SECONDARY OPERATIONS IN K-THEORY AND APPLICATIONS TO METASTABLE HOMOTOPY

BY

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1. Introduction

There has been considerable research into the general properties of secondary and higher order cohomology operations in K-theory [1], [6]. Most of this has been directed towards the construction of a K-theoretic version of the Adams spectral sequence.

In the present paper we calculate explicitly with particular operations to prove the following two theorems:

THEOREM I. *Let $n + 1 \equiv 2^r\ (2^{r+1})$, let $n > 0$ and let $\theta \in \pi_{4n-k}(S^{2n-k})$ be such that $E^{k+1}\theta = [\iota_{2n+1}, \iota_{2n+1}]$. Then:*

(a) $2^q \mid o(\theta)$ *where* $q = \min\left([(k + 2)/2], n - r - 1\right)$.

(b) *If* $r = 0$ *and* $n \neq 2, k = 0$.

If $r \not\equiv 2\ (4), r > 0,$ *and* $n \neq 5, 7, k \leq 2r + 1$.

If $r \equiv 2\ (4), k \leq 2r + 2$.

If $n = 7, k \leq 9$.

If $n = 5, k \leq 5$.

If $n = 2, k \leq 1$.

THEOREM II. *Suppose* $\gamma \in \pi_{4n+8j-2}(S^{2n}), H(\gamma) = \rho_j, 8j < 2n - 3, 2^r \mid n + 4j$ and $r \geq 1$. Then $2^{r+2-k} \mid o(E^{2k}\gamma)$.

Part (b) of Theorem I is equivalent to a theorem about k-frames on S^n which differs from the best possible result, due to Adams [2], in that the condition $k \leq 2r$ should be replaced by $k \leq 2r - 1$ if $r \equiv 0, 1 \pmod 4$ and the exceptions for $n = 2, 5, 7$ should be removed. Our proof differs from Adams' in that it is essentially unstable and independent of the topology of stunted projective spaces. It seems worth noting that Theorem Ib is equivalent to the best result obtainable by Adams' method without the use of KO-theory except for the cases $n = 2, 5, 7$.

Theorem II is of interest when r is large compared to j. In these cases Mahowald's results on metastable homotopy [5] leave open the essentiality of $[\iota_{2n-1}, \rho_j]$.

Theorems I and II are both applications of our principal result, Theorem 5.3, which characterizes desuspensions of $[\iota_{2n+1}, \iota_{2n+1}]$ in terms of the operations defined in §2.

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2. The operations $\phi^n_{p,q}$

We work with the functor $\tilde{k}(X) = [X, BU]$ which may be identified with $\tilde{K}(X)$ when X has the homotopy type of a finite CW-complex. By taking limits over finite skeleta, we may define maps

$$f^n_p : BU \to BU$$

which extend the operations $\psi_p - p^n$ to the functor $\tilde{k}$. We define

$$A^n_{p,q} : BU \to BU \times BU \quad \text{and} \quad B^n_{p,q} : BU \times BU \to BU$$

by the formulae

$$A^n_{p,q}(x) = (f^n_p(x), f^n_q(x)) \quad \text{and} \quad B^n_{p,q}(y, z) = (f^n_q(y) - f^n_p(z)).$$

The minus sign refers to the commutative H-structure on BU.

Since the operations, $(\psi_p - p^n)$ and $(\psi_q - q^n)$, commute, $B^n_{p,q} \circ A^n_{p,q}$ is null-homotopic on any finite sub-complex of BU. Since BU has nontrivial homotopy only in even dimensions and a CW-structure without odd dimensional cells, it follows by elementary obstruction theory that $B^n_{p,q} \circ A^n_{p,q}$ is null-homotopic.

If $g \in \tilde{k}(X)$ is such that $f^n_p \circ g$ and $f^n_q \circ g$ are null homotopic, the secondary composition,

$$\phi^n_{p,q}(g) = \langle B^n_{p,q}, A^n_{p,q}, g\rangle,$$

is defined in the sense of Toda [7]. $\phi^n_{p,q}(g) \subset \tilde{k}(EX)$ and is a coset of

$$\tilde{k}(EBU) \circ Eg + B^n_{p,q} \circ [EX, BU \times BU].$$

We note that $\tilde{k}(EBU) = 0$ and

$$B^n_{p,q} \circ [EX, BU \times BU] = f^n_p \circ \tilde{k}(EX) + f^n_p \circ \tilde{k}(EX).$$

In particular, if X has the homotopy type of a finite CW-complex, $\phi^n_{p,q}$ may be regarded as a secondary operation on $\tilde{K}(X)$, defined on

$$\ker (\psi_p - p^n) \cap \ker (\psi_q - q^n)$$

and taking values in $\tilde{K}(EX)/\mathrm{Im}\ (\psi_p - p^n) + \mathrm{Im}\ (\psi_q - q^n)$.

The Bott isomorphism $\beta : \tilde{K}(X) \to \tilde{K}(E^2X)$ is extended to the functor, $\tilde{k}$, by a homotopy equivalence, $\beta : BU \to \Omega^2 BU$, so that $\beta(g)$ is adjoint to $\beta \circ g$. Since the operations, $\psi_p \circ \beta$ and $p\beta \circ \psi_p$, are identical on $\tilde{K}(X)$, it follows by another elementary obstruction theoretic argument that

$$\Omega^2 f^{n+1}_p \circ \beta = p\beta \circ f^n_p : BU \to \Omega^2 BU.$$

Multiplication by p on the right hand side of the equality refers to the H-structure on $\Omega^2 BU$.

PROPOSITION 2.1. (a) *If $g \in \tilde{k}(X)$ is in the domain of $\phi^n_{p,q}$, then $\beta(g)$ is in the domain of $\phi^{n+1}_{p,q}$ and $pq\beta(\phi^n_{p,q}(g)) \subset \phi^{n+1}_{p,q}(\beta(g))$.*

(b) *$\phi^n_{p,q}$ is additive on its domain.*

(c) *If $f : X' \to X$ then $(Ef')(\phi^n_{p,q}(g)) \subset \phi^n_{p,q}(f'(g))$.*

Proof. Consider the following commutative diagram:

$$\begin{array}{ccccccc} X \xrightarrow{g} & BU & \xrightarrow{A^n_{p,q}} & BU \times BU & \xrightarrow{B^n_{p,q}} & BU \\ & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle p\beta \times q\beta} & & \downarrow{\scriptstyle pq\beta} \\ & \Omega^2 BU & \xrightarrow{\Omega^2 A^{n+1}_{p,q}} & \Omega^2 BU \times \Omega^2 BU & \xrightarrow{\Omega^2 B^{n+1}_{p,q}} & \Omega^2 BU \end{array}$$

We have made implicit use of the fact that the loop-space is distributive through Cartesian products. Dropping subscripts, we have (see [7])

$$\begin{aligned} pq\beta \circ \langle B^n, A^n, g\rangle \subset \langle pq\beta \circ B^n, A^n, g\rangle &= \langle \Omega^2 B^{n+1} \circ (p\beta \times q\beta), A^n, g\rangle \\ &\subset \langle \Omega^2 B^{n+1}, (p\beta \times q\beta) \circ A^n, g\rangle = \langle \Omega^2 B^{n+1}, \Omega^2 A^{n+1} \circ \beta, g\rangle \\ &= \langle \Omega^2 B^{n+1}, \Omega^2 A^{n+1}, \beta \circ g\rangle. \end{aligned}$$

But the first term is adjoint to $pq\beta(\phi^n_{p,q}(g))$ and the last to $\phi^{n+1}_{p,q}(\beta(g))$. (b) is proved similarly from a diagram which reflects the fact that $A^n_{p,q}$ and $B^n_{p,q}$ are H-maps. We shall not use part (b) in the sequel and the details are left to the reader. (c) is merely a restatement of one of the elementary properties of secondary compositions. ∎

3. The complexes X_n and $\hat{X}_n$

Let $n > 1$ and let $\alpha \in \pi_{4n}(S^{2n})$ be such that $E\alpha = [\iota_{2n+1}, \iota_{2n+1}]$. Let X_n be the complex, $S^{2n} \cup e^{4n} \cup e^{4n+1}$ where e^{4n} and e^{4n+1} are attached to S^{2n} by $[\iota_{2n}, \iota_{2n}]$ and α respectively. We write h and j for the respective inclusions of $C_{[\iota_{2n},\iota_{2n}]}$ and C_α in X_n, and i for that of S^{2n} in $C_{[\iota_{2n},\iota_{2n}]}$. We note that the homotopy type of X_n is independent of the choice of α subject to the above condition.

LEMMA 3.1. *There is a map*

$$f : S^{4n+1} \to X$$

and a map

$$G : E^2 C_f, E^2 X \to C_{[\iota_{2n+2},\iota_{2n+2}]}, S^{2n+2}$$

such that

(a) $H^{4n+2}(C_f) = Z_2$

(b) *G induces isomorphisms*

$$H^{2n+2}(S^{2n+2}) \to H^{2n+2}(E^2 X)$$

and

$$H^{4n+4}(C_{[\iota_{2n+2},\iota_{2n+2}]}, S^{2n+2}) \to H^{4n+4}(C_f, X).$$

Proof. It is well known that $C_{[\iota_{2n},\iota_{2n}]}$ has the homotopy type of a $(6n-1)$-dimensional skeleton of ΩS^{2n+1}. Under this identification $\iota_*(\alpha)$ evidently corresponds to $\partial[\iota_{2n+1}, \iota_{2n+1}]$. It is easily proved by means of the Serre spectral sequence that the non-trivial cohomology of the pair $(\Omega^2 S^{2n+2}, \Omega S^{2n+1})$, in dimensions less than $6n$, under the map

$$\Omega\, \partial \iota_{2n+2} : \Omega S^{2n+1} \to \Omega^2 S^{2n+2},$$

is given by $H^{4n+2}(\Omega^2 S^{2n+2}, \Omega S^{2n+1}) = Z$. By the exactness of the *EHP* sequence, the kernel of

$$(\Omega\, \partial \iota_{2n+2})_* : \pi_{4n}(\Omega S^{2n+1}) \to \pi_{4n}(\Omega^2 S^{2n+2})$$

is generated by $\partial[\iota_{2n+1}, \iota_{2n+1}]$. It follows that $C_{\iota_*(\alpha)} = X_n$ has the homotopy type of a $(6n-1)$-dimensional skeleton of $\Omega^2 S^{2n+2}$. We choose f to be in the homotopy class corresponding to $\partial^2[\iota_{2n+2}, \iota_{2n+2}]$ under this identification. The above argument may be iterated to show that C_f has the homotopy type of a $(6n-1)$-dimensional skeleton of $\Omega^3 S^{2n+3}$. (a) is now immediate and for the map of (b) we choose the restriction to C_f of a cellular approximation to the map from

$$(E\,\Omega^3 S^{2n+3}, E\,\Omega^2 S^{2n+2}) \quad \text{to} \quad (\Omega S^{2n+3}, S^{2n+2})$$

which is adjoint to the identity map of the pair $(\Omega^3 S^{2n+3}, \Omega^2 S^{2n+2})$. ∎

We shall write $\hat{X}_n$ for C_f and $k : X \to \hat{X}$ for the natural inclusion.

4. Computation of $\tilde{K}$

In the following discussion μ_n will denote a generator of $\tilde{K}(S^{2n})$ and $\bar{\mu}_n$, $\tilde{\mu}_n$ or $\hat{\mu}_n$ an element of infinite order in $\tilde{K}(Y)$ for some space, Y, which has the property that $(\psi_p - p^n)\bar{\mu}_n$ has strictly higher filtration than $\bar{\mu}_n$ for each integer p.

LEMMA 4.1. *$\tilde{K}(C_{[\iota_{2n},\iota_{2n}]})$ is generated by $\tilde{\mu}_n$ and $\tilde{\mu}_{2n}$ where $\tilde{\mu}_n$ may be chosen to satisfy*

$$\psi_k(\tilde{\mu}_n) = k^n \tilde{\mu}_n, \quad \textit{for all } k, \qquad \lambda^2(\tilde{\mu}_n) = -2^{n-1}\tilde{\mu}_n + \tilde{\mu}_{2n}.$$

Proof. It is immediate from the exact sequence of the co-fibration

$$S^{2n} \to C_{[\iota_{2n},\iota_{2n}]} \to S^{4n}$$

that $\tilde{K}(EC_{[\iota_{2n},\iota_{2n}]} = 0$ and that there is a short exact sequence

$$0 \to \tilde{K}(S^{4n}) \to \tilde{K}(C_{[\iota_{2n},\iota_{2n}]}) \to \tilde{K}(S^{2n}) \to 0.$$

Since the Whitehead product is stably trivial, this sequence is stably split over the operations ψ_k. Since $\tilde{K}(C_{[\iota_{2n},\iota_{2n}]})$ is without torsion, a stable splitting determines an unstable splitting.

Since the image of the squaring operation in integral cohomology has index 2 in $H^{4n}(C_{[\iota_{2n},\iota_{2n}]})$ it follows from the multiplicative properties of the Chern character that $\tilde{\mu}_n^2 = 2\tilde{\mu}_{2n}$. $\lambda^2(\tilde{\mu}_n)$ may now be evaluated from the equation $\psi_2(\tilde{\mu}_n) = \tilde{\mu}_n^2 - 2\lambda^2(\tilde{\mu}_n)$ [4]. ∎

LEMMA 4.2. *$h^!$ maps $\tilde{K}(X_n)$ isomorphically onto $\tilde{K}(C_{[\iota_{2n},\iota_{2n}]})$. $\tilde{K}(EX_n)$ is generated by $\bar{\mu}_{2n+1}$.*

Proof. Since $d_c(\alpha)$ is trivial, so is $d_c(\iota_*\,\alpha) : \tilde{K}(C_{[\iota_{2n},\iota_{2n}]}) \to \tilde{K}(S^{4n})$. Lemma 4.2 is now immediate from the exact sequence of the co-fibration

$$C_{[\iota_{2n},\iota_{2n}]} \to C_{\iota_*\alpha} = X_n \to S^{4n+1}.$$ ∎

LEMMA 4.3. *There is a split exact sequence*

$$0 \to Z_2 \to \tilde{K}(\hat{X}_n) \xrightarrow{h^! \circ k^!} \tilde{K}(C_{[\iota_{2n}, \iota_{2n}]}) \to 0.$$

Proof. $C_{k \circ h}$ has the homotopy type of $S^{4n+1} \cup_2 e^{4n+2}$. Hence $\tilde{K}(C_{k\circ h}) = Z_2$ and $\tilde{K}(EC_{k\circ h}) = 0$. The exact sequence of Lemma 4.3 follows. It is split because $\tilde{K}(C_{[\iota_{2n}, \iota_{2n}]})$ is free. ∎

We write $\gamma \in \tilde{K}(\hat{X}_n)$ for the non-trivial element in the kernel of $h^! \circ k^!$. We write $\hat{\mu}_n$ and $\hat{\mu}_{2n}$ for the images of $\tilde{\mu}_n$ and $\tilde{\mu}_{2n}$ under some splitting of the exact sequence of Lemma 4.3.

LEMMA 4.4. *For odd* p, $\psi_p(\hat{\mu}_n) = p^n \hat{\mu}_n$. $\psi_2(\hat{\mu}_n) = 2^n \hat{\mu}_n + \gamma$.

Proof. Let $G : E^2 \hat{X}_n \to C_{[\iota_{2n+2}, \iota_{2n+2}]}$ be the map of Lemma 3.1. Then $G^!(\tilde{\mu}_{n+1}) = \beta(\hat{\mu}_n)$ (or $-\beta(\hat{\mu}_n)$, but the sign is immaterial to the argument) and $G^!(\tilde{\mu}_{2n+2}) = \beta(\gamma)$. Then

$$\lambda^2 \circ \beta(\hat{\mu}_n) = G^!(\lambda^2(\tilde{\mu}_{n+1})) = G^!(-2^n \tilde{\mu}_{n+1} + \tilde{\mu}_{2n+2}) = -2^n \beta(\hat{\mu}_n) + \beta(\gamma).$$

But $\lambda^2 \circ \beta = -\beta \circ \psi_2$. Hence $\psi_2 \hat{\mu}_n = 2^n \hat{\mu}_n + \gamma$.

$E^2(C_{[\iota_{2n+2}, \iota_{2n+2}]})$ is coreducible. $E^2 G$ followed by a coreducing map for $E^2 C_{[\iota_{2n+2}, \iota_{2n+2}]}$ is a coreducing map for $E^4 \hat{X}$. Hence $\psi_k \beta^2 \hat{\mu}_n = k^{n+2} \beta^2 \hat{\mu}_n$ for all k. Since $\tilde{K}(\hat{X})$ lacks odd torsion, it follows that $\psi_p \hat{\mu}_n = p^n \hat{\mu}_n$ for all odd p. ∎

5. Evaluation of $(\phi^n_{p,2})_\alpha$

LEMMA 5.1. $(\psi_p - p^n)_k(\gamma) \in \phi^n_{p,2}(\bar{\mu}_n)$ *for any odd* p.

Proof. We may interpret $A^n_{p,q}$ and $B^n_{p,q}$ as operations from $\tilde{k}$ to $\tilde{k} \oplus \tilde{k}$ and back again respectively. Then by the Peterson-Stein formula,

$$(B^n_{p,2})_k(A^n_{p,2}(\hat{\mu}_n)) \subset \phi^n_{3,2}(k^!(\hat{\mu}_n)) = \phi^n_{3,2}(\bar{\mu}_n).$$

But

$$A^n_{p,2}(\hat{\mu}_n) = (0, \gamma) \quad \text{and} \quad (B^n_{p,2})_k(0, \gamma) = (\psi_2 - 2^n)_k(0) + (\psi_p - p^n)_k(\gamma).$$

Since $0 \in (\psi_2 - 2^n)_k(0)$, the lemma follows. ∎

LEMMA 5.2. $\frac{1}{2}(p^{2n+1} - p^n)\bar{\mu}_{2n+1} \in (\psi_p - p^n)_k(\gamma)$.

Proof. The cofibration $X_n \to \hat{X}_n \to S^{4n+2}$ induces the exact sequence

$$\tilde{K}(E\hat{X}_n) = 0 \to \tilde{K}(EX_n) \to \tilde{K}(S^{4n+2}) \to \tilde{K}(\hat{X}_n) \xrightarrow{k^!} \tilde{K}(X_n) \to 0.$$

It is evident that $\mu_{2n+1} \in \tilde{K}(S^{4n+2})$ maps to γ and that $\bar{\mu}_{2n+1} \in \tilde{K}(EX_n)$ maps to $2\mu_{2n+1}$ (up to an unimportant sign). Here $\frac{1}{2}(p^{2n+1} - p^n)\bar{\mu}_{2n+1}$ maps to

$$(p^{2n+1} - p^n)\mu_{2n+1} = (\psi_p - p^n)\mu_{2n+1}.$$

Thus by definition $\frac{1}{2}(p^{2n+1} - p^n)\bar{\mu}_{2n+1} \in (\psi_p - p^n)_k(\gamma)$.

THEOREM 5.3. *$\frac{1}{2}(p^{2n+1} - p^n)\mu_{2n+1} \in (\phi^n_{p,2})_\alpha(\mu_n)$ for all odd p.*

Proof. By Lemmas 5.1 and 5.2,

$$\tfrac{1}{2}(p^{2n+1} - p^n)\bar{\mu}_{2n+1} \in \phi^n_{p,2}(\bar{\mu}_n).$$

By Prop. 2.1 c,

$$\tfrac{1}{2}(p^{2n+1} - p^n)Ej^!(\bar{\mu}_{2n+1}) \in \phi^n_{p,2}(j^!\bar{\mu}_n) \quad \text{in } \tilde{K}(C_\alpha).$$

Theorem 5.3 follows by definition. ∎

Remark. Since there is no indeterminacy involved in the identification of $\tilde{K}(C_\alpha)$ and $\tilde{K}(EC_\alpha)$ with $\tilde{K}(S^{2n})$ and $\tilde{K}(S^{4n+2})$ respectively, the indeterminacy of the functional secondary operation $(\phi^n_{p,2})_\alpha$ corresponds under this identification with the indeterminacy of the operation $\phi^n_{p,q}$ on $\tilde{K}(C_\alpha)$ and is generated by

$$(\psi_p - p^n)\mu_{2n+1} \quad \text{and} \quad (\psi_2 - 2^n)\mu_{2n+1}.$$

It follows that the indeterminacy of $(\phi^n_{p,2})_\alpha : \tilde{K}(S^{2n}) \to \tilde{K}(S^{4n+2})$ is the cyclic subgroup of $\tilde{K}(S^{4n+2})$ generated by

$$GCD(p^n(p^{n+1} - 1), 2^n(2^{n+1} - 1))\mu_{2n+1}.$$

$\frac{1}{2}(p^{2n+1} - p^n)\mu_{2n+1}$ is not an element of this subgroup unless

$$2^n \mid \tfrac{1}{2}(p^{2n+1} - p^n).$$

In particular, if $p \equiv 3, 5 \pmod 8$ $(\phi^n_{p,2})_\alpha$ is non-trivial unless $n = 0, 1$ or 3. This is the translation into the language of secondary cohomology operations of the Adams-Atiyah proof [3] of the non-existence of elements of Hopf invariant 1.

6. An important lemma

LEMMA 6.1. *Let $0 < k < n - 2$ and $\gamma \in \pi_{4n+2k}(S^{2n})$. Let p be odd and suppose $2^n \nmid p^n - 1$. Then $(\phi^n_{p,2})_\gamma(\mu_n)$ contains odd multiples of μ_{2n+k+1} iff $(\psi_p - p^{2n})_{EH(\gamma)}(\mu_{2n})$ does.*

Proof. By an argument similar to those of §3, $\Omega^2 C_\gamma$ has a skeleton of the form $Y = X_{n-1} \cup e^{4n+2k-1}$ which admits a map $F : E^2 Y \to C_\gamma$ inducing cohomology isomorphisms in dimensions $2n$ and $2n + 2k + 1$. If is evident that $Y/C_{[\iota_{2n-2}, \iota_{2n-2}]}$ has the homotopy type of $C_{E^{-1}H(\gamma)}$. We can choose generators $\bar{\mu}_{n-1}$, $\bar{\mu}_{2n-2}$ for $\tilde{K}(Y)$; $\bar{\mu}_{2n-1}$, $\bar{\mu}_{2n+k}$ for $\tilde{K}(EY)$, $\tilde{\mu}_n$ for $\tilde{K}(C_\gamma)$; and $\tilde{\mu}_{2n+k+1}$ for $\tilde{K}(EC_\gamma)$ such that

$$F^!(\tilde{\mu}_n) = \beta(\bar{\mu}_{n-1}), \quad (EF)^!(\tilde{\mu}_{2n+k+1}) = \beta\bar{\mu}_{2n+k},$$

$\bar{\mu}_{n-1}$ is in the domain of $\phi^{n-1}_{p,2}$ and $\tilde{\mu}_n$ is in the domain of $\phi^n_{p,2}$.

By the results of §5 and the naturality of the operation, $\phi^{n-1}_{p,2}(\bar{\mu}_{n-1})$ contains an element of the form $\frac{1}{2}(p^{2n-1} - p^{n-1})\bar{\mu}_{2n-1} + m\bar{\mu}_{2n+k}$ for some integer m. By Prop. 2.1 a,

$$(p^{2n} - p^n)\beta\bar{\mu}_{2n-1} + 2pm\beta\bar{\mu}_{2n+k} \in \phi^n_{p,2}(\beta\bar{\mu}_{n-1}).$$

We note that

$$(\psi_p - p^n)\beta\bar{\mu}_{2n-1} = (p^{2n} - p^n)\beta\bar{\mu}_{2n-1} + a\beta\bar{\mu}_{2n+k}$$

where $a\mu_{2n+k+1} \in (\psi_p - p^n)_{EH(\gamma)}(\mu_{2n})$. Hence

$$(2pm - a)\beta\bar{\mu}_{2n+k} \in \phi^n_{p,2}(\beta\bar{\mu}_{n-1}).$$

Let $c\tilde{\mu}_{2n+k+1} \in \phi^n_{p,2}(\tilde{\mu}_n)$. Then $c\beta\bar{\mu}_{2n+k} \in \phi^n_{p,2}\beta(\bar{\mu}_{n-1})$. Then

$$(2pm - a - c)\beta\bar{\mu}_{2n+k} \in \operatorname{Im}(\psi_p - p^n) + \operatorname{Im}(\psi_2 - 2^n).$$

Under the hypothesis on p, it is not difficult to verify by computation that $2pm - a - c$ must be even. It follows that $(\psi_p - p^n)_{EH(\gamma)}\,\mu_{2n}$ and $(\phi^n_{p,2})_\gamma\,\mu_n$ have the same parity. ∎

7. Proofs of Theorems I and II

Proof of Theorem II. Since ρ_j is defined for $j \geq 1$, it follows that $n \geq 6$. Therefore $2^n \nmid 3^n - 1$. Since

$$\rho_j : S^{4n+8j-1} \to S^{4n}$$

has the property that $(\psi_3 - 3^{2n})_{\rho_j}\,\mu_{2n}$ contains odd multiples of μ_{4n+8j}, it follows by Lemma 6.1 that the same is true of $(\phi^n_{3,2})_\gamma\,\mu_n$. Thus $(\phi^{n+k}_{3,2})_{E^{2k}\gamma}\,\mu_{n+k}$ contains odd multiples of $2^k\mu_{2n+4j+k}$. The indeterminacy subgroup of the operation $(\phi^{n+k}_{3,2})_{E^{2k}\gamma}$ is the cyclic group generated by

$$GCD\,(2^{n+k}(2^{n+4j} - 1),\ 3^{n+k}(3^{n+4j} - 1))\mu_{2n+4j+k}.$$

Under the hypotheses of Theorem II, $2^{r+2} \mid (3^{n+4j} - 1)$ and $n \geq r + 2$. Hence odd multiples of $2^q\mu_{2n+4j+k}$ are not in the indeterminacy of the operation for $q < r + 2$ and Theorem II is proved.

Proof of Theorem I. Let $\alpha = E^k\theta$. Then α is as in §§3–5. By Theorem 5.3,

$$\tfrac{1}{2}p^n(p^{n+1} - 1)\mu_{2n+1} \in (\phi^n_{p,2})_\alpha\,\mu_n.$$

If we ignore the exceptional cases $n = 1, 3$ in which Theorem I is vacuously true, then $(\phi^n_{3,2})_\alpha\,\mu_n$ consists of odd multiples of μ_{2n+1} if $r = 0$, and $(\phi^n_{5,2})_\alpha\,\mu_n$ consists of odd multiples of $2^{r+1}\mu_{2n+1}$ if $r > 0$.

In the case $r = 0$, the proof of part (a) consists of the observation that

$$2 \mid o(\phi^n_{3,2})_\alpha(\mu_n).$$

Provided $n \neq 2$, we may apply Lemma 6.1 to conclude that $(\psi_3 - 3^{2n})_{EH(\alpha)}(\mu_{2n})$ contains odd multiples of μ_{2n+1}. It follows that $2 \mid o(H(\alpha))$ from which we may conclude that α does not admit a desuspension, proving part (b).

In the case $r > 0$, we may assume without loss of generality that k is even. Let

$$m\mu_{2n+1-k/2} \in (\phi^{n-k/2}_{5,2})_\theta(\mu_{n-k/2}).$$

Then by Prop. 1a $2^{k/2}m$ is an odd multiple of 2^{r+1}. Hence m is an odd multiple of $2^{r+1-k/2}$. As in the proof of Theorem II, the indeterminacy subgroup of the operation $(\phi_{5,2}^{n-k/2})_\theta$ does not contain odd multiples of $2^q \mu_{2n+1-k/2}$ unless $q \geq \min(r+2, n-k/2)$. This proves part (a) of Theorem I.

Suppose now that $k = 2r+2$. Then $(\phi_{5,2}^{n-k/2})_\theta(\mu_{n-k/2})$ contains odd multiples of $\mu_{2n+1-k/2}$. Provided $n \neq 5, 7$, θ satisfies the hypotheses of Lemma 6.1 and we may conclude that

$$EH(\theta) \in \pi_{4n-2r-1}(S^{4n-4r-4})$$

is such that $(\psi_5 - 5^{(n-r-2)})_{EH(\theta)}\ \mu_{2(n-r-2)}$ contains odd multiples of μ_{2n-r}. This is not possible unless $r \equiv 2\ (4)$ as one may prove by using the commutation of ψ_5 with ψ_3 if r is odd and by introducing KO and complexification if $r \equiv 0\ (4)$.

It follows that the element θ cannot exist if $r \not\equiv 2\ (4)$ and does not admit a desuspension if $r \equiv 2\ (4)$. This completes the proof of Theorem 1 except for the cases $n = 2, 5, 7$. In the case $n = 7$ we simply note that if $k = 10$, $2^5 m$ must be an odd multiple of 2^4. The cases $n = 2$ and $n = 5$ are similar.

Note. It seems very likely that a more detailed study of the space $\Omega^2 S^{2n}$ will lead to a considerable relaxation of the hypothesis on the degree of γ in Lemma 6.1. This would improve Theorem II and eliminate from Theorem I the exceptions for $n = 2, 5, 7$.

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