\lambda^2 t, \pm x)\,\frac{dx}{1+x} - (\log\lambda)\,\ell(\lambda^2 t, 0)\right\} \xrightarrow{\mathcal{L}} S^\pm(m^{(1)}; t).$$

Similarly, if

$$m(dx) = \frac{dx}{(1+x)\{\log(1+x)\}^c} \quad (c > 0),$$

then

$$\{m(0,\lambda x] - m(0,\lambda]\}\cdot(\log\lambda)^c \to \log x, \quad \lambda\to\infty,$$

and the assumptions are satisfied with $L(\lambda) = (\log(1+\lambda))^{-c}$. Note that $m(dx)$ is finite if $c > 1$.

Proof of Theorem 3.5. The proof can be carried out in a similar way as that of the previous theorem: Define $m_\lambda \in \mathbb{M}\,(\lambda > 0)$ by

$$m_\lambda(x) = \frac{1}{L(\lambda)}(m(\lambda x) - m(\lambda)).$$

Then,

$$\left(\frac{1}{\lambda L(\lambda)}\big(S^\pm(m;\lambda^2 t) - m(\lambda)\ell(\lambda^2 t, 0)\big), \frac{1}{\lambda}B(\lambda^2 t)\right) \stackrel{(d)}{=} \big(S^\pm(m_\lambda; t), B(t)\big),$$

and hence the proof is reduced to the continuity theorem (Theorem 2.5). Of course, in order to show that $m_\lambda \to m^{(1)}$ in $\mathbb{M}$, it remains to make sure that (m.2) is satisfied, i.e.,

$$\lim_{\delta\to 0+}\limsup_{\lambda\to\infty}\int_0^\delta m_\lambda(x)^2 dx = 0. \tag{3.8}$$

However, it is known that, under our assumptions, there exists, for every $\epsilon > 0$, a constant $A_\epsilon > 0$ such that

$$\frac{|m(x) - m(y)|}{L(x)} \le A_\epsilon \max\left\{(y/x)^\epsilon, (y/x)^{-\epsilon}\right\}$$

for every $x, y > 0$ (see [3, p. 172]). This implies that

$$|m_\lambda(x)| \le A_\epsilon x^{-\epsilon}, \quad 0 < x < 1,$$

and hence we see

$$\int_0^\delta m_\lambda(x)^2 dx \le A_\epsilon^2 \int_0^\delta x^{-2\epsilon} dx,$$

which proves (3.8). □

In Theorem 3.3 we studied the case $1 < \alpha < 2$, and in Theorem 3.5 we considered the case $\alpha = 1$. Since the case $0 < \alpha < 1$ is much simpler, we shall not go into details (see [11]). We now proceed to discuss the case which corresponds to $\alpha = 2$.

THEOREM 3.7 (Central limit theorem). *Let $m(dx)$ be a finite Borel measure on $(-\infty, \infty)$ such that*

$$\int_0^\lambda \big(m(x,\infty)\big)^2\, dx \sim \sigma_+^2 L(\lambda), \quad \lambda \to \infty, \tag{3.9}$$

$$\int_0^\lambda \big(m(-\infty,-x)\big)^2\, dx \sim \sigma_-^2 L(\lambda), \quad \lambda \to \infty, \tag{3.10}$$

for some slowly varying $L(\lambda) > 0$. Then putting $m_+(x) = m(0,x]$, $m_-(x) = m[-x,0)$, $m_+ = m(0,\infty)$, $m_- = m(-\infty,0)$, we have

$$\left(\frac{1}{\sqrt{\lambda L(\lambda)}}\Big(S^{\pm}(m_\pm;\lambda^2 t) - m_\pm\, \ell(\lambda^2 t, 0)\Big), \frac{1}{\lambda}B(\lambda^2 t)\right) \xrightarrow{\mathcal{L}} \Big(\sqrt{2}\sigma_\pm B^{(\pm)}(\ell(t,0)), B(t)\Big) \tag{3.11}$$

and

$$\left(\frac{1}{\sqrt{\lambda L(\lambda)}}\Big(T^{\pm}(m_\pm;\lambda t) - m_\pm\, \lambda t\Big), \frac{1}{\lambda}B(\lambda^2 t)\right) \xrightarrow{\mathcal{L}} \Big(\sqrt{2}\,\sigma_\pm B^{(\pm)}(t), B(t)\Big), \tag{3.12}$$

where $(B^{(+)}, B^{(-)}, B)$ and ℓ are the same as in Theorem 2.9.

Proof. Put

$$m_\lambda^+(x) = \sqrt{\lambda/L(\lambda)}\, m(\lambda x, \infty), \quad m_\lambda^-(x) = \sqrt{\lambda/L(\lambda)}\, m(-\infty, -\lambda x), \quad x > 0.$$

Then, as in the proof of Theorem 3.3, we have

$$\left(\frac{1}{\sqrt{\lambda L(\lambda)}}\big(S^{\pm}(m;\lambda^2 t) - m_\pm \ell(\lambda^2 t, 0)\big), \frac{1}{\lambda}B(\lambda^2 t)\right) \stackrel{(d)}{=} \big(S^{\pm}(m_\lambda; t), B(t)\big).$$

The functions $m_\lambda^\pm(x)$ satisfy the assumption of Theorem 2.9: For example, for $m_\lambda^+(x)$ we have

$$\int_0^\delta m_\lambda^+(x)^2 dx = \frac{1}{L(\lambda)}\int_0^{\delta\lambda} m(x,\infty)^2 dx \sim \frac{L(\delta\lambda)}{L(\lambda)}\sigma_+^2 \sim \sigma_+^2 \quad \text{as}\, \lambda \to \infty,$$

and this implies that, for any $0 < \alpha < \beta$, $(\beta - \alpha) m_\lambda(\beta)^2 \le \int_\alpha^\beta m_\lambda^+(x)^2 dx \to 0$ as $\lambda \to \infty$, proving that $m_\lambda^+(x) \to 0$ for all $x > 0$. The result then follows by applying Theorem 2.9. □

In the special case where we can choose $L(x)$ to be constant, this theorem can be rewritten as follows:

Corollary 3.8. *Let $m(dx)$ be a Borel measure on $(-\infty, \infty)$ such that*

$$\sigma_+^2 = \int_0^\infty m(x,\infty)^2 dx < \infty, \quad \sigma_-^2 = \int_0^\infty m(-\infty,-x)^2 dx < \infty.$$

Then, as $\lambda \to \infty$,

$$\begin{aligned}(3.13)\qquad &\left(\frac{1}{\sqrt{\lambda}}\big(S^{\pm}(m_\pm;\lambda^2 t) - m_\pm\, \ell(\lambda^2 t,0)\big), \frac{1}{\lambda}B(\lambda^2 t)\right)\\ &\qquad\qquad \xrightarrow{\mathcal{L}} \left(\sqrt{2}\sigma_\pm B^{(\pm)}(\ell(t,0)), B(t)\right)\end{aligned}$$

and

$$(3.14)\quad \left(\frac{1}{\sqrt{\lambda}}\big(T^{\pm}(m_\pm;\lambda t) - m_\pm\, \lambda t\big), \frac{1}{\lambda}B(\lambda^2 t)\right) \xrightarrow{\mathcal{L}} \left(\sqrt{2}\sigma_\pm B^{(\pm)}(t), B(t)\right).$$

Example 3.9. Let

$$m(dx) = \frac{dx}{(1+|x|)^{3/2}}.$$

Then, $m_+ = m_- = 2$ and

$$\int_0^\lambda \big(m(x,\infty)\big)^2\, dx = \int_0^\lambda \big(m(-\infty,-x)\big)^2\, dx \sim 4\log\lambda, \quad \lambda \to \infty.$$

Thus Theorem 3.7 is applicable with $L(\lambda) = \log\lambda$ and $\sigma_\pm = 2$. We also note that, if we add the two terms, we have

$$\frac{1}{\sqrt{\lambda\log\lambda}}\left(\int_{-\infty}^\infty \ell(\lambda^2 t,x)\, m(dx) - 4\,\ell(\lambda^2 t,0)\right) \xrightarrow{\mathcal{L}} 2\sqrt{2}\,\widetilde{B}(\ell(t,0)), \quad \lambda \to \infty,$$

where $\widetilde{B}$ is an independent copy of B.

4. Application to the generalized arcsine law

Let $X = \{X(t)\}_{t\ge 0}$ be a conservative and recurrent diffusion process on the real line. As is well known, with a suitable change of the scale, we may assume that Feller's canonical representation of the generator of X is of the form $\frac{d}{dm}\frac{d}{dx}$, where $m(dx)$, which is referred to as the speed measure, is a non zero Radon measure on $(-\infty,\infty)$. We need not assume that $m(dx)$ is positive everywhere so that generalized or gap diffusions (including birth and death processes) and reflecting diffusions on sub-intervals are also allowed. Of course, in this case the state space of X is the support of the speed measure $m(dx)$. In what follows we are concerned with the long time asymptotics of the occupation time on the half line $(0,\infty)$,

$$\Gamma(t) := \int_0^t 1_{(0,\infty)}(X(s))\, ds, \qquad t \ge 0.$$

It is known that the class of possible limit random variables in law of $\Gamma(t)/t$ as $t \to \infty$ coincides with that of Lamperti's random variables $Y_{p,\alpha}$, $0 \le p \le$

1, $0 \le \alpha \le 1$: $Y_{p,\alpha}$ is a $[0,1]$–valued random variable with the Stieltjes transform given by

$$E\left(\frac{1}{\lambda + Y_{p,\alpha}}\right) = \frac{p(\lambda+1)^{\alpha-1} + (1-p)\lambda^{\alpha-1}}{p(\lambda+1)^{\alpha} + (1-p)\lambda^{\alpha}}, \qquad \lambda > 0. \tag{4.1}$$

Also, a sufficient condition for the convergence, which turns out to be necessary and sufficient when $0 < p < 1$, can be given in terms of $m(dx)$. For details, we refer to Watanabe [11] and also to Kasahara and Watanabe [7] for some refinements in the case $\alpha = 0$.

In the present section we are interested in the case $\alpha = 1$. Notice that in this case $Y_{p,\alpha}$ degenerates to a constant: $Y_{p,1} \equiv p$. Here, we exclude the trivial case of $p = 1$ or $p = 0$ and so we confine ourselves to the case where

$$\frac{1}{\lambda}\Gamma(\lambda) \xrightarrow{P} p \in (0,1), \quad \lambda \to \infty. \tag{4.2}$$

A necessary and sufficient condition for (4.2) to hold is that $x \mapsto m([0,x])$ and $x \mapsto m([-x,0])$ $(x > 0)$ are slowly varying at ∞ with a balancing condition

$$\lim_{x\to\infty} \frac{m(0,x)}{m([-x,x])} = p. \tag{4.3}$$

A particular, and the most typical, case is the positively recurrent case, where $m(\mathbb{R}) < \infty$ with the balancing condition

$$\frac{m(0,\infty)}{m(\mathbb{R})} = p \in (0,1). \tag{4.4}$$

Our assumption that m is slowly varying means that we are here interested in positively recurrent diffusions or similar processes. Also we note that the convergence (4.2) can be strengthened to almost sure convergence if and only if the diffusion is positively recurrent, i.e., $m(\mathbb{R}) < \infty$ (see Bertoin [2]).

Now, the aim of this section is to apply the results in the preceding section to evaluate the fluctuation

$$\frac{1}{\lambda}\Gamma(\lambda t) - pt.$$

Let $m^{(\alpha)}$ and $T^{\pm}(m^{(\alpha)};t)$ be as before. Thus $T^{+}(m^{(\alpha)};t)$ and $T^{-}(m^{(\alpha)};t)$ are independent α–stable Lévy processes as we have seen in Section 2.

Theorem 4.1 (The case $1 < \alpha < 2$). *Assume that $m(dx)$ is finite, i.e., $m(\mathbb{R}) < \infty$, with the balancing condition (4.4) and assume in addition that*

$$m(x,\infty) \sim c_+ x^{-\beta} L(x), \quad m(-\infty,-x) \sim c_- x^{-\beta} L(x), \quad x \to \infty,$$

for some $c_\pm > 0$, $0 < \beta < 1/2$, and slowly varying $L(x)$. Then, letting $\alpha = 1/(1-\beta)$ so that $1 < \alpha < 2$, we have

$$\frac{m(\mathbb{R})^{1/\alpha}}{\lambda^{1/\alpha} L(\lambda)}\Big(\Gamma(\lambda\, t) - p\,\lambda\, t\Big) \xrightarrow{f.d.} (1-p)\, c_+ T^{+}(m^{(\alpha)};t) - p\, c_- T^{-}(m^{(\alpha)};t).$$

Here and throughout, $\xrightarrow{f.d.}$ *denotes the convergence of all finite-dimensional marginal distributions.*

Recall that the limiting process is an α–stable Lévy process. Also note that the convergence here cannot be strengthened to convergence with respect to the Skorohod topology, which does not allow continuous processes to converge to a discontinuous one.

THEOREM 4.2 (The case $\alpha = 1$). *We assume that* $m(dx)$ *satisfies the slowly varying property and the balancing condition* (4.3)*. We further assume that, as* $\lambda \to \infty$,

$$\frac{m(0,\lambda x] - m(0,\lambda]}{L(\lambda)} \longrightarrow c_+ \log x, \quad (\forall x > 0),$$
$$\frac{m[-\lambda x, 0) - m[-\lambda, 0)}{L(\lambda)} \longrightarrow c_- \log x, \quad (\forall x > 0),$$

for some positive constants c_+, c_- *and slowly varying function* $L(x)$ *such that* $L(x)$ *and* $1/L(x)$ *are locally bounded on* $[0,\infty)$*. Then, putting* $p(\lambda) = m(0,\lambda]/m([-\lambda,\lambda])$ *and* $q(\lambda) = \lambda\, m([-\lambda,\lambda])$*, we have*

$$\frac{1}{\lambda L(\lambda)}\Big(\Gamma(q(\lambda)\, t) - p(\lambda)\, q(\lambda)\, t\Big) \xrightarrow{f.d.} (1-p)\, c_+ T^+(m^{(\alpha)}; t) - p\, c_- T^-(m^{(\alpha)}; t).$$

EXAMPLE 4.3. Let $m(dx) = dx/(1+|x|)$, $-\infty < x < \infty$. Then, $m(0,x] = m[-x,0) = \log(1+x)$ and hence the assumptions are satisfied with $L(\lambda) = 1$, $p(\lambda) = p = 1/2$, $c_\pm = 1$ and $q(\lambda) = 2\lambda\log\lambda$. Therefore,

$$\frac{\log\lambda}{\lambda}\Big(\Gamma(\lambda\, t) - \frac{1}{2}\,\lambda\, t\Big) \xrightarrow{f.d.} T^+(m^{(1)}; t) - T^-(m^{(1)}; t).$$

Here, we used the fact that $\frac{1}{2}\lambda/\log\lambda$ is an asymptotic inverse of $q(\lambda) = 2\lambda\log\lambda$. We note that the limiting process is a usual symmetric Cauchy process.

THEOREM 4.4 (Central limit theorem; the case $\alpha = 2$). *Assume that* $m(\mathbb{R}) < \infty$ *with the balancing condition* (4.4)*. Assume further that*

$$\int_0^\lambda m(x,\infty)^2 dx \sim \sigma_+^2 L(\lambda), \qquad \int_0^\lambda m(-\infty,-x)^2 dx \sim \sigma_-^2 L(\lambda),$$

as $\lambda \to \infty$*, for some* $\sigma_\pm > 0$ *and slowly varying function* $L(\lambda) > 0$*. Then we have*

$$\sqrt{\frac{m(\mathbb{R})}{\lambda L(\lambda)}}(\Gamma(\lambda t) - p\lambda t) \xrightarrow{f.d.} \sqrt{2}\,(1-p)\,\sigma_+ B^{(+)}(t) - \sqrt{2}\,p\,\sigma_- B^{(-)}(t),$$

as $\lambda \to \infty$*, where* $(B^{(+)}, B^{(-)})$ *is a two-dimensional Brownian motion starting at the origin.*

Before we proceed to the proofs, we recall the well known fact on constructing the diffusion process $X(t)$ from $B(t)$ via time change: Let

$$S(t) = \int_{-\infty}^{\infty} \ell(t,x)\, m(dx), \quad t \geq 0. \tag{4.5}$$

Then it is well known that $\{B(S^{-1}(t))\}_t$ is equivalent in law to $\{X(t)\}_t$. Therefore, we may assume that $X(t) = B(S^{-1}(t))$ without loss of generality. Next, define

$$A(t) = \int_{x>0} \ell(t,x)\, m(dx). \tag{4.6}$$

Then, by changing variables, it is not difficult to see that

$$\Gamma(t) = \int_0^t 1_{(0,\infty)}(X(s))\, ds = \int_0^t 1_{(0,\infty)}(B(S^{-1}(s))\, ds = A(S^{-1}(t)), \tag{4.7}$$

and hence our problem is reduced to the study of the joint asymptotic behavior of $A(t)$ and $S(t)$ as $t \to \infty$ (see Kasahara and Watanabe [7]).

Proof of Theorem 4.1. Define $m^{\pm} \in \mathbb{M}$ by $m^{+}(x) = -m(x,\infty)$ and $m^{-}(x) = m(-\infty,-x)$ for $x > 0$. Then, by Theorem 3.3, we have

$$\left(\frac{1}{\lambda^{1/\alpha}L(\lambda)} S^{\pm}(m^{\pm};\lambda^2 t), \frac{1}{\lambda}B(\lambda^2 t)\right) \xrightarrow{\mathcal{L}} \big(c_{\pm}S^{\pm}(m^{(\alpha)};t), B(t)\big). \tag{4.8}$$

Recall Lemma 2.4. Then, it is easy to verify the equality

$$A(t) - p\,S(t) = (1-p)\,S^{+}(m^{+};t) - p\,S^{-}(m^{-};t).$$

Notice here that the balancing condition (4.4) implies that $\ell(t,x)$ does not appear explicitly on the right-hand side. Therefore, (4.8) implies that

$$\begin{aligned}&\left(\frac{1}{\lambda^{1/\alpha}L(\lambda)}\Big\{A(\lambda^2 t) - p\,S(\lambda^2 t)\Big\}, \frac{1}{\lambda}B(\lambda^2 t)\right)\\ &\qquad \xrightarrow{\mathcal{L}} \Big((1-p)\,c_{+}S^{+}(m^{(\alpha)};t) - p\,c_{-}S^{-}(m^{(\alpha)};t), B(t)\Big).\end{aligned} \tag{4.9}$$

On the other hand, by the scaling property, we have

$$\frac{1}{\lambda}S(\lambda^2 t) = \int_{-\infty}^{\infty} \frac{1}{\lambda}\ell(\lambda^2 t, x)\, m(dx) \overset{(d)}{=} \int_{-\infty}^{\infty} \ell(t, x/\lambda)\, m(dx),$$

and it is easy to deduce from this that

$$\left(\frac{1}{\lambda}S(\lambda^2 t), \frac{1}{\lambda}B(\lambda^2 t)\right) \xrightarrow{\mathcal{L}} \left(m(\mathbb{R})\,\ell(t,0), B(t)\right), \quad \lambda \to \infty. \tag{4.10}$$

Noting that the inverse process of $t \mapsto \frac{1}{\lambda}S(\lambda^2 t)$ is $t \mapsto \frac{1}{\lambda^2}S^{-1}(\lambda t)$, and the inverse of $t \mapsto m(\mathbb{R})\,\ell(t,0)$ is $t \mapsto \ell^{-1}(t/m(\mathbb{R}),0)$, we substitute these inverse processes into t of the first components in both sides of (4.9), respectively. Then we obtain

$$\text{(4.11)}\quad \frac{1}{\lambda^{1/\alpha}L(\lambda)}\Big(A(S^{-1}(\lambda t)) - p\,\lambda t\Big) \xrightarrow{f.d.} (1-p)\,c_+T^+\Big(m^{(\alpha)}; \frac{t}{m(\mathbb{R})}\Big) - p\,c_-T^-\Big(m^{(\alpha)}; \frac{t}{m(\mathbb{R})}\Big),$$

as $\lambda \to \infty$. (For arguments of this kind see, e.g., Kasahara and Kotani [6].) Since $\Gamma(t) = A(S^{-1}(t))$, replacing t by $m(\mathbb{R})\,t$ and λ by $\lambda/m(\mathbb{R})$, we obtain the assertion of the theorem. □

We next prove Theorem 4.2. The proof is essentially the same as the one above. However, we need a small modification to deal with the case when $m(\mathbb{R}) = \infty$.

Let $m^+(x) = m(0,x]$, $m^-(x) = m[-x,0]$, $x > 0$. Then $A(m;t) = S^+(m;t)$, and hence

$$\begin{aligned} S^+(m;\lambda^2 t) - p(\lambda)S(m;\lambda^2 t) &\\ = (1-p(\lambda))\{S^+(m^+;\lambda^2 t) - m^+(\lambda)\ell(\lambda^2 t, 0)\}&\\ - p(\lambda)\{S^-(m^-;\lambda^2 t) - m^-(\lambda)\ell(\lambda^2 t, 0)\}.& \end{aligned}$$

Thus we obtain:

Lemma 4.5. *As $\lambda \to \infty$,*

$$\Big(\frac{1}{\lambda L(\lambda)}\Big(S^+(m;\lambda^2 t) - p(\lambda)S(m;\lambda^2 t)\Big), \frac{1}{\lambda}B(\lambda^2 t)\Big) \xrightarrow{\mathcal{L}} \Big((1-p)\,c_+S^+(m^{(\alpha)};t) - p\,c_-S^-(m^{(\alpha)};t), B(t)\Big).$$

We next prove a result that corresponds to (4.10).

Lemma 4.6. *Under the assumptions of Theorem 4.2, it holds that*

$$\Big(\frac{1}{q(\lambda)}S(m;\lambda^2 t), \frac{1}{\lambda}B(\lambda^2 t)\Big) \xrightarrow{\mathcal{L}} \big(\ell(t,0), B(t)\big).$$

Proof. We first see that $dm(\lambda x)/m[-\lambda,\lambda]$ converges vaguely to the Dirac function: For every $0 < \delta < A$, we have

$$\begin{aligned} \frac{m(\lambda\delta,\lambda A]}{m[-\lambda,\lambda]} &\le \frac{m(\lambda\delta,\lambda A]}{m(\lambda\epsilon,\lambda]}\\ &= \frac{m[0,\lambda A] - m[0,\lambda\delta]}{L(x)}\Big/\frac{m[0,\lambda] - m[0,\lambda\epsilon]}{L(x)}\\ &\longrightarrow (\log A - \log\delta)/\log(1/\epsilon), \quad \lambda\to\infty, \end{aligned}$$

for every $\epsilon > 0$. Since $\epsilon > 0$ is arbitrary, this implies

$$m(\lambda\delta, \lambda A]/m[-\lambda, \lambda] \longrightarrow 0, \quad \lambda \to \infty.$$

Similarly,

$$m[-\lambda A, -\lambda\delta]/m[-\lambda, \lambda] \longrightarrow 0, \quad \lambda \to \infty.$$

Thus $dm(\lambda x)/m[-\lambda, \lambda] \to \delta(dx)$. Since $q(\lambda) = \lambda\, m[-\lambda, \lambda]$, we have

$$\begin{aligned}&\left(\frac{1}{q(\lambda)}\int \ell(\lambda^2 t, x)\, dm(x), \frac{1}{\lambda}B(\lambda^2 t)\right)\\ &\quad \overset{(d)}{=} \left(\frac{1}{m[-\lambda, \lambda]}\int \ell(t, x)\, dm(\lambda x), B(t)\right) \longrightarrow \big(\ell(t,0), B(t)\big), \quad a.s. \qquad \square\end{aligned}$$

We are now ready to prove Theorem 4.2. Since $A(t) = S^+(m; t)$ in the present case, we have from Lemmas 4.5 and 4.6 that

$$\begin{aligned}&\left(\frac{1}{\lambda L(\lambda)}\Big(A(\lambda^2 t) - p(\lambda) S(m; \lambda^2 t)\Big), \frac{1}{q(\lambda)} S(m; \lambda^2 t)\right)\\ &\qquad \xrightarrow{\mathcal{L}} \Big((1-p)\, c_+ S^+(m^{(\alpha)}; t) - p\, c_- S^-(m^{(\alpha)}; t),\ \ell(t, 0)\Big).\end{aligned}$$

The rest of the proof is the same as in the proof of Theorem 4.1.

The proof of Theorem 4.4, which is based on Theorem 3.7, is similar to that of Theorem 4.1 and therefore it is omitted.

The different features of the above theorems are illustrated by the following example:

Example 4.7. Let

$$m(dx) = (1 + |x|)^{\frac{1}{\alpha} - 2} dx, \quad -\infty < x < \infty.$$

It is known that if $0 < \alpha < 1$, then, $\Gamma(\lambda)/\lambda$ converges in law to Lamperti's random variable $Y_{p,\alpha}$ with $p = 1/2$, and if $\alpha \geq 1$, then it converges to $1/2$ in probability. In the latter case, we further have the following from the results in the above. If $\alpha = 1$, then the law of $\frac{\log \lambda}{\lambda}(\Gamma(\lambda) - \lambda/2)$ converges to a symmetric Cauchy distribution; if $1 < \alpha < 2$, then the law of $\frac{1}{\lambda^{1/\alpha}}(\Gamma(\lambda) - \lambda/2)$ converges to a symmetric α-stable law; and if $\alpha = 2$ (or $\alpha > 2$), then the law of $\frac{1}{\sqrt{\lambda \log \lambda}}(\Gamma(\lambda) - \lambda/2)$ (or $\frac{1}{\sqrt{\lambda}}(\Gamma(\lambda) - \lambda/2)$, resp.) converges to a centered normal distribution.

5. A remark on Lamperti's distribution

Let $Y_{p,\alpha}$ be Lamperti's random variable as in the previous section. Then, it is easy to see from (4.1) that the law of $Y_{p,\alpha}$ is continuous in (p, α). In particular, for every fixed $0 < p < 1$, it holds that $Y_{p,\alpha} \xrightarrow{\mathcal{L}} Y_{p,1} \equiv p$ as $\alpha \uparrow 1$. The aim of this section is to refine this convergence.

THEOREM 5.1. *Let $0 < p < 1$ and put $r(\alpha) = p^{1/\alpha}/\{p^{1/\alpha} + (1-p)^{1/\alpha}\}$. Then, as $\alpha \uparrow 1$,*

$$\frac{1}{1-\alpha}\left(Y_{p,\alpha} - r(\alpha)\right) \xrightarrow{\mathcal{L}} p(1-p)\left(T^+(m^{(1)};1) - T^-(m^{(1)};1)\right).$$

Proof. The proof can be carried out in a similar way as in Section 4: Let $X_{p,\alpha} = \{X_{p,\alpha}(t)\}_{t\geq 0}$ be a diffusion on the real line with Feller generator $\frac{d}{dm_{p,\alpha}(x)}\frac{d}{dx}$, where

$$m_{p,\alpha}[0,x] = p^{\frac{1}{\alpha}}\, x^{\frac{1}{\alpha}-1}, \quad m_{p,\alpha}[-x,0) = (1-p)^{\frac{1}{\alpha}}\, x^{\frac{1}{\alpha}-1}, \qquad x > 0.$$

This process is called the *skew Bessel diffusion process.* Let

$$\Gamma_\alpha(t) = \int_0^t 1_{(0,\infty)}(X_{p,\alpha}(u))\, du.$$

Then it is known ([1], [11]) that $\Gamma_\alpha(t)/t$ is distributed like $Y_{p,\alpha}$ for every $t > 0$. Now define

$$S_\alpha^+(t) = \int_{[0,\infty)} \ell(t,x)\, m_{p,\alpha}(dx), \quad S_\alpha^-(t) = \int_{(-\infty,0)} \ell(t,x)\, m_{p,\alpha}(dx),$$

and let

$$S_\alpha(t) = S_\alpha^+(t) + S_\alpha^-(t).$$

Then, as we have seen in the previous section, it holds

$$\Gamma_\alpha(t) = S_\alpha^+\left(S_\alpha^{-1}(t)\right). \tag{5.1}$$

By Theorem 3.2, we have, as $\alpha \uparrow 1$,

$$\begin{aligned}\left(\frac{1}{1-\alpha}\left(S_\alpha^+(t) - p^{1/\alpha}\ell(t,0)\right), \frac{1}{1-\alpha}\left(S_\alpha^-(t) - (1-p)^{1/\alpha}\ell(t,0)\right)\right)& \\ \xrightarrow{P} \left(pS^+(m^{(1)};t), (1-p)S^-(m^{(1)};t)\right).&\end{aligned} \tag{5.2}$$

Consequently, we have

$$\begin{aligned}\frac{1}{1-\alpha}\Big(S_\alpha^+(t) - r(\alpha)\, S_\alpha(t)\Big) &= \frac{1}{1-\alpha}\Big(\left(1-r(\alpha)\right)S_\alpha^+(t) - r(\alpha)\, S_\alpha^-(t)\Big) \\ &\xrightarrow{P} p(1-p)\left(S^+(m^{(1)};t) - S^-(m^{(1)};t)\right),\end{aligned} \tag{5.3}$$

as well as

$$S_\alpha(t) = S_\alpha^+(t) + S_\alpha^-(t) \xrightarrow{P} p\,\ell(t,0) + (1-p)\,\ell(t,0) = \ell(t,0). \tag{5.4}$$

Substituting the inverse process of (5.4) into (5.3), we derive

$$\frac{1}{1-\alpha}\Big(S_\alpha^+(S_\alpha^{-1}(t)) - r(\alpha)\, t\Big) \xrightarrow{f.d.} p(1-p)\left(T^+(m^{(1)};t) - T^-(m^{(1)};t)\right).$$

In view of (5.1), this proves the assertion of the theorem. □

6. Complete monotonicity of the Lévy measures

A function $f(x)$ on $(0,\infty)$ is said to be completely monotone if it possesses derivatives of all orders such that

$$(-1)^n f^{(n)}(x) \geq 0, \quad x > 0,$$

and it is well known that this is equivalent to the condition that

$$f(x) = \int_0^\infty e^{-x\xi}\, \sigma(d\xi), \quad x > 0,$$

for some Radon measure $\sigma(d\xi)$. (See, e.g., Feller [4].) When $f(x)$ is defined on $(-\infty, 0)$, it is said completely monotone if $f(-x)$ is completely monotone. Now let $m \in \mathbb{M}$. Then, as we have seen in Section 2, $T^+(m;t)$ and $T^-(m;t)$ are Lévy processes without discontinuities of negative jumps. Let $\psi(s) = \psi(m;s)$ be its exponent, i.e.,

$$E[e^{-sT^+(m;t)}] = E[e^{-sT^-(m;t)}] = e^{-t\psi(s)}, \quad s > 0, t \geq 0.$$

Then, $\psi(s)$ can be represented as

$$\psi(s) = c_0 + c_1 s - \frac{c^2}{2}s^2 + \int_{(0,\infty)} (1 - e^{-sx} - sx1_{(0,1)}(x))\, n(dx) \tag{6.1}$$

for some constants c_0, c_1, c and a Borel measure $n(dx)$, which is referred to as Lévy measure, such that

$$\int_0^\infty \min\{1, x^2\} n(dx) < \infty.$$

THEOREM 6.1. *Let $m \in \mathbb{M}$. Then the Lévy measure $n(dx)$ of $T^\pm(m;t)$ can be expressed as*

$$n[x,\infty) = \int_{(0,\infty)} e^{-x\xi} \sigma(d\xi),$$

where $\sigma(d\xi)$ is a Borel measure on $(0,\infty)$ such that

$$\int_{(0,\infty)} \frac{\sigma(d\xi)}{1+\xi^2} < \infty. \tag{6.2}$$

REMARK 6.2. After the first draft of this paper was prepared, S. Kotani proved analytically that the constant c in (6.1) vanishes. He also proved that, conversely, if a Borel measure $\sigma(d\xi)$ satisfying (6.2) is given, there exists an $m \in \mathbb{M}$ that corresponds to it and, furthermore, the correspondence is one-to-one in some sense. This means that a Lévy process can be realized in the form $T^+(m^+;t) - T^-(m^-;t)$ ($m^\pm \in \mathbb{M}$) if and only it has no Gaussian part and the Lévy measure is completely monotone. His theory also provides us with an analytical method of computing the exponent $\psi(s)$. In particular, Theorem 3.1 can be proved analytically.

Proof of Theorem 6.1. We first show the complete monotonicity of the Lévy measure. If $m(0+) \geq 0$, it is known that $\psi(s)$ can be expressed as

$$\psi(s) = sh^*(s), \quad h^*(s) = a + \int_{[0,\infty)} \frac{\sigma(d\xi)}{\xi + s}, \quad s > 0,$$

where $\sigma(d\xi)$ is a Radon measure on $[0,\infty)$ such that

$$\int_{[0,\infty)} \frac{\sigma(d\xi)}{1+\xi} < \infty.$$

In fact, $\sigma(d\xi)$ is the spectral measure of the *dual string* $m^*(x) = \inf\{u > 0; m(u) > x\}$ of m. (See [9] for details.) Hence, in this case, using Fubini's theorem, we easily obtain

$$n[x,\infty) = \int_{(0,\infty)} e^{-x\xi}\sigma(d\xi), \quad x > 0.$$

To extend this fact to general $m \in \mathbb{M}$, recall Corollary 2.6. Since the convergence in law of Lévy processes implies that of the Lévy measures, we see that the Lévy measure of $T(m;t)$ is a limit of completely monotone functions and, hence, is completely monotone, too. It remains to prove (6.2), which is in fact a special case of the next theorem. $\square$

THEOREM 6.3. *Let $n(dx)$ be a Borel measure on $(0,\infty)$ such that*

$$n[x,\infty) = \int_{(0,\infty)} e^{-x\xi}\sigma(d\xi), \quad x > 0,$$

for a Borel measure $\sigma(d\xi)$ on $(0,\infty)$. Then,

$$\int_{(0,\infty)} \frac{u^2}{1+u^2} n(du) < \infty \quad \textit{iff} \quad \int_{(0,\infty)} \frac{\sigma(d\xi)}{1+\xi^2} < \infty,$$

and

$$\int_{(0,\infty)} \frac{u}{1+u} n(du) < \infty \quad \textit{iff} \quad \int_{(0,\infty)} \frac{\sigma(d\xi)}{1+\xi} < \infty.$$

Proof. We can prove the assertion in a more general form: For $k \geq j \geq 0$, there exist absolute constants $0 < c_{k,j} < C_{k,j}$ such that

$$c_{k,j} \int_{(0,\infty)} \frac{\xi^{j-k}}{1+\xi^j}\sigma(d\xi) \leq \int_{(0,\infty)} \frac{u^k}{1+u^j} n(du) \leq C_{k,j} \int_{(0,\infty)} \frac{\xi^{j-k}}{1+\xi^j}\sigma(d\xi).$$

Indeed, since

$$\begin{aligned}\int_{(0,\infty)} \frac{u^k}{1+u^j} n(du) &= \int_{(0,\infty)} \int_{(0,\infty)} \frac{u^k}{1+u^j} du\, \xi e^{-u\xi}\sigma(d\xi) \\ &= \int_{(0,\infty)} V(\xi)\, \frac{\xi^{j-k}}{1+\xi^j}\, \sigma(d\xi),\end{aligned}$$

where

$$V(\xi) = \int_0^\infty \frac{(1+\xi^j)y^k}{\xi^j + y^j} e^{-y}\,dy,$$

it suffices to find constants such that $0 < c_{k,j} \le V(\xi) \le C_{k,j}$. Since

$$\frac{(1+\xi^j)y^k}{\xi^j + y^j} \le y^k + y^{k-j},$$

we have $V(\xi) \le \Gamma(k+1) + \Gamma(k-j+1)$, while

$$V(\xi) = \int_0^\infty \frac{(1+\xi^j)y^k}{\xi^j + y^j} e^{-y}\,dy \ge \int_0^\infty \frac{y^k}{1+y^j} e^{-y}\,dy. \qquad \square$$

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