THE SINGULARITIES, S_1^q

BY
HAROLD I. LEVINE[1]

Introduction

In this paper all manifolds and maps are either real C^∞ or complex analytic. A submanifold is always a regularly embedded submanifold, that is, the inclusion map into the ambient manifold is a homeomorphism into (real C^∞ or complex analytic).

Let V and M be manifolds of dimensions n and p respectively, and let $s = \min(n, p)$. If f is a map of V in M, let $S_1(f)$ be the set of all $v \in V$ such that rank $f_* = s - 1$ at v; here f_* means the induced map on tangent spaces. If $S_1(f)$ is a submanifold of V, we define $S_1^2(f)$ to be $S_1(f \mid S_1(f))$. In this way, for "sufficiently nice" maps, we may proceed letting $S_1^q(f) = S_1(f \mid S_1^{q-1}(f))$. This is the definition of Thom [7].

In Theorem 1, S_1^q are described "universally" independent of the map. That is, S_1^q are submanifolds of J^q, the space of q-jets at the origin of maps of n-space in p-space, such that if f maps V in M and the induced jet mapping $J^q(f) : V \to J^q(V, M)$ is transversal to all the $S_1^q(V, M)$, then

$$S_1^q(f) = (J^q(f))^{-1}(S_1^q(V, M)).$$

Here $J^q(V, M)$ is the bundle over $V \times M$ with fibre J^q and group the group of q-jets of coordinate changes in n-space and p-space; $S_1^q(V, M)$ is the subbundle of $J^q(V, M)$ induced by the inclusion $S_1^q \subset J^q$. Jet normal forms are given which show that whenever S_1^q is nonempty, then S_1^q either is the orbit of a single point if $n \leqq p$, or is the orbit of $[(n - p)/2] + 1$ distinct points if $n \geqq p$. The codimensions of S_1^q in J^q and the local equations of $S_1^q(f)$ are given. The proof of Theorem 1 for $n \geqq p$ is given in Section 3. The proof for the case $n < p$ is omitted since it parallels but is somewhat simpler than the proof for $n \geqq p$.

Suppose now that V and M are both n-dimensional manifolds, and that f maps V in M with rank $f_* \geqq n - 1$ everywhere. Further assume that $J^q(f)$ is transversal to the singularities $S_1^q(V, M)$ for all q. The object of Section 2 is to prove that under these conditions, the total characteristic class (Stiefel-Whitney class (mod 2) in the real case, and Chern class in the complex case) of V, $c(V)$, and the "pulled back" total characteristic class of M, $f^*c(M)$, are related by

$$c(V) = f^*c(M) - \textstyle\sum_{q=1}^{n} (j_q)_{\#}\, c(S_1^q(f)),$$

where j_q is the inclusion of $S_1^q(f)$ in V and $(j_q)_{\#}$ is the Gysin homomorphism of the cohomology of $S_1^q(f)$ into that of V.

Received September 21, 1962.

[1] This work was supported in part by the National Science Foundation.

This result is along the same lines as Theorem 5.5 of [4] in which holomorphic maps of V into complex projective space are studied. There the dimension of the projective space is strictly larger than that of V, and the expected dependence of $c(V)$ on the Chern classes of the singular manifolds does not appear explicitly; the assumption on the maps is that their induced first order jet maps are transversal to the first order singularities.

Except for 2.3, Section 2 may be read without reference to Sections 1 and 3. In 2.3, we refer to Theorem 1 for the existence of the singularities S_1^q, and for the jet-normal form of f at points of $S_1^q(f)$.

1. The singularities, S_1^q

Let $A = R$ or C. Using the notation of [3], we let J^q denote the space of q-jets at the origin of (real C^∞ or complex analytic) maps of A^n into A^p which take the origin into the origin. The group of q-jets of germs of (real C^∞ or complex analytic) diffeomorphisms at the origin of the source, A^n, and the target, A^p, leaving the respective origins fixed, acts on J^q by the "chain rule". For $r \leqq q$, let $\pi_{q,r}$ be the projection of J^q onto J^r.

Given any map F from A^n into A^p we let F^q be the induced map of A^n into J^q. The components of F^q are computed relative to fixed product coordinate systems in the source and target. Also given any element $f \in J^q$, we let P_f be the map of A^n into A^p taking the origin into the origin such that $(P_f)^q(0) = f$; the components of P_f are polynomials of degree at most q.

If S is a submanifold of J^q, we let ${}_TS \subset J^{q+1}$ be the set of all $(q+1)$-jets at the origin of maps F of A^n into A^p such that $F^q(0) \in S$, and such that F^q is transversal to S at 0. By $S(F)$ we mean $(F^q)^{-1}(S)$.

Following Thom (see [3], [7], and [9]), we propose to define the q^{th} order singularity, S_1^q, in J^q as follows:

(1) $f \in S_1^1 = S_1$ if and only if $\operatorname{rank}(P_f)_*(0) = \min(n, p) - 1$.

Assuming S_1^{q-1} is defined and is a submanifold of J^{q-1},

(2) $f \in S_1^q$ if and only if $f \in {}_TS_1^{q-1}$ and

$$\operatorname{rank}\,(P_f \mid S_1^{q-1}(P_f))_*(0) = \min\,(p, \dim S_1^{q-1}(P_f)) - 1,$$

where the inferior asterisk means the induced mapping of tangent spaces.

A priori it is not clear that this definition for S_1^q makes sense for $q > 2$, since we must know that S_1^{q-1} is a submanifold of J^{q-1}. In [3] it is proved that all $S_h\,S_k$ are submanifolds of J^2, so in particular $S_1\,S_1 = S_1^2$ is. Thus we know that S_1^q are defined for $q = 1, 2, 3$ and are submanifolds for $q = 1, 2$.

THEOREM 1. *S_1^q are submanifolds of J^q for all q.*

A. *For* $n = p + t$,

(i) *If* $q > p$, *then* $S_1^q = \emptyset$.

(ii) *If* $q \leqq p$, *then* codim $S_1^q = q + n - p$.

(iii) *If* $q \leqq p$, *then* $f \in {}_TS_1^q$ *and* $f \notin S_1^{q+1}$ *if and only if it is in the orbit (under the group defined by the diffeomorphisms of neighborhoods of the origins in the* *source and target) of the $(q+1)$-jet at the origin of one of the maps, F, given by*

$$
\begin{aligned}
&U \circ F(x, y, u) = u, \\
(*) \quad &Y \circ F(x, y, u) = \textstyle\sum_{i=1}^{t} \pm y_i^2 + \sum_{i=1}^{q-1} x^i u_i / i! \\
&\qquad\qquad + x^{q+1}/(q+1)! + R(x, u),
\end{aligned}
$$

where the order of R is greater than $q+1$, and

$$(x, y_1, \cdots, y_t, u_1, \cdots, u_{p-1}) = (x, y, u), \qquad (Y, U_1, \cdots, U_{p-1}) = (Y, U)$$

are coordinate systems in the source and target.

(iv) *For a map F given by $(*)$, the submanifold $S_1^q(F)$ is defined in a neighborhood of* 0 *by the equations:*

$$\frac{\partial Y \circ F}{\partial y_j} = 0, \quad 1 \leqq j \leqq t, \quad \text{and} \quad \frac{\partial^i Y \circ F}{\partial x^i} = 0, \quad 1 \leqq i \leqq q.$$

B. *For $p = n + m - 1$,*

(i) *If $(q-1)(p-n+1) \geqq n$, then $S_1^q = \emptyset$, and if $q(p-n+1) > n$, then ${}_T S_1^q = \emptyset$.*

(ii) *If $(q-1)(p-n+1) < n$, then* codim $S_1^q = q(p-n+1)$.

(iii) *If $q(p-n+1) \leqq n$, then $g \in {}_T S_1^q$ and $g \notin S_1^{q+1}$ if and only if it is in the orbit of the $(q+1)$-jet at the origin of a map G given by*

$$
\begin{aligned}
&U \circ G(x, u) = u, \\
(**) \quad &Y_j \circ G(x, u) = \textstyle\sum_{i=0}^{q-1} (x^{i+1}/(i+1)!) u_{j+im} + R_j(x, u), \\
&\qquad\qquad 1 \leqq j \leqq m-1, \\
&Y_m \circ G(x, u) = \textstyle\sum_{i=1}^{q-1} (x^i/i!) u_{im} + x^{q+1}/(q+1)! + S(x, u),
\end{aligned}
$$

where the orders of R_j and S are greater than $q+1$, and

$$(x, u_1, \cdots, u_{n-1}) = (x, u) \quad \text{and} \quad (Y_1, \cdots, Y_m, U_1 \cdots, U_{n-1}) = (Y, U)$$

are coordinate systems in the source and target respectively.

(iv) *For a map G given by $(**)$, the submanifold $S_1^q(G)$ is defined in a neighborhood of the origin by the equations:*

$$\frac{\partial^j Y_k \circ G}{\partial x^j} = 0, \quad 1 \leqq j \leqq q, \quad 1 \leqq k \leqq m.$$

The codimensions of S_1^q are those given by Whitney [9], and the forms for the $(q+1)$-jets have been stated by Haefliger [2].

It is easy to see that if F maps a neighborhood of 0 in A^n into A^p, and if $F(0) = 0$, $F^2(0) \in {}_T S_1$, and $F^2(0) \notin S_1^2$, then in a neighborhood of 0 we can choose coordinates so that either

$$
\begin{aligned}
&U \circ F(x, y, u) = u, \\
&Y \circ F(x, y, u) = \textstyle\sum_{i=1}^{t} \pm y_i^2 + x^2/2 \qquad \text{if} \quad n = p + t,
\end{aligned}
$$

or

$$\begin{aligned} U \circ F(x, u) &= u, \\ Y_j \circ F(x, u) &= xu_j, \quad 1 \leqq j \leqq m-1, \\ Y_m \circ F(x, u) &= x^2/2 \qquad \text{if} \quad p = n + m - 1. \end{aligned}$$

In part A of the theorem, the remainder term is independent of the y-coordinates. This suggests that, at least for $n \geqq p$, to obtain polynomial forms locally for mappings displaying singularities of type S_1^q transversally, it suffices to consider the case $n = p$ for the smallest value of n at which such a mapping exists. For example, with minor variations in the proof of Whitney [8] for the case $n = p = 2$, it can be shown that if F maps a neighborhood of 0 in A^n into A^p with $n = p + t$, and if $F(0) = 0$, $F^3(0) \in {}_TS_1^2$, and $F^3(0) \notin S_1^3$, then in a neighborhood of 0 we can choose coordinates so that

$$\begin{aligned} U_j \circ F(x, y, u) &= u_j, \quad 1 \leqq j \leqq p-1, \\ Y \circ F(x, y, u) &= \textstyle\sum_{i=1}^t \pm y_i^2 + xu_1 + x^3/3!\,. \end{aligned}$$

2. Banal vector bundle homomorphisms

In this section we again consider both the real C^∞ and complex analytic cases and will distinguish between them when necessary. In the complex case, two vector bundles over the same manifold are called equivalent if they are real C^∞ equivalent and if the isomorphisms of the fibres given by the equivalence are complex. Thus in both the real and complex cases a short exact sequence of vector bundles

$$0 \longrightarrow \alpha \longrightarrow \beta \longrightarrow \gamma \longrightarrow 0$$

gives the equivalence of β with $\alpha \oplus \gamma$.

2.1. Let ξ and η be n- and p-vector bundles over a manifold V. In Hom $(\xi, \eta) = \eta \otimes \xi^*$, let S_k be the submanifold of elements of rank equal to min $(n, p) - k$. If $\phi : \xi \longrightarrow \eta$ is a bundle homomorphism, then let $Z_\phi : V \longrightarrow \eta \otimes \xi^*$ be the section that takes $x \in V$ to ϕ_x, where ϕ_x is the homomorphism obtained by restricting ϕ to the fibre of ξ over x, $\phi_x : \xi_x \longrightarrow \eta_x$.

Definition. A homomorphism $\phi : \xi \longrightarrow \eta$ is called *banal* if

(1) rank $\phi_x \geqq$ min $(n, p) - 1$, for all $x \in V$,

(2) $S(\phi) = \{x \in V \mid \text{rank}\, \phi_x = \min(n, p) - 1\}$ is a submanifold of V, and if $x \in S(\phi)$, then dim $((Z_\phi)_*(V_x) + (S_1)_{Z_\phi(x)}) \geqq$ dim $S_1 + 1$, where V_x is the tangent space to V at x and $(S_1)_{Z_\phi(x)}$ is the tangent space to S_1 at $Z_\phi(x)$.

A special case of a banal homomorphism is that of a homomorphism ϕ which satisfies condition (1) above and has the property that Z_ϕ is transversal to S_1.

LEMMA 2.1. (i) *Let $\phi : \xi \to \eta$ be a banal homomorphism such that $S(\phi)$ has codimension 1 in V; then there exist vector bundles $\hat{\xi}$ and $\hat{\eta}$ and homomorphisms ϕ_1, ϕ_2, σ, τ such that*

$$\begin{array}{ccc} \hat{\xi} & \xrightarrow{\phi_1} & \eta \\ \sigma \uparrow & \overset{\phi}{\nearrow} & \uparrow \tau \\ \xi & \xrightarrow[\phi_2]{} & \hat{\eta} \end{array}$$

commutes, and $S(\sigma) = S(\tau) = S(\phi)$, and $\operatorname{rank} \phi_1 = \operatorname{rank} \phi_2 = \min(n, p)$.

(ii) *Denote by a prime restriction to $S(\phi)$. Let λ be the normal line bundle of $S(\phi)$ in V. Then*

(a) $$\ker \sigma' = \ker \phi', \quad \textit{and} \quad \ker \tau' = \lambda^* \otimes \operatorname{coker} \phi'.$$

Let ζ be defined by the exactness of

$$0 \to \ker \phi' \to \xi' \to \zeta \to 0, \qquad 0 \to \zeta \to \eta' \to \operatorname{coker} \phi' \to 0.$$

Then the following sequences are also exact:

(b) $$0 \to \zeta \to \hat{\xi}' \to \lambda \otimes \ker \phi' \to 0,$$

(c) $$0 \to \lambda^* \otimes \operatorname{coker} \phi' \to \hat{\eta}' \to \zeta \to 0.$$

Remark. This lemma is essentially a special case of [4, Theorem 3.2]. There however the construction of the new bundles may be a little obscure since it is done not on V but on $\hat{V}$, a manifold obtained from V by sigma process; also the new bundles are compared with the original ones lifted to $\hat{V}$. Therefore we repeat the proof in this simplified setting. If $n = p$, and if $\operatorname{rank} \phi_x \geqq n - 1$ and Z_ϕ were transversal to S_1, this lemma would be a special case of the above-mentioned theorem. At present the author does not know the appropriate full generalization.

Proof. It suffices to prove the lemma in case $n \leqq p$. The other case can be obtained from this one by duality.

We will work with coordinate bundles representing ξ and η. Suppose then that we are given an open covering of V by coordinate neighborhoods $\{U_\alpha, \alpha \in \mathfrak{A}\}$, $\mathfrak{A}$ some index set, such that ϕ is defined by the diagram:

$$\begin{array}{ccc} \pi_\xi^{-1}(U_\alpha) & \xrightarrow{\phi} & \pi_\eta^{-1}(U_\alpha) \\ \downarrow & & \downarrow \\ U_\alpha \times A^n & \xrightarrow{\phi_\alpha} & U_\alpha \times A^p \end{array}$$

where the vertical arrows are the coordinate maps for the coordinate bundles representing ξ and η. It is no restriction to assume, for $x \in U_\alpha$ and t a column

n-vector, that $\phi_\alpha(x, t) = (x, H_\alpha(x) \cdot t)$, where $H_\alpha(x)$ is a $p \times n$ matrix of the form

$$\begin{pmatrix} I_{n-1} & 0 \\ 0 & h_\alpha(x) \end{pmatrix},$$

where I_{n-1} is the $n - 1 \times n - 1$ identity matrix and $h_\alpha(x)$ is a $(p - n + 1)$-column vector.

Let $\mathfrak{A}_1 = \mathfrak{A} - \mathfrak{A}_0$, where $\alpha \in \mathfrak{A}_0$ if and only if $U_\alpha \cap S(\phi) = \emptyset$. By condition (2) for banality of ϕ, for any $x \in S(\phi)$, $dh_\alpha(x) \neq 0$. We assume that for all $\alpha \in \mathfrak{A}_1$, U_α are chosen small enough so that at least one of the differentials of dh_α is nonzero throughout U_α. We may, without loss of generality, assume that ${}^t h_\alpha = (x_\alpha, 0, \cdots, 0)$. We may further assume that in U_α for $\alpha \in \mathfrak{A}_0$, ${}^t h_\alpha = (1, 0, \cdots, 0)$; we let $x_\alpha = 1$ for $\alpha \in \mathfrak{A}_0$. Thus in each U_α, the defining equation for $S(\phi) \cap U_\alpha$ is $x_\alpha = 0$.

Let $d_1 = n$ and $d_2 = p$. We define maps N_α^i of U_α into the $d_i \times d_i$ matrices by

$$N_\alpha^i = \begin{pmatrix} I_{n-1} & 0 \\ 0 & x_\alpha I_{(d_i - n+1)} \end{pmatrix}, \qquad i = 1, 2.$$

Let ${}^t K_\alpha$ be the constant map which takes all of U_α into the $n \times p$ matrix $(I_n \, 0)$. Thus on U_α

$$H_\alpha = K_\alpha N_\alpha^1 = N_\alpha^2 K_\alpha . \tag{1}$$

Suppose $E_{\alpha\beta}$ and $F_{\alpha\beta}$ are the transition functions for the coordinate bundles we have taken to represent ξ and η. Then

$$H_\alpha E_{\alpha\beta} = F_{\alpha\beta} H_\beta . \tag{2}$$

If we drop all the indices and write the transition functions in blocks,

$$E = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \text{and} \quad F = \begin{pmatrix} G & P \\ J & M \end{pmatrix},$$

where A and G are $n - 1 \times n - 1$. We see that on $S(\phi)$, B and J vanish identically. Thus we may write

$$E_{\alpha\beta} = \begin{pmatrix} A_{\alpha\beta} & x_\beta \hat{B}_{\alpha\beta} \\ C_{\alpha\beta} & D_{\alpha\beta} \end{pmatrix} \quad \text{and} \quad F_{\alpha\beta} = \begin{pmatrix} G_{\alpha\beta} & P_{\alpha\beta} \\ x_\alpha \hat{J}_{\alpha\beta} & M_{\alpha\beta} \end{pmatrix} \quad \text{in} \quad U_\alpha \cap U_\beta . \tag{3}$$

Note that restricted to $S(\phi)$ the following are transition functions for the indicated bundles: $A_{\alpha\beta}$ for ζ, $D_{\alpha\beta}$ for $\ker \phi'$, $M_{\alpha\beta}$ for $\operatorname{coker} \phi'$.

Let $\hat{\xi}$ and $\hat{\eta}$ be represented by coordinate bundles which are defined by their transition functions

$$\hat{E}_{\alpha\beta} = \begin{pmatrix} A_{\alpha\beta} & \hat{B}_{\alpha\beta} \\ x_\alpha C_{\alpha\beta} & L_{\alpha\beta} D_{\alpha\beta} \end{pmatrix} \quad \text{and} \quad \hat{F}_{\alpha\beta} = \begin{pmatrix} G_{\alpha\beta} & x_\beta P_{\alpha\beta} \\ \hat{J}_{\alpha\beta} & L_{\beta\alpha} M_{\alpha\beta} \end{pmatrix} \tag{4}$$

on $U_\alpha \cap U_\beta$, respectively, where $L_{\alpha\beta} = (x_\alpha/x_\beta)$. The $L_{\alpha\beta} \mid S(\phi)$ are the transition functions for a coordinate bundle representing the normal bundle to $S(\phi)$ in V, λ. We see that the functions defined by (4) are the transition functions of coordinate bundles, for, suppressing as many indices as possible, we have formally from (3) and (4)

$$\hat{E} = (N^1)(E)(N^1)^{-1} \quad \text{and} \quad \hat{F} = (N^2)^{-1}(F)(N^2). \tag{5}$$

Also from (1) and (2) we have

$$HE = FH = KN^1E = FKN^1 = N^2KE = FN^2K. \tag{6}$$

To define the homomorphisms it suffices to do so locally. Both ϕ_i, $i = 1, 2$, are defined by

$$(\phi_i)_\alpha : U_\alpha \times A^n \to U_\alpha \times A^p : (x, t) \to (x, K_\alpha(x)\cdot t).$$

The last two pairs of equal terms of (6) together with the defining equations (5) show that we have well-defined homomorphisms between appropriate bundles. Both of the thus-defined homomorphisms have rank n. The homomorphisms σ and τ are defined for $i = 1, 2$, respectively by

$$U_\alpha \times A^{d_i} \to U_\alpha \times A^{d_i} : (x, t) \to (x, N^i_\alpha(x)t).$$

That these local homomorphisms piece together correctly is immediate from (5). The commutativity of the diagram of conclusion (i) is just a restatement of (1), and that $S(\sigma) = S(\tau) = S(\phi)$ is trivial from the definition of σ and τ. All of the parts of conclusion (ii) follow by inspection of (4).

Remark 1. If in the preceding lemma, $n = p$, then $\hat{\xi}$ is equivalent to η, and $\hat{\eta}$ is equivalent to ξ. If ϕ, ξ, η, and V are holomorphic, then the equivalences are also holomorphic.

Remark 2. The $\hat{\xi}$ of the lemma is unique for $n \leqq p$. In particular let $G_n(\eta)$ be the bundle associated with η with fibre the Grassmann manifold of n-planes in A^p, $G_n(A^p)$, and let $\Gamma_n(\eta)$ be the n-vector bundle over $G_n(\eta)$ whose points are pairs (X, v) where $X \in G_n(\eta)$ and $v \in X$. Suppose that

$$\psi : V - S(\phi) \to G_n(\eta) : x \to (\text{range of } \phi_x).$$

The existence of $\hat{\xi}$ yields an extension $\hat{\psi} : V \to G_n(\eta)$, a section in $G_n(\eta)$. Let $\gamma = \hat{\psi}^{-1}(\Gamma_n(\eta))$; clearly γ is equivalent to $\hat{\xi}$. In the obvious way γ is a subbundle of η, and $\phi : \xi \to \eta$ can be factored through γ, i.e., there is a map $\theta : \xi \to \gamma$ which satisfies the hypothesis of the lemma such that $\phi = i \circ \theta$, where i is the injection of γ in η. Since $\hat{\psi}$ is unique, so is γ.

Remark 3. On $S(\phi)$ we have a map analogous to ψ, say $\psi' : S(\phi) \to G_{n-1}(\eta)$ which takes a point x to the range of ϕ_x. That $\psi'^{-1}(\Gamma_{n-1}(\eta))$ is equivalent to ζ is obvious since the map of ξ' into $\psi'^{-1}(\Gamma_{n-1}(\eta))$ is onto and has kernel = $\ker \phi'$.

2.2 *Notation.* Given a map f, of X into Y, X and Y manifolds, we let $f_\#$ be the Gysin homomorphism from the cohomology of X into that of Y.

Here the coefficients for cohomology are Z_2 in the real case and Z in the complex case.

For a vector bundle α, we let $c(\alpha)$ be the total Stiefel-Whitney class (mod 2) in the real case and the total Chern class in the complex case.

THEOREM 2.2 (see [4, §3.3]). *Let ϕ be a banal homomorphism of an n-vector bundle ξ into a p-vector bundle η, $n \leqq p$, both bundles over a manifold V. Suppose that $S(\phi)$ has codimension 1 in V. If ζ and $\hat{\xi}$ are as in the preceding lemma, then*

$$c(\xi) = c(\hat{\xi}) - j_{\#} c(\zeta),$$

where j is the inclusion of $S(\phi)$ in V.

This theorem is a consequence of the Atiyah-Hirzebruch-Grothendieck-Riemann-Roch Theorem [1].

LEMMA (Porteous, [6]). *Suppose that α and β are vector bundles of the same rank over a manifold X, and let ψ be a homomorphism from α to β such that*

(a) *Except on a closed submanifold $j : Y \subset X$ of codimension 1, ψ is an isomorphism.*

(b) *$\psi' = \psi \mid (\alpha \mid Y)$ is of constant rank.*

(c) *For each $P \in X$, if x is the germ of a function defining Y at P, and if s is a germ of a section in β at P, then xs is in the image under ψ of a germ of a section in α.*

If $\mathfrak{P}$ and $\mathfrak{P}'$ are the corresponding sheaf homomorphisms, then

$$\operatorname{coker} \mathfrak{P} = (\operatorname{coker} \mathfrak{P}')^0, \quad \textit{and} \quad \operatorname{coker} \psi' = \lambda \otimes \ker \psi',$$

where $(\operatorname{coker} \mathfrak{P}')^0$ *is the sheaf* coker $\mathfrak{P}'$ *extended by zero to $X - Y$, and λ is the normal line bundle of Y in X.*

Applying the AHGRR theorem [1, Theorem 3.1, Theorem 5.1, §6] to the sheaf conclusion of the Porteous lemma we have

$$c(\beta - \alpha) = 1 + j_{\#}\left(\frac{1}{v}\left\{\frac{c(\operatorname{coker} \psi')}{c(\lambda^* \otimes \operatorname{coker} \psi')} - 1\right\}\right), \tag{1}$$

where $c(\lambda) = 1 + v$, and λ^* is the line bundle dual to λ. If we denote by α' and β' the restrictions to Y of α and β, we have the exact sequence of bundles:

$$0 \to \ker \psi' \to \alpha' \xrightarrow{\psi'} \beta' \to \operatorname{coker} \psi' \to 0.$$

Let $\gamma = \operatorname{coker}(\ker \psi' \to \alpha') = \ker(\beta' \to \operatorname{coker} \psi')$. Then substituting in (1) we obtain

$$c(\beta - \alpha) = 1 + j_{\#}(c(\gamma - \alpha')\{(1/v)(c(\lambda \otimes \ker \psi') - c(\ker \psi'))\}). \tag{2}$$

Multiplying both sides of (2) by $c(\alpha)$ and using the fact that

$$c(\alpha) j_{\#}(\ ) = j_{\#}[c(\alpha')(\ )]$$

we have proved

$$(3) \qquad c(\beta) = c(\alpha) + j_{\#}\left(c(\gamma)\left\{\frac{c(\lambda \otimes \ker \psi') - c(\ker \psi')}{v}\right\}\right).$$

If further $\ker \psi'$ were a line bundle, then $c(\lambda \otimes \ker \psi') - c(\ker \psi') = v$, and so we have

$$c(\beta) = c(\alpha) + j_{\#} c(\gamma).$$

The verification that the bundles ξ, $\hat{\xi}$ and the homomorphism σ satisfy the conditions on α, β, ψ of the lemma is immediate from the definition of $\hat{\xi}$ and σ (see preceding section).

Using Remarks 2 and 3 above, we have in the situation of Theorem 2.2

$$c(\xi) = \hat{\psi}^* c(\Gamma_n(\eta)) - j_{\#} \psi'^* c(\Gamma_{n-1}(\eta)).$$

2.3. Let V and M be manifolds of dimensions n and p respectively, and let f map V in M. Suppose that $f_* : T(V) \to f^{-1}T(M)$ is banal, where $T(V)$ and $T(M)$ are the tangent bundles. Notice that the dual map,

$$f^* : f^{-1}(T(M))^* \to T^*(V)$$

is also banal, and that $S(f^*) = S(f_*)$. Call this singular set simply $S(f)$. We apply Lemma 2.1 and Theorem 2.2 to f_* and f^* when $n \leqq p$ and $n \geqq p$ respectively. By Remark 2 of 2.1 we have maps

$$\hat{T}_{f_*} : V \to G_n(T(M)) \quad \text{and} \quad \hat{T}_{f^*} : V \to G_p(T^*(V)),$$

which map a point $x \in V$ to the range of $(f_*)_x$ and $(f^*)_x$. Here the map $\hat{T}_{f_*}$ is the composite of the map given by the remark into $G_n(f^{-1}(T(M)))$ followed by the obvious map into $G_n(T(M))$. If we let $(f_*)'$ and $(f^*)'$ denote the restrictions to $S(f)$, we have

$$T_{(f_*)'} : S(f) \to G_{n-1}(T(M)) \quad \text{and} \quad T_{(f^*)'} : S(f) \to G_{p-1}(T^*(V)).$$

Letting j be the inclusion of $S(f)$ in V and assuming that $S(f)$ has codimension 1 in V, we have by Theorem 2.2

(a) If $n \leqq p$, $c(V) = (\hat{T}_{f_*})^* c(\Gamma_n(T(M))) - j_{\#}(T_{(f_*)'})^* c(\Gamma_{n-1}(T(M)))$.

(b) If $n \geqq p$,

$$f^* c(T^*(M)) = (\hat{T}_{f^*})^* c(\Gamma_p(T^*V)) - j_{\#}(T_{(f^*)'})^* c(\Gamma_{p-1}(T^*(V))).$$

Since we will only apply formula (a) above, we restrict our attention now to the case $n \leqq p$. Let $g = f \mid S(f)$, and assume that g_* is again banal with $S(g)$ of codimension 1 in $S(f)$. Since $T_{g_*} \mid S(f) - S(g)$ agrees with $T_{(f_*)'} \mid S(f) - S(g)$, $T_{(f_*)'} = \hat{T}_{g_*}$. Thus we get

(c) $c(S(f)) = (T_{(f_*)'})^* c(\Gamma_{n-1}(T(M))) - k_{\#}(T_{(g_*)'})^* c(\Gamma_{n-2}(T(M)))$,

where k is the inclusion of $S(g)$ in $S(f)$. Thus (a) and (c) collapse to give

(d) $j_{\#}c(S(f)) + c(V)$

$$= (\hat{T}_{f_*})^* c(\Gamma_n(T(M))) - j_{\#} k_{\#}(T_{(g_*)'})^* c(\Gamma_{n-2}(T(M))).$$

If we were so fortunate that now $f \mid S(g) = h$ had the property that h_* were banal and $S(h)$ had codimension 1 in $S(g)$, we could apply the same argument again. We would, if the banal and codimension 1 conditions were satisfied every time we restricted the map of a singular set to its singular set, eventually obtain

$$\sum_i (j_i)_{\#} c(S(f_i)) + c(V) = (\hat{T}_{f_*})^* c(\Gamma_n(T(M))),$$

where $f_i = f \mid S(f_{i-1})$, $f_0 = f$, and j_i is the inclusion of $S(f_i)$ in V.

A case in which this simple situation does in fact occur is given by the following theorem.

THEOREM 2.3. *Let V and M be n-manifolds, and let f be a map of V in M such that $S_i(f) = \emptyset$ for $i > 1$. If $J^q(f)$ is transversal to the singularities S_1^q for $q = 1, \cdots, n$, then*

$$c(V) = f^* c(M) - \sum_{q=1}^{n} (j_q)_{\#} c(S_1^q(f)),$$

where j_q is the inclusion of $S_1^q(f)$ in V and $S_1^{q+1}(f) = S_1(f \mid S_1^q(f))$.

Proof. Since in this case $G_n(T(M)) = M$ and $\Gamma_n(T(M)) = T(M)$, we have

$$\hat{T}_{f_*} = f \quad \text{and} \quad (\hat{T}_{f_*})^* c(\Gamma_n(T(M))) = f^* c(M).$$

To complete the proof it suffices to show that if $f^i = f \mid S_1^i(f)$, then f_*^i is banal, since we already know that $S(f^i) = S_1^{i+1}(f)$ is of codimension 1 in $S_1^i(f)$ or is empty. But that f_*^i is banal is trivial since the hypotheses that $S_i(f) = \emptyset$ for $i > 1$ and that $J^q(f)$ are transversal to S_1^q imply respectively that conditions (1) and (2) of the definition of banal homomorphism are satisfied for f_*^i.

3. Proof of Theorem 1.

Since the proofs of parts A and B are similar, we will just prove the theorem in case $n \geqq p$, i.e., part A. In J^1, let W^1 be a neighborhood of the 1-jet of the mapping given by

$$U \circ F(x, y, u) = u, \qquad Y \circ F(x, y, u) = 0,$$

where the notation is as in the statement of the theorem. Further $f \in W^1$ if and only if $((\partial U_i(P_f)/\partial u_j)(0))$ is nonsingular, $1 \leqq i, j \leqq p - 1$.

Let $\mathfrak{W}$ be the set of all germs at the origin of maps F of A^n in A^p taking the origin into the origin such that the germ of F is in $\mathfrak{W}$ if and only if $F^1(0) \in W^1$. In the following we will use the same notation for the germ of a mapping and the mapping itself; this abuse of notation should lead to no confusion.

We define a map θ of $\mathfrak{W}$ into itself by giving for each $F \in \mathfrak{W}$ a diffeomorphism of a neighborhood of the origin in the source leaving the origin fixed; $\boldsymbol{\theta}F$ is defined by composing the diffeomorphism with F. Such a map, $\boldsymbol{\theta}$, induces a map of $W^q = \pi_{q,1}^{-1}(W^1)$ into itself, say θ^q, defined by the equation $\theta^q f = (\boldsymbol{\theta}P_f)^q(0)$. If $F \in \mathfrak{W}$, then F has the form

$$(1)\quad U \circ F(x, y, u) = U^*(x, y, u), \qquad Y \circ F(x, y, u) = Y^*(x, y, u),$$

where $((\partial U_i^*/\partial u_j)(0))$ is nonsingular. By virtue of the nonsingularity condition of (1) we can define a diffeomorphism of a neighborhood of 0 in A^n into itself which takes a point with coordinates (x, y, u) into one with coordinates $(x, y, C(x, y, u))$, where $U^*(x, y, C(x, y, u)) = u$. We let $\boldsymbol{\theta}F$ be the composition of F with this diffeomorphism:

$$(2)\quad U \circ (\boldsymbol{\theta}F)(x, y, u) = u, \qquad Y \circ (\boldsymbol{\theta}F)(x, y, u) = Y^*(x, y, C(x, y, u)).$$

Note that whenever $(\partial U_i^*/\partial u_j)$ is nonsingular, the partials of C with respect to x, y, u depend only on the partials of U^*.

Given a map F from A^n to A^p, the coordinates of the jet $F^q(0)$ are given by the partial derivatives of orders up to and including the q^{th} of $U \circ F$ and $Y \circ F$ with respect to x, y, u at 0. These coordinates will be denoted by the corresponding partial derivative symbols, e.g.,

$$\frac{\partial^j Y}{\partial x^j}(F^q(0)) \quad \text{means} \quad \frac{\partial^j (Y \circ F)}{\partial x^j}(0), \qquad \text{for} \quad j \leqq q.$$

For $f \in W^q$, $q \geqq 2$, let $K(f) = (\partial^2 Y/\partial y_i\, \partial y_{i'})(\theta^q f)$, and for each $j = 2, \cdots, q$, let $L_j(f) = (\partial^r Y/\partial x^{r-1} \partial u_k)(\theta^q f)$ if $j \leqq p$, and the zero matrix otherwise, and let

$$M_j(f) = \begin{pmatrix} K(f) & \dfrac{\partial^2 Y}{\partial y_i\, \partial u_k}(\theta^q f) \\ \dfrac{\partial^r Y}{\partial x^{r-1} \partial y_{i'}} & L_j(f) \end{pmatrix},$$

if $j \leqq p$, and the zero matrix otherwise. Here the indices range as follows: $2 \leqq r \leqq j$, $1 \leqq k \leqq j - 1$, $1 \leqq i, i' \leqq t$.

Define open sets $N^q \subset W^q$ as follows: $N^1 = W^1$, and for $q \geqq 2$

$$N^q = \{f \in W^q \mid K(f), L_j(f), \text{ and } M_j(f) \text{ are nonsingular for all } j = 2, \cdots, q\}.$$

Clearly if $N^q \neq \emptyset$, $\pi_{q,r}(N^q) = N^r$, for $r \leqq q$. Let

$$T^q = \left\{ f \in N^q \;\middle|\; \frac{\partial Y}{\partial y_i}(\theta^q f) = 0, 1 \leqq i \leqq t; \text{ and } \frac{\partial^j Y}{\partial x^j}(\theta^q f) = 0, 1 \leqq j \leqq q \right\}.$$

LEMMA 3.1. *T^q is a submanifold of N^q and* codim $T^q = n - p + q$ *if $q \leqq p$.*

Proof. If $q > p$, $N^q = \emptyset$, and there is nothing to prove. Suppose that $q \leqq p$. It suffices to prove the following: Let $F = P_f$ for $f \in N^q$; then

$$\frac{\partial Y(\boldsymbol{\theta} F)}{\partial y_i}(0) \equiv \frac{\partial Y \circ F}{\partial y_i}(0), \qquad 1 \leqq i \leqq t,$$
$$\frac{\partial^j Y(\boldsymbol{\theta} F)}{\partial x^j}(0) \equiv \frac{\partial^j Y \circ F}{\partial x^j}(0), \qquad 1 \leqq j \leqq q,$$

where congruence means equality modulo a function of the partials of $Y \circ F$ other than those listed and of the partials of $U \circ F$. The proof of this is trivial using (2). In this proof no use is made of the fact that we are working inside N^q rather than W^q. The restriction to N^q is for later convenience, since we will show that S_1^q is the orbit of T^q. If we had defined T^q in W^q by the same equations, although T^q would be submanifolds of W^q, the T^q would contain points not in S_1^q.

If $F \in \mathfrak{W}$ and $F^q(0) \in N^q$, then for P sufficiently close to 0, $F^q(P) \in N^q$, and $\theta^q F^q(P) = (\boldsymbol{\theta} F)^q(P)$, where $F^q(P)$ is the q-jet at 0 of the map $T_{-F(P)} \circ F \circ T_P$, where $T_{-F(P)}$ is the translation in A^p taking $F(P)$ to 0, and T_P is the translation taking 0 to P in A^n. Thus in a neighborhood of 0, the equations of $T^q(F)$ are

$$\frac{\partial Y(\boldsymbol{\theta} F)}{\partial y_i} = 0, \quad 1 \leqq i \leqq t \quad \text{and} \quad \frac{\partial^j Y(\boldsymbol{\theta} F)}{\partial x^j} = 0, \quad 1 \leqq j \leqq q.$$

If further $F^{q+1} \in N^{q+1}$ and $F = \boldsymbol{\theta} F$ in a neighborhood of 0, we can choose a neighborhood of 0 so that $M_j(F^j)$ has rank $(j - 1 + t)$ there, for $j = 2, \cdots, q + 1$. Thus in this neighborhood, ${}_T T^q(F)$ is defined by the equations defining $T^q(F)$.

Suppose $F \in \mathfrak{W}$ and $F^q(0) \in N^q$, $q \geqq 2$, and $F = \boldsymbol{\theta} F$; then the equations for F are

$$U \circ F(x, y, u) = u, \qquad Y \circ F(x, y, u) = Y^*(x, y, u).$$

Expanding Y^* in powers of x yields

$$(3) \qquad \begin{aligned} &Y^*(x, y, u) \\ &\quad = G_0(y, u) + \textstyle\sum_{i=1}^q (x^i/i!)(a_i + G_i(y, u)) + x^{q+1} R(x, y, u), \end{aligned}$$

where $a_i = (\partial^i Y^*/\partial x^i)(0)$. Since we have assumed that $F^q(0) \in N^q$, i.e., that $N^q \neq \emptyset$, we have $q \leqq p$, and that the matrices K, L_j, and M_j are nonsingular for $j = 2, \cdots, q$. In the notation of (3),

$$L_j(F^q(0)) = \left(\frac{\partial G_r}{\partial u_k}\right)(0); \qquad 1 \leqq r, k \leqq j - 1.$$

Thus we may define new coordinates in the source and target by

$$(4) \qquad \begin{aligned} \tilde{u}_i &= G_i(0, u), & \tilde{U}_i &= G_i(0, U), \\ \tilde{u}_j &= u_j, & \tilde{U}_j &= U_j, \\ \tilde{y}_k &= y_k, \qquad \text{and} & \tilde{Y} &= Y - Y^*(0, 0, U), \\ \tilde{x} &= x, \end{aligned}$$
$$1 \leqq i \leqq q - 1; \quad q \leqq j \leqq p - 1; \quad 1 \leqq k \leqq t.$$

By letting $H_i(\tilde{y}, \tilde{u}) = G_i(y, u) - G_i(0, u), i = 0, \cdots, q-1$, the equations defining the mapping F become, after dropping the tildes of the substitutions (4),

$$U \circ F(x, y, u) = u,$$

$$Y \circ F(x, y, u) = H_0(y, u) + \textstyle\sum_{i=1}^{q-1} (x^i/i!)(a_i + u_i + H_i(y, u)) + (x^q/q!)(a_q + S(x, y, u)),$$

where the order of S is greater than zero. Define b_i and J by

$$\textstyle\sum_{i=1}^{t} b_i y_i + J(x, y, u) = H_0(y, u) + \sum_{i=1}^{q-1} (x^i/i!)H_i(y, u) + (x^q/q!)(S(x, y, u) - S(x, 0, u));$$

b_i are constants, order $J \geqq 2$, and $J(x, 0, u) = 0$. Letting $R(x, u) = S(x, 0, u)$ we have

$$Y \circ F(x, y, u) = \textstyle\sum_{i=1}^{t} b_i y_i + J(x, y, u) + \sum_{i=1}^{q-1} (x^i/i!)(a_i + u_i) + (x^q/q!)(a_q + R(x, u)).$$

Note that

$$K(F^q(0)) = \left(\frac{\partial^2 G_0}{\partial y_i\, \partial y_j}\right)(0) = \left(\frac{\partial^2 H_0}{\partial y_i\, \partial y_j}\right)(0) = \left(\frac{\partial^2 J}{\partial y_i\, \partial y_j}\right)(0)$$

is nonsingular.

LEMMA 3.2. *Let A be a function of $p+t$ variables $(z_1, \cdots, z_t, w_1, \cdots, w_p)$ such that $(A_{z_i z_j})(0)$ is nonsingular. Then there are functions $\tilde{z}_i$, $i = 1, \cdots, t$ defined in a neighborhood of 0 such that $(\tilde{z}, w)$ form a coordinate system there, and such that*

$$A(z, w) = h(w) + \textstyle\sum_{i=1}^{t} b_i g_i(\tilde{z}, w) + \sum_{i=1}^{t} \pm \tilde{z}_i^2,$$

where $g_i(0, w) = 0$ and $((g_i)_{\tilde{z}_j})$ is nonsingular at 0 and $b_i = A_{z_i}(0)$.

Proof. Write $A(z, w) = f(w) + \sum_{i=1}^{t} b_i z_i + J(z, w)$, where $f(w) = A(0, w)$ and $b_i = A_{z_i}(0)$. Let $J_{z_i} = J_i$; $J_i(0) = 0$, and $((J_i)_{z_j})(0)$ is nonsingular. Thus $J_i(z, w) = 0$ can be solved for z in terms of w, say $z_i = \phi_i(w)$, with $\phi_i(0) = 0$, is the solution of this system. Set $z_i = z_i' + \phi_i(w)$. Thus

$$\begin{aligned} A(z, w) &= f(w) + \textstyle\sum_{i=1}^{t} b_i \phi_i(w) + J(\phi(w), w) \\ &\qquad + \textstyle\sum_{i=1}^{t} b_i z_1' + [J(z' + \phi(w)) - J(\phi(w), w)] \\ &= h(w) + \textstyle\sum_{i=1}^{t} b_i z_i' + K(z', w). \end{aligned}$$

$K(0, w) = 0$, $K_{z_i'}(0, w) = J_i(\phi(w), w) = 0$, and

$$(K_{z_i' z_j'})(0, w) = (J_{z_i z_j})(\phi(w), w)$$

is nonsingular for sufficiently small w. To this function K we apply the theorem of Morse [5]. That is, there are new coordinates $(\tilde{z}, w)$ such that $((\tilde{z}_i)_{z'_j})(0)$ is nonsingular and $K(z', w) = \sum_{i=1}^t \pm\tilde{z}_i^2$.

Applying Lemma 3.2 to the function $Y \circ F$ we have, dropping the tildes,

$$Y \circ F(x, y, u) = \sum_{i=1}^t b_i\, g_i(x, y, u) + \sum_{i=1}^t \pm y_i^2 + h(x, u)$$
$$+ \sum_{i=1}^{q-1} (x^i/i!)(a_i + u_i) + (x^q/q!)(a_q + R(x, u)).$$

Let $h(x, u) = \sum_{i=1}^{q-1} (x^i/i!)h_i(u) + (x^q/q!)h_q(x, u)$. Since $L_q(F^q(0))$ is nonsingular, we may take as new coordinates

$$\tilde{u}_i = u_i + h_i(u) - h_i(0), \qquad \tilde{U}_i = U_i + h_i(U) - h_i(0),$$
$$i = 1, \cdots, q-1,$$
$$\tilde{Y} = Y - Y^*(0, 0, U);$$

all others remain the same. This yields finally, by letting $k_i(x, y, \tilde{u}) = g_i(x, y, u)$ and dropping the tildes,

$$U \circ F(x, y, u) = u,$$
$$(5) \quad Y \circ F(x, y, u) = \sum_{i=1}^t b_i\, k_i(x, y, u) + \sum_{i=1}^t \pm y_i^2$$
$$+ \sum_{i=1}^{q-1} (x^i/i!)(c_i + u_i) + (x^q/q!)(c_q + S(x, u)),$$

where $c_i = a_i + h_i(0)$, $i = 1, \cdots, q$ and ord $S \geqq 1$. Note that $b_i = 0$ for all $i = 1, \cdots, t$ if and only if all $(\partial Y \circ F/\partial y_j)(0) = 0$, $j = 1, \cdots, t$.

The transformations used to obtain (5) define a map of the set of θF for $F \in \mathfrak{W}$ such that $F^q(0) \in N^q$ into itself. We call ψ, the composition of θ followed by this map; the induced map of N^q into itself we call ψ^q. The equations for the germ of ψF are given by (5). Notice that for $F = \psi F$, $F^q(0) \in T^q$ f and only if

$$(6) \qquad b_i = 0 \quad \text{and} \quad c_j = 0; \qquad 1 \leqq i \leqq t, \ \ 1 \leqq j \leqq q.$$

Thus we see that T^q is contained in the orbit of the q-jets of mappings (*) given in the statement A of the theorem.

LEMMA 3.3. $S_1^q \cap N^q = T^q$.

Proof. The proof goes by induction on q and is trivial for $q = 1$. Suppose the lemma proved up to but not including q. As usual we set $n = p + t$. We may assume that $q \leqq p$. Let $f \in \psi^q N^q$ and $F = P_f$. Then F has the form (5). We must show that $f \in S_1^q$ if and only if (6) holds. By our induction assumption $F^{q-1}(0) \in S_1^{q-1}$ if and only if $b_i = 0$, $1 \leqq i \leqq t$, and $c_j = 0$, $1 \leqq j \leqq q-1$. The equations for $S_1^{q-1}(F)$ in a small neighborhood of 0 are

$$\frac{\partial Y \circ F}{\partial y_i} = 0, \quad 1 \leqq i \leqq t \quad \text{and} \quad \frac{\partial^j Y \circ F}{\partial x^j} = 0, \quad 1 \leqq j \leqq q-1.$$

These equations become, in this case,

$$y_i = 0, \quad 1 \leqq i \leqq t \quad \text{and} \quad \sum_{i=j}^{q-1} (x^{i-j}/(i-j)!)u_i + (x^{q-j}/(q-j)!)c_q = 0, \quad 1 \leqq j \leqq q-1.$$

These equations can easily be solved for the u's in terms of x, and $S_1^{q-1}(F)$ is defined in a small neighborhood of the origin by

$$(7) \qquad y_i = 0, \quad 1 \leqq i \leqq t \quad \text{and} \quad u_j = \phi_j(x), \quad 1 \leqq j \leqq q-1.$$

For convenience let $u_j = v_j$ and $U_j = V_j$ for $j = q, \cdots, p-1$. Restricting F to $S_1^{q-1}(F)$ in a neighborhood of 0 gives

$$\begin{aligned} U \circ F(x, 0, \phi(x), v) &= \phi(x), \\ V \circ F(x, 0, \phi(x), v) &= v, \\ Y \circ F(x, 0, \phi(x), v) &= \hat{Y}(x, v). \end{aligned}$$

At 0 this map has rank $(p-q)$, i.e., $F^q(0) \in S_1^q$ if and only if

$$\frac{\partial \phi}{\partial x}(0) = 0 \quad \text{and} \quad \frac{\partial \hat{Y}}{\partial x}(0) = 0.$$

Since

$$\frac{\partial \hat{Y}}{\partial x}(0) = \frac{\partial Y \circ F}{\partial x}(0) + \sum_{i=1}^{q-1} \frac{\partial Y \circ F}{\partial u_i}(0) \cdot \frac{\partial \phi_i}{\partial x}(0)$$

and $(\partial Y \circ F/\partial x)(0) = 0$ since $F^1(0) \in S_1$, we see that $F^q(0) \in S_1^q$ if and only if $(\partial \phi/\partial x)(0) = 0$. Further we know that on $S_1^{q-1}(F)$,

$$\frac{\partial^j Y \circ F}{\partial x^j}(x, 0, \phi(x), v) = 0, \qquad j = 1, \cdots, q-1.$$

Thus on $S_1^{q-1}(F)$

$$0 = \frac{\partial}{\partial x}\left(\frac{\partial^j Y \circ F}{\partial x^j}(x, 0, \phi(x), v)\right) = \frac{\partial^{j+1} Y \circ F}{\partial x^{j+1}}(x, 0, \phi(x), v) + \sum_{i=1}^{q-1} \frac{\partial^{j+1} Y \circ F}{\partial x^j \partial u_i}(x, 0, \phi(x), v)\frac{\partial \phi_i}{\partial x}(x)$$

for $j = 1, \cdots, q-1$. Since by assumption $L_q(F^q(0))$ is nonsingular, we see that $(\partial \phi/\partial x)(0) = 0$ if and only if $c_q = 0$.

LEMMA 3.4. *Suppose* $F^{q+1}(0) \in {}_T T^q$ *but* $F^{q+1}(0) \notin T^{q+1}$; *then we can choose coordinates at the respective origins so that*

$$\begin{aligned} U \circ F(x, y, u) &= u, \\ (8) \quad Y \circ F(x, y, u) &= \textstyle\sum_{i=1}^{t} \pm y_i^2 + \sum_{i=1}^{q-1} (x^i/i!)u_i \\ &\quad + (x^{q+1}/(q+1)!) + R(x, u), \end{aligned}$$

where ord $R > q + 1$.

Proof. Since $F^q(0) \in T^q$, we may assume that F has the form given by (5) with (6) holding. That is,

$$
\begin{aligned}
& U \circ F(x, y, u) = u, \\
(9) \quad & Y \circ F(x, y, u) = \textstyle\sum_{i=1}^{t} \pm y_i^2 + \sum_{i=1}^{q-1} (x^i/i!)u_i \\
& \qquad + (x^q/q!)(L(u) + ex/(q+1) + S(x, u)),
\end{aligned}
$$

where ord $S > 1$ and L is linear in the u's and e is a constant. Since $F^{q+1}(0) \in {}_TT^q$, we know that

$$
(10) \qquad \begin{pmatrix} \left(\dfrac{\partial^2 Y}{\partial y_i \, \partial y_{i'}}\right) & \left(\dfrac{\partial^2 Y}{\partial y_i \, \partial u_k}\right) & \left(\dfrac{\partial^2 Y}{\partial y_i \, \partial x}\right) \\ \left(\dfrac{\partial^{j+1} Y}{\partial x^j \partial y_{i'}}\right) & \left(\dfrac{\partial^{j+1} Y}{\partial x^j \partial u_k}\right) & \left(\dfrac{\partial^{j+1} Y}{\partial x^{j+1}}\right) \end{pmatrix}(0)
$$

has rank $t + q$ $(1 \leqq j \leqq q; \;\; 1 \leqq i, i' \leqq t; \;\; 1 \leqq k \leqq p - 1)$.

For our map this matrix becomes

$$
(10') \qquad \begin{pmatrix} E_t & 0 & 0 & 0 \\ 0 & I_{q-1} & 0 & 0 \\ 0 & ((\partial L/\partial u_j)(0)) & & e \end{pmatrix},
$$

where E_t is a $t \times t$ matrix with ± 1's on the diagonal and zeros elsewhere, I_{q-1} is the $q - 1$ identity matrix, and $((\partial L/\partial u_j)(0))$ is a $1 \times (p-1)$ matrix. Since $F^{q+1}(0) \notin T^{q+1}$, we know that $e \neq 0$, so we may assume $e = 1$. Since the lemma merely states that the order of the remainder is greater than $q + 1$, it suffices to prove the result without carrying the remainder along if we make coordinate changes which keep the origins fixed and which do not change the y-coordinates.

For P any linear function of u,

$$
\frac{(x - P(u))^q}{q!}\left(P(u) - \frac{(x - P(u))}{q+1}\right) = \frac{x^{q+1}}{(q+1)!} + \sum_{j=1}^{q} \frac{x^{q-j}}{(q-j)!} P_j(u),
$$

where ord $P_j \geqq j + 1$. Thus if we replace x by $x - L(u)$ in (9) we obtain

$$
\begin{aligned}
U \circ F(x, y, u) &= u, \\
Y \circ F(x, y, u) &= \textstyle\sum_{i=1}^{t} \pm y_i^2 + \sum_{i=0}^{q-1} (x^i/i!)Q_i(u) + x^{q+1}/(q+1)!\,.
\end{aligned}
$$

By the rank condition of (10) we may take as new coordinates

$$
\tilde{u}_i = Q_i(u), \quad \tilde{U}_i = Q_i(U), i = 1, \cdots, q-1, \quad \tilde{Y} = Y - Y \circ F(0, 0, U).
$$

F now has the desired form.

Let B be any subset of J^q. By $O(B)$ we mean the orbit of B under the group of q-jets of diffeomorphisms at the origin of the source and target.

LEMMA 3.5. $S_1^q = O(T^q)$.

Proof. Since $S_1^q \cap N^q = T^q$, we know that $O(T^q) \subset S_1^q$. For $q = 1$, the assertion of the lemma is trivial. Suppose $S_1^q = O(T^q)$; we show that

$$
S_1^{q+1} = O(T^{q+1}).
$$

By our induction hypothesis $O({}_TT^q) = {}_TS^q$. To prove the lemma it suffices to show that $S_1^{q+1} \cap {}_TT^q \subset O(T^{q+1})$. Suppose F is such that $F^{q+1}(0) \in {}_TT^q$ and $F = P_{F^{q+1}(0)}$. Since $F^q(0) \in T^q$, we may apply ψ to F. Call the resulting map G; we know that $G^{q+1}(0)$ is in the orbit of $F^{q+1}(0)$. Let $Y^* = Y \circ G$. We know that

$$\frac{\partial Y^*}{\partial y_i}(0) = 0, \quad i = 1, \cdots, t; \quad \text{and} \quad \frac{\partial^j Y^*}{\partial x^j}(0) = 0, \quad j = 1, \cdots, q.$$

Since $G^{q+1}(0) \in {}_TT^q$, if it is in the orbit of N^{q+1}, we are done. If $G^{q+1}(0) \notin O(N^{q+1})$, we may apply Lemma 3.4 and assume that G has the form given by (8) without remainder. The equations for $S_1^q(G)$ assume a very simple form as in (7):

$$y_i = 0, \quad i = 1, \cdots, t; \qquad u_j = 0, \quad j = 1, \cdots, q-1; \qquad x = 0.$$

Restricting G to $S_1^q(G)$ we see that at 0

$$\operatorname{rank} G = (p - q) = \dim S_1^q(G).$$

Thus $G^{q+1}(0) \notin S_1^{q+1}$.

Applying Lemma 3.5 to Lemma 3.4, we obtain conclusion (iii) of the theorem. Conclusion (iv) follows since the equations given there defining $S_1^q(F)$ are the defining equations of $T^q(F)$. We obtain (i) and (ii) also since the corresponding statements hold for T^q and ${}_TT^q$.

References

1. M. F. Atiyah and F. Hirzebruch, *The Riemann-Roch theorem for analytic embeddings*, Topology, vol. 1 (1962), pp. 151–166.
2. A. Haefliger, *Sur le self-intersections des applications différentiables* (Colloques Internationaux, C.N.R.S., 89, Lille, 1959), Bull. Soc. Math. France, vol. 87 (1959), pp. 351–359.
3. H. I. Levine, *Singularities of differentiable mappings I*, Bonn, Mathematisches Institut der Universität Bonn, 1959.
4. ———, *A generalization of a formula of Todd*, Summa Brasil. Math., to appear.
5. M. Morse, *The calculus of variations in the large*, Amer. Math. Soc. Colloquium Publications, vol. 18, 1934.
6. I. R. Porteous, *On the Chern classes of algebraic varieties related by a regular algebraic correspondence*, University of Liverpool, England (mimeographed).
7. R. Thom, *Un lemme sur les applications différentiables*, Bol. Soc. Mat. Mexicana (2), vol. 1 (1956), pp. 59–71.
8. H. Whitney, *On singularities of mappings of Euclidean spaces. I. Mappings of the plane into the plane*, Ann. of Math. (2), vol. 62 (1955), pp. 374–410.
9. ———, *Singularities of mappings of Euclidean spaces*, Symposium Internacional de Topología Algebraica, México, Universidad Nacional Autónoma de México y la UNESCO, 1958, pp. 285–301.

Brandeis University
Waltham, Massachusetts