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EXACT SEQUENCES IN ALGEBRAIC K-THEORY

BY

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In this paper we provide a general framework for producing long exact sequences in algebraic K-theory.

Recall the space $S\mathcal{M}$, constructed in [8, p. 181] for any exact category $\mathcal{M}$, in terms of which the K-groups can be defined: $K_i\mathcal{M} = \pi_{i+1}S\mathcal{M}$. Suppose $F\colon \mathcal{P} \to \mathcal{M}$ is an exact functor such that for each admissible monomorphism $M \rightarrowtail N$ of $\mathcal{M}$ there exists an admissible monomorphism $M \rightarrowtail N'$ and a commutative diagram of admissible monomorphisms

$$\begin{array}{ccccc} M & = L_0 \rightarrowtail & \cdots & \rightarrowtail L_s \\ \downarrow & & & \| \\ N\amalg_M N' & = N_0 \rightarrowtail & \cdots & \rightarrowtail N_q \end{array}$$

in which each N_i/N_{i-1} or L_i/L_{i-1} is isomorphic to an object in the image of F. We prove that a certain square

$$\begin{array}{ccc} 0|SF & \to & S\mathcal{P} \\ \downarrow & & \downarrow SF \\ pt \sim 0|S\mathcal{M} & \to & S\mathcal{M} \end{array}$$

is homotopy cartesian, thereby yielding a long exact sequence

$$\cdots\; K_i\mathcal{M} \to \pi_i(0|SF) \to K_{i-1}\mathcal{P} \to K_{i-1}\mathcal{M} \;\cdots .$$

The proof is based on the methods of [5].

It turns out that the criterion formulated above for F is often satisfied and the space $0|SF$ can often be brought into homotopy equivalence with other interesting spaces. In this way we provide new proofs of the following theorems:

(i) the cofinality theorem of Grayson and Waldhausen [4.1.1], [1]:
(ii) the dévissage theorem of Quillen [6, Theorem 4];
(iii) the resolution theorem of Quillen [6, Theorem 3];
(iv) the localization theorem for projection modules of Quillen [2], [3].

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The advantage of these new proofs is that they share substantial common ground, and the unique portions are thereby reduced in complexity.

In the final section, we apply the main theorem to produce an infinite sequence of definitions, for higher algebraic K-theory of an exact category, which were suggested by H. Gillet, and should enable one to define λ-operations on the higher K-groups of any exact category equipped with exterior powers.

I thank Henri Gillet for useful discussions.

1. The main theorem

We assume known all the terminology of [5].

We fix exact categories $\mathscr{P}$ and $\mathscr{M}$ and zero objects called 0 in each.

Let $F\colon \mathscr{P} \to \mathscr{M}$ be an exact functor such that $F(0) = 0$, and consider the map

$$SF\colon S\mathscr{P} \to S\mathscr{M}.$$

For $A \in \Delta$ and $\overline{M} \in S\mathscr{M}(A)$ we consider the right fiber $\overline{M}|SF$; a q-simplex W of it can be illustrated by the following diagram, which doesn't show the choices for the various quotients:

$$W = \begin{pmatrix} & 0 = P_0 \rightarrowtail \cdots \rightarrowtail P_q \\ & \overline{\overline{}} \\ 0 = M_0 \rightarrowtail \cdots \rightarrowtail M_a \rightarrowtail & N_0 \rightarrowtail \cdots \rightarrowtail N_q \end{pmatrix}$$

Here the horizontal arrows are admissible monomorphisms,

$$A = [a] = \{0 < 1 < \cdots < a\},$$
$$\overline{M} = (0 = M_0 \rightarrowtail \cdots \rightarrowtail M_a),$$

the top row represents a q-simplex of $S\mathscr{P}$, the bottom row represents a $q + a + 1$ simplex of $S\mathscr{M}$, and the double line represents the identity

$$\begin{array}{ccccc} 0 = FP_0 & \rightarrowtail & \cdots & \rightarrowtail & FP_q \\ \| & & & & \| \\ 0 = N_0/N_0 & \rightarrowtail & \cdots & \rightarrowtail & N_q/N_0 \end{array}$$

between the right hand q-face of the bottom row and F applied to the top row. Here (and later) the notation N_i/N_j refers to the chosen quotient implicitly provided as part of the data constituting W.

In $\mathscr{P}$ and $\mathscr{M}$ we choose direct sums $M \oplus N$ for all M and N, and choose pushouts $L \amalg_M N$ for all pairs $L \leftarrowtail M \rightarrowtail N$ of admissible monomorphisms.

We ensure the identities $M \oplus 0 = M = 0 \oplus M$, $L \amalg_M M = L = M \amalg_M L$, and $L \amalg_0 M = L \oplus M$. For each admissible monomorphism $M \rightarrowtail N$ we choose a cokernel $\operatorname{coker}(M \rightarrowtail N)$, ensuring that $\operatorname{coker}(0 \rightarrowtail M) = M$ and $\operatorname{coker}(1_M) = 0$.

It is an easy exercise to replace $\mathscr{M}$ and $\mathscr{P}$ by equivalent categories and provide direct sum operations which are strictly associative, have a strict identity, and for which F is strictly additive. It is a little bit harder (but still possible) to do the same thing for pushouts. Still, we will use some natural isomorphisms later that can't be made into identities, so we don't bother to make any of them into identities. Nevertheless, with our choice for direct sum, we get a map

$$+: S\mathscr{M} \times S\mathscr{M} \to S\mathscr{M}$$

which makes $S\mathscr{M}$ into a homotopy associative H-space, with 0 as (strict) identity.

We use the pushouts chosen above to define an operation

$$+: \overline{M}|SF \times \overline{M}|SF \to \overline{M}|SF$$

by setting

$$W + W' := \left(\frac{\qquad\qquad\qquad\qquad 0 = P_0 \oplus P_0' \rightarrowtail \cdots \rightarrowtail P_q \oplus P_q'}{0 = M_0 \rightarrowtail \cdots \rightarrowtail M_a \rightarrowtail N_0 \amalg_{M_a} N_0' \rightarrowtail \cdots \rightarrowtail N_q \amalg_{M_a} N_q'} \right)$$

and specifying the undisplayed quotients as follows:

$$\frac{P_i \oplus P_i'}{P_j \oplus P_j'} := \frac{P_i}{P_j} \oplus \frac{P_i'}{P_j'}, \quad \frac{N_i \amalg_{M_a} N_i'}{N_j \amalg_{M_a} N_j'} := F\left(\frac{P_i \oplus P_i'}{P_j \oplus P_j'}\right),$$

$$\frac{N_i \amalg_{M_a} N_i'}{M_a} := \frac{N_i}{M_a} \oplus \frac{N_i'}{M_a},$$

$$\frac{N_i \amalg_{M_a} N_i'}{M_j} := \operatorname{coker}\left(M_j \to N_i \amalg_{M_a} N_i'\right) \quad \text{for } 1 \le j < a.$$

We see that $+$ is a simplicial map. The base-change map $\overline{M}|SF \to 0|SF$ which factors out M_a preserves $+$ strictly. The projection map π: $\overline{M}|SF \to S\mathscr{P}$ preserves $+$ strictly. As in [5], we use natural transformations to see that $\overline{M}|SF$ is a homotopy associative and homotopy commutative H-space. This gives $\pi_0(\overline{M}|SF)$ an operation $+$ and a homotopy identity element 0 which makes it into a (commutative) monoid. The simplex which represents 0 is:

$$0 := \left(\begin{array}{c} \qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{\underline{\;0 = P_0\;}} \\ 0 = M_0 \rightarrowtail \cdots \rightarrowtail M_a \overset{1}{\rightarrowtail} \quad M_a \end{array} \right)$$

THEOREM C′. *Suppose for all $A \in \Delta$ and all $\overline{M} \in S\mathcal{M}(A)$ that the monoid $\pi_0(\overline{M} \mid SF)$ is a group. Then the square*

$$\begin{array}{ccc} 0 \mid SF & \rightarrow & S\mathcal{P} \\ \downarrow & & \downarrow \\ pt \sim 0 \mid S\mathcal{M} & \rightarrow & S\mathcal{M} \end{array}$$

is homotopy cartesian.

Proof. By Theorem B' of [5], we need only show for any $f\colon A' \to A$ in Δ, that the base change map $f^*\colon \overline{M} \mid SF \to f^*\overline{M} \mid SF$ is a homotopy equivalence.

Let $g\colon [0] \to A'$ be any map in Δ. It suffices to show that $g^*f^* = (fg)^*$ and g^* are homotopy equivalences. Both fg and g have $[0]$ as source, so replacing f by either of them, we may assume that f is the map $f_i\colon [0] \to A$, defined for $i \in A$ by $f_i(0) = i$. Thus f_i^* is a map $\overline{M} \mid SF \to 0 \mid SF$ (because 0 is the only vertex of $S\mathcal{M}$).

We define a map $H\colon 0 \mid SF \to \overline{M} \mid SF$ by direct sum with $\overline{M}$, i.e., by

$$\begin{array}{c} \left(\begin{array}{c} \qquad\quad \underline{\underline{\;0 = P_0 \rightarrowtail \cdots \rightarrowtail P_q\;}} \\ 0 \rightarrowtail \qquad N_0 \rightarrowtail \cdots \rightarrowtail N_q \end{array} \right) \\ \Big\downarrow{\scriptstyle H} \\ \left(\begin{array}{c} \qquad\qquad\qquad\qquad\qquad\qquad \underline{\underline{\;0 = P_0 \rightarrowtail \cdots \rightarrowtail \qquad\quad P_q\;}} \\ 0 = M_0 \rightarrowtail \cdots \rightarrowtail M_a \rightarrowtail M_a \oplus N_0 \rightarrowtail \cdots \rightarrowtail M_a \oplus N_q \end{array} \right) \end{array}$$

The undisplayed quotients here are defined as follows.

$$\frac{M_a \oplus N_j}{M_a \oplus N_k} := \frac{N_j}{N_k} \quad \big(= F(P_j/P_k)\big),$$

$$\frac{M_a \oplus N_j}{M_a} := N_j, \quad \frac{M_a \oplus N_j}{M_i} := \frac{M_a}{M_i} \oplus N_j.$$

The map $f_i^* \circ H\colon 0|SF \to 0|SF$ is depicted thus:

$$\left(\begin{array}{c} 0 = P_0 \rightarrowtail \cdots \rightarrowtail P_q \\ \\ 0 \rightarrowtail N_0 \rightarrowtail \cdots \rightarrowtail N_q \end{array}\right)$$

$$\Big\downarrow f_i^* \circ H$$

$$\left(\begin{array}{c} 0 = P_0 \rightarrowtail \cdots \rightarrowtail \qquad P_q \\ \hline\hline 0 \rightarrowtail \dfrac{M_a}{M_i} \oplus N_0 \rightarrowtail \cdots \rightarrowtail \dfrac{M_a}{M_i} \oplus N_q \end{array}\right)$$

for $0 \le i \le a$, and $f_a^* \circ H$ is the identity. Thus the map $f_i^* \circ H$ is the translation on the H-space $0|SF$ obtained by adding the vertex

$$\left(\begin{array}{c} 0 \\ \hline\hline 0 \rightarrowtail \dfrac{M_a}{M_i} \end{array}\right);$$

since its homotopy class is in the image of the monoid map $\pi_0(0|SF) \to [0|SF, 0|SF]$, and $\pi_0(0|SF)$ is a group, we see that $f_i^* \circ H$ is a homotopy equivalence. (Here $[-,-]$ denotes homotopy classes of maps.)

If we show H is a homotopy equivalence, then it will follow that f_i^* is a homotopy equivalence for each i. In order to show that H is a homotopy equivalence, it is enough to show that $H \circ f_a^*$ is homotopic to the identity map 1 of $\overline{M}|SF$. The desired homotopy results from the natural isomorphism

$$1 + H \circ f_a^* \cong 1 + 1,$$

which in turn amounts to the natural isomorphism

$$N_j \amalg_{M_a} \left(M_a \oplus \frac{N_j}{M_a} \right) \cong N_j \amalg_{M_a} N_j. \qquad \text{Q.E.D.}$$

2. Interpretation

In this section we interpret the hypothesis of Theorem C', and say that F is *dominant* if the hypothesis is satisfied. The method is to describe $\pi_0(\overline{M}|SF)$ in terms of generators and relations.

An admissible monomorphism of $\mathscr{M}$ whose cokernel is isomorphic to an object in the image of F will be called an F-mono.

We start with the vertices as generators, and all 1-simplices as relations. If two vertices are isomorphic, then they are joined by a 1-simplex. Thus we may forget about the choices for quotients, and take as generators all admissible monomorphisms $(M_a \rightarrowtail N)$ (here N being variable, and M_a, as part of $\overline{M}$, being fixed). We take as *elementary* relations the following:

$$(M_a \rightarrowtail N_0) \sim (M_a \rightarrowtail N_1)$$

if there is an F-mono i: $N_0 \rightarrowtail N_1$ such that

$$\begin{array}{ccc} M_a & \rightarrowtail & N_0 \\ \| & & \downarrow \\ M_a & \rightarrowtail & N_1 \end{array}$$

commutes. The operation $+$ is expressed by

$$(M_a \rightarrowtail N_0) + (M_a \rightarrowtail N_1) := (M_a \rightarrowtail N_0 \amalg_{M_a} N_1),$$

and respects the relations.

If $L \twoheadleftarrow N \rightarrowtail M$ are F-monos, then so are all the arrows in the corresponding pushout diagram. Thus we may convert any chain of elementary equivalences into a chain where the arrows of the same sense are grouped together:

$$\begin{array}{ccccccccccccc} M_a & = & M_a & = & \cdots & = & M_a & = & M_a & = & \cdots & = & M_a & = & M_a \\ \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\ N_0 & \rightarrowtail & N_1 & \rightarrowtail & \cdots & \rightarrowtail & N_i & = & N_j' & \twoheadleftarrow & \cdots & \twoheadleftarrow & N_1' & \twoheadleftarrow & N_0'. \end{array}$$

Thus we arrive at the following theorem.

THEOREM 2.1. *The exact functor F is dominant (and thus Theorem C′ applies to it) if and only if for each admissible monomorphism $M \rightarrowtail N$ of $\mathcal{M}$ there exists an admissible monomorphism $M \rightarrowtail N'$ and a commutative diagram*

$$\begin{array}{ccccccc} M & = & L_0 & \rightarrowtail & \cdots & \rightarrowtail & L_s \\ \downarrow & & & & & & \| \\ N \amalg_M N' & = & N_0 & \rightarrowtail & \cdots & \rightarrowtail & N_q \end{array}$$

in which the horizontal arrows are F-monos.

Remark. Theorem C' is asymmetric, in the sense that it may apply to F^{op} when it doesn't apply to F. A similar theorem which is symmetric is desirable.

Remark. Edgewise subdivision of the square in Theorem C' gives a homotopy cartesian square

$$\begin{array}{ccc} E' & \longrightarrow & E \\ \downarrow & & \downarrow \\ Q\mathscr{P} & \longrightarrow & Q\mathscr{M} \end{array}$$

of categories, where E is the extension construction of $\mathscr{M}$, and E' is the pullback. This fact can be deduced (as pointed out by Swan [7, p. 152]) from the proof in [2], [3] that the square

$$\begin{array}{ccc} S^{-1}E' & \rightarrow & S^{-1}E \\ \downarrow & & \downarrow \\ Q\mathscr{P} & \rightarrow & Q\mathscr{M} \end{array}$$

is homotopy cartesian, but that proof depends on the presence of the localization S^{-1}, for which one needs the additional hypothesis that every object in $\mathscr{M}$ have projective dimension ≤ 1 within $\mathscr{M}$.

Define $\operatorname{im} F$ to be the set of those objects of $\mathscr{M}$ isomorphic to $F(P)$ for some $P \in \operatorname{obj} \mathscr{P}$.

COROLLARY 2.2. *If* $\operatorname{im} F = \operatorname{obj} \mathscr{M}$ *(i.e., F is surjective on isomorphism classes of objects) then F is dominant.*

Call $\operatorname{im} F$ *cofinal* in $\operatorname{obj} \mathscr{M}$ if for any $T \in \operatorname{obj} \mathscr{M}$ there is a $T' \in \operatorname{obj} \mathscr{M}$ with $T \oplus T' \in \operatorname{im} F$.

COROLLARY 2.3. *If* $\operatorname{im} F$ *is cofinal in* $\operatorname{obj} \mathscr{M}$, *then F is dominant.*

Proof. Given $M \rightarrowtail N$ as in the theorem, we find T' so

$$\frac{N}{M} \oplus T' \in \operatorname{im} F.$$

Set $N' = M \oplus T'$, so $N \coprod_M N' = N \oplus T'$. The inclusion $M \rightarrowtail N \coprod_M N'$ is itself an F-mono. Q.E.D.

Call $\operatorname{im} F$ *closed under direct summand* if $M \oplus N \in \operatorname{im} F$ implies $M \in \operatorname{im} F$.

Call $\operatorname{im} F$ *closed under extension* if exactness of

$$0 \rightarrow F(P') \rightarrow M \rightarrow F(P'') \rightarrow 0$$

in $\mathscr{M}$ implies $M \in \operatorname{im} F$.

Call $\operatorname{im} F$ *closed under cokernel* if exactness of

$$0 \rightarrow F(P') \rightarrow F(P) \rightarrow M'' \rightarrow 0$$

in $\mathscr{M}$ implies $M'' \in \operatorname{im} F$. Similarly for *closed under kernel.*

COROLLARY 2.4. *If* im F *is closed under direct summand, extension, and cokernel, then F is dominant if and only if every $M \in$* obj $\mathscr{M}$ *fits into an exact sequence* $0 \to M \to F(P) \to F(P'') \to 0$.

Proof. Since im F is closed under extension, the composition of two F-monos is again an F-mono (to see this, apply the ker-coker exact sequence for a composition).

Assuming F is dominant, we consider the mono $0 \rightarrowtail M$ and find $M' \in$ obj $\mathscr{M}$ and F-monos $0 \rightarrowtail L \leftarrowtail M \oplus M'$. Thus we have an exact sequence

$$0 \to M \oplus M' \xrightarrow{(f,g)} F(P) \to F(P'') \to 0.$$

Consider the ker-coker exact sequence for the composition

$$M \oplus M' \xrightarrow{\begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix}} F(P) \oplus F(P) \overset{+}{\twoheadrightarrow} F(P);$$

it reduces to

$$0 \to F(P) \to \frac{F(P)}{M} \oplus \frac{F(P)}{M'} \to F(P'') \to 0,$$

and allows us to deduce that $F(P)/M \in$ im F, which is what we wanted.

Now we try to prove F is dominant, assuming the other hypothesis. Starting with $M \rightarrowtail N$ as in the theorem we find exact sequences

$$0 \to M \to F(P) \to F(P'') \to 0, \quad 0 \to N \to F(Q) \to F(Q'') \to 0$$

and

$$0 \to M \to F(Q) \to L \to 0.$$

Now form the pushout K as in the diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & M & \to & F(P) & \to & F(P'') & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & F(Q) & \to & K & \to & F(P'') & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & L & = & L & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Exactness of the middle row shows $K \in \operatorname{im} F$, and exactness of the middle column shows $L \in \operatorname{im} F$. The commutative diagram

$$\begin{array}{ccc} M & \rightarrowtail & F(Q) \\ \downarrow & & \| \\ N & \rightarrowtail & F(Q) \end{array}$$

in which the horizontal arrows are F-monos shows F is dominant.

COROLLARY 2.5. *If* $\operatorname{im} F$ *is closed under extension and cokernel, and every* $M \in \operatorname{obj} \mathcal{M}$ *fits into an exact sequence* $0 \to M \to F(P) \to F(P'') \to 0$, *then* F *is dominant.*

Proof. Same as half of the proof of 2.4. Q.E.D.

Call $\operatorname{im} F$ *closed under subobject* if $0 \to M' \to F(P) \to M'' \to 0$ being exact implies $M' \in \operatorname{im} F$. Similarly for *closed under quotient object.*

COROLLARY 2.6. *Suppose* $\mathcal{M}$ *is an abelian category, and* $\operatorname{im} F$ *is closed under subobject. Then* F *is dominant if and only if every object of* $\mathcal{M}$ *has a finite filtration whose consecutive quotients are in* $\operatorname{im} F$.

Proof. First assume the condition about filtrations. Then given $M \rightarrowtail N$ an admissible mono as in 2.1, we may find such a filtration for N/M, which expresses $M \rightarrowtail N$ as a composition of F-monos, showing F is dominant.

Now assume F is dominant. Then by 2.1, for each $M \in \mathcal{M}$ we may find an M' and F-monos $M \oplus M' = N_0 \rightarrowtail \cdots \rightarrowtail N_q = L_s \leftarrowtail \cdots \leftarrowtail L_0 = 0$. We have

$$\frac{L_i \cap M}{L_{i-1} \cap M} \subset \frac{L_i}{L_{i-1}} \in \operatorname{im} F,$$

so the consecutive quotients of the filtration

$$0 = L_0 \cap M \subset \cdots \subset L_q \cap M = M$$

are in $\operatorname{im} F$. Q.E.D.

3. Application

Call an exact subcategory $\mathcal{P}$ of an exact category $\mathcal{M}$ *closed under exact sequences* if any exact sequence of $\mathcal{M}$ whose members are objects of $\mathcal{P}$, is also an exact sequence of $\mathcal{P}$.

THEOREM 3.1. *Assume $\mathcal{P} \subset \mathcal{M}$ is a full exact subcategory closed under exact sequences and under extension. Let F: $\mathcal{P} \to \mathcal{M}$ denote the inclusion. Then $0|SF$ is homotopy equivalent to the subcategory $\mathcal{C}_F$ of $\mathcal{M}$ whose arrows are all the admissible monomorphisms of $\mathcal{M}$ with cokernel isomorphic to an object of $\mathcal{P}$.*

Proof. The set of arrows mentioned is closed under composition because $\mathcal{P}$ is closed under extension in $\mathcal{M}$, and thus $\mathcal{C}_F$ is a category. The obvious map g: $0|SF \to \mathcal{C}_F$ depicted by

$$\begin{array}{c}
\left(\begin{array}{c}
0 = P_0 \rightarrowtail \ \cdots \ \rightarrowtail P_q \\
\hline\hline
0 \rightarrowtail M_0 \rightarrowtail \ \cdots \ \rightarrowtail M_q
\end{array}\right) \\
\Big\downarrow{\scriptstyle g} \\
(M_0 \rightarrowtail \ \cdots \ \rightarrowtail M_q)
\end{array}$$

simply forgets the choices for the quotients M_j/M_i. Choosing cokernels in $\mathcal{P}$ for all F-monos gives a map h: $\mathcal{C}_F \to 0|SF$ the other way, with $g \circ h = 1$. The natural isomorphism $h \circ g \cong 1$ provides a homotopy $h \circ g \sim 1$. Q.E.D.

4. The resolution theorem

Define $i\mathcal{M}$ to be the subcategory of $\mathcal{M}$ whose arrows are the admissible monomorphisms of $\mathcal{M}$. It has 0 as initial object, so is contractible.

THEOREM 4.1. *Assume $\mathcal{P} \subset \mathcal{M}$ is a full exact subcategory closed under exact sequences, extension, and cokernel. Assume that any $M \in \mathcal{M}$ has a resolution*

$$0 \to M \to P \to P'' \to 0$$

with P, $P'' \in \mathcal{P}$. Then the map $S\mathcal{P} \to S\mathcal{M}$ is a homotopy equivalence.

Proof. By 3.1 and 2.5, it is enough to show that $\mathcal{C}_F$ is contractible, where F: $\mathcal{P} \to \mathcal{M}$ is the inclusion. It is enough to show that the inclusion map G: $i\mathcal{P} \to \mathcal{C}_F$ is a homotopy equivalence. For this we will use Theorem A of [6], so for any $M \in \mathcal{M}$ we consider the fiber $M|G$ (or M/G in the notation of Quillen), and show $M|G$ is contractible.

Choose a resolution $0 \to M \to P_0 \to P_0'' \to 0$ with P_0', $P_0'' \in \mathcal{P}$; the mono $M \rightarrowtail P_0$ is an object of $M|G$. Suppose $M \rightarrowtail P$ is any other object of $M|G$.

Pushout gives a diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \to & M & \to & P & \to M'' \to & 0 \\ & & \downarrow & & \downarrow & \| & \\ 0 & \to & P_0 & \to & P_0 \amalg_M P & \to M'' \to & 0 \\ & & \downarrow & & \downarrow & & \\ & & P_0'' & = & P_0'' & & \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}.$$

from which we see that $P_0 \amalg_M P$ and hence M'' are in $\mathscr{P}$. The arrows

$$(M \rightarrowtail P) \to (M \rightarrowtail P_0 \amalg_M P) \leftarrow (M \rightarrowtail P_0)$$

in $M|G$ constitute natural transformations which show that the identity map on $M|G$ is homotopic to a constant map. Q.E.D.

5. The Dévissage theorem

THEOREM 5.1 [6]. *Suppose $\mathscr{M}$ is an abelian category, and $\mathscr{P} \subset \mathscr{M}$ is a full (abelian) subcategory closed under direct sum, subobject, and quotient object. If every object of $\mathscr{M}$ has a finite filtration whose consecutive quotients are in $\mathscr{P}$, then the map $S\mathscr{P} \to S\mathscr{M}$ is a homotopy equivalence.*

Proof. Let F: $\mathscr{P} \to \mathscr{M}$ be the inclusion. By 2.6 it is enough to show that $0|SF$ is contractible. Since $\mathscr{P}$ is a subcategory of $\mathscr{M}$, we see (by the same argument as in the proof of 3.1) that $0|SF$ is homotopy equivalent to the simplicial set $\mathscr{D}_F$, whose q-simplices are all chains $M_0 \rightarrowtail \cdots \rightarrowtail M_q$ of admissible monomorphisms in $\mathscr{M}$, such that M_q/M_0 is in $\mathscr{P}$.

It will suffice to show that the inclusion map G: $\mathscr{D}_F \to i\mathscr{M}$ is a homotopy equivalence. For this we will use Theorem A' of [5], so for any $M \in \mathscr{M}$ we still show the fiber $G|M$ is contractible. Associating an admissible monomorphism $N \rightarrowtail M$ with the corresponding subobject of M gives a map g: $G|M \to \mathscr{B}_M$, where $\mathscr{B}_M$ is the simplicial set whose q-simplices are all chains $N_0 \subset \cdots \subset N_q \subset M$ of subobjects of M such that N_q/N_0 is in $\mathscr{P}$. The map g is a homotopy equivalence, as it has an inverse up to natural isomorphism.

Now choose a filtration $0 = M_0 \subset \cdots \subset M_s = M$ of M such that $M_i/M_{i-1} \in \mathscr{P}$ for all i. Define a map F_i: $\mathscr{B}_M \to \mathscr{B}_M$ by

$$F_i\colon (N_0 \subset \cdots \subset N_q \subset M) \mapsto (N_0 + M_i \subset \cdots \subset N_q + M_i \subset M).$$

Since $N_q + M_i/N_0 + M_i$ is a quotient of N_q/N_0, it is in $\mathscr{P}$ and the latter chain is a simplex of $\mathscr{B}_M$.

The inclusions $N_j + M_{i-1} \subset N_j + M_i$ give a homotopy from F_{i-1} to F_i because

$$\frac{N_q + M_i}{N_0 + M_{i-1}}$$

is a quotient of

$$\frac{N_q}{N_0} \oplus \frac{M_i}{M_{i-1}},$$

and thus is in $\mathscr{P}$.

Since $F_0 = 1_{\mathscr{B}_M}$ and F_s is constant, we see that $\mathscr{B}_M$ is contractible. Q.E.D.

6. Cofinality

THEOREM 6.1 [4, 1.1], [1]. *Assume $\mathscr{P} \subset \mathscr{M}$ is a full exact subcategory closed under exact sequences and extension. If* obj $\mathscr{P}$ *is cofinal in* obj $\mathscr{M}$, *then $K_i\mathscr{P} \to K_i\mathscr{M}$ is an isomorphism for $i \geq 1$, and a monomorphism for $i = 0$.*

Proof. By 3.1, 2.3, and Theorem C', we see that it is enough to show that $\pi_i(\mathscr{C}_F) = 0$ for $i \geq 1$, where F denotes the inclusion $\mathscr{P} \to \mathscr{M}$. Let $\mathscr{B}$ be the connected component of 0 in $\mathscr{C}_F$.

If M is an object of $\mathscr{B}$, then as an object of $\mathscr{M}$, its class in $K_0\mathscr{M}$ lies in the image of the map $K_0\mathscr{P} \to K_0\mathscr{M}$. By an elementary argument from [4, Section 1] there exists $P \in \mathscr{P}$ so $M \oplus P \in \mathscr{P}$.

Now suppose $\mathscr{A} \subset \mathscr{B}$ is any sub-simplicial set with a finite number of objects. By summing the P's found above for each $M \in \mathscr{A}$, we may find a $P \in \mathscr{P}$ so that $M \oplus P \in \mathscr{P}$ for all $M \in \mathscr{A}$. This P can be used to construct a null-homotopy for $\mathscr{A} \hookrightarrow \mathscr{B}$ from the following pair of simplicial homotopies:

$$\begin{array}{c}(M_0 \rightarrowtail \cdots \rightarrowtail M_q)\\ \downarrow \\ (M_0 \oplus P \rightarrowtail \cdots \rightarrowtail M_q \oplus P)\\ \uparrow \\ (0 \rightarrowtail \cdots \rightarrowtail 0)\end{array} \quad .$$

Checking that these maps are actually simplicial homotopies amounts to the assertion that $(M_q \oplus P)/M_0$ and $M_q \oplus P$ are in $\mathscr{P}$, which is what we have arranged above.

It follows now that $\pi_i(\mathscr{C}_F) = \pi_i(\mathscr{B}) = 0$ for $i \geq 1$. Q.E.D.

7. Localization for projective modules

Let R be a ring, and $S \subset R$ a multiplicative set of central nonzerodivisors. As usual $\mathscr{P}_R$ denotes the exact category of finitely generated projective (left) R-modules, and $\mathscr{M}_R$ denotes the exact category of finitely generated R-modules. We let $\mathscr{M} \subset \mathscr{P}_{S^{-1}R}$ be the full subcategory whose objects are those isomorphic to $S^{-1}P$ for some $P \in \mathscr{P}_R$. It is closed under extensions (because all extensions are split), so becomes an exact category in a natural way.

Let $\mathscr{P} \subset \mathscr{M}_R$ denote the full subcategory whose objects are those P with projective R-dimension ≤ 1, and with $S^{-1}P \in \mathscr{M}$. It is closed under extension, so is an exact category.

Let F: $\mathscr{P} \to \mathscr{M}$ send P to $S^{-1}P$.

Let $\mathscr{H} \subset \mathscr{P}$ be the full subcategory whose objects are those P with $S^{-1}P \cong 0$. It is closed under extensions, and is an exact category. For convenience, we may assume $\mathscr{M}$ has only one zero object, so in fact $S^{-1}P = 0$.

THEOREM 7.1. *The square*

$$\begin{array}{ccc} S\mathscr{H} & \to & S\mathscr{P} \\ \downarrow & & \downarrow {\scriptstyle SF} \\ 0 & \to & S\mathscr{M} \end{array}$$

is homotopy cartesian, and thus there is a long exact sequence

$$\cdots \to K_{i+1}S^{-1}R \to K_i\mathscr{H} \to K_iR \to K_iS^{-1}R \to \cdots$$

which ends at $K_0S^{-1}R$.

Proof. The cofinality theorem applied to $\mathscr{M} \subset \mathscr{P}_{S^{-1}R}$ justifies the "and thus" part.

The functor F is dominant because it is surjective on isomorphism classes of objects. We will apply Theorem C' to F^{op}, which is also dominant. We get a commutative diagram

$$\begin{array}{ccccc} S\mathscr{H}^{\mathrm{op}} & \xrightarrow{G} & 0|SF^{\mathrm{op}} & \to & S\mathscr{P}^{\mathrm{op}} \\ \downarrow & & \downarrow & & \downarrow {\scriptstyle SF^{\mathrm{op}}} \\ 0 & \xrightarrow{\sim} & 0|S\mathscr{M}^{\mathrm{op}} & \to & S\mathscr{M}^{\mathrm{op}} \end{array}$$

in which the right hand square is homotopy cartesian, and the map marked $\sim$ is a homotopy equivalence.

In the opposite of an exact category, the epis and monos interchange roles, so we may depict a typical q-simplex t of $0|SF^{\mathrm{op}}$ by the following diagram of admissible epis:

$$t = \left(\begin{array}{c} 0 = P_0 \twoheadleftarrow \cdots \twoheadleftarrow P_q \\ \hline\hline 0 \twoheadleftarrow M_0 \twoheadleftarrow \cdots \twoheadleftarrow M_q \end{array}\right).$$

The map G is then described by

$$\begin{array}{c} (0 = H_0 \twoheadleftarrow \cdots \twoheadleftarrow H_q) \\ G \Big\downarrow \\ \left(\begin{array}{c} 0 = H_0 \twoheadleftarrow \cdots \twoheadleftarrow H_q \\ \hline\hline 0 \twoheadleftarrow 0 \twoheadleftarrow \cdots \twoheadleftarrow 0 \end{array}\right). \end{array}$$

where $H_i \in \mathscr{H}$.

It is enough to prove that G is a homotopy equivalence, and for this we will use Theorem A′ of [5]; so for each t as above it is enough to show that the fiber $G|t$ is contractible. A typical p-simplex u of it can be depicted by the diagram

$$u = \left(\begin{array}{ccccccccccc} & & & & & 0 = P_0 & \twoheadleftarrow P_1 & \twoheadleftarrow & \cdots & \twoheadleftarrow & P_q \\ & & & & & \downarrow & \square \downarrow \square & & & \square & \downarrow \\ & 0 = H_0 \twoheadleftarrow & \cdots & H_p & \twoheadleftarrow & P_0' & \twoheadleftarrow P_1' & \twoheadleftarrow & \cdots & \twoheadleftarrow & P_q' \\ \hline\hline 0 \twoheadleftarrow & 0 \twoheadleftarrow & \cdots & 0 & \twoheadleftarrow & M_0 & \twoheadleftarrow M_1 & \twoheadleftarrow & \cdots & \twoheadleftarrow & M_q \end{array}\right)$$

The second and third rows portray a $p + q + 1$-simplex of $0|SF^{\mathrm{op}}$ whose lefthand p-face is in the image of G. The cartesian squares marked $\square$ are a way of displaying some of the choices for kernels implicit in the data making up a simplex, but not usually drawn: we do this so we can indicate that the right hand q-face of this simplex is t. The first and third rows are fixed, and the second is variable. Notice that because $p \geq 0$, the simplex u includes isomorphisms $S^{-1}P_i' \cong M_i$ as part of its data.

The simplicial set $G|t$ is easily seen to be homotopy equivalent to the category $\mathscr{C}$ where an object is a diagram

$$V' = \left(\begin{array}{ccccccc} 0 & \twoheadleftarrow P_1 & \twoheadleftarrow & \cdots & \twoheadleftarrow & P_q \\ \downarrow & \square \downarrow \square & & & \square & \downarrow \\ P_0' & \twoheadleftarrow P_1' & \twoheadleftarrow & \cdots & \twoheadleftarrow & P_q' \\ * \downarrow & * \downarrow & & & & * \downarrow \\ M_0 & \twoheadleftarrow M_1 & \twoheadleftarrow & \cdots & \twoheadleftarrow & M_q \end{array}\right)$$

in which the arrows marked $*$ are R-homomorphisms inducing isomorphisms $S^{-1}P_i' \xrightarrow{\sim} M_i$. An arrow $V'' \to V'$ of $\mathscr{C}$ is a diagram

$$\left(\begin{array}{ccccccc}
0 & \twoheadleftarrow & P_1 & \twoheadleftarrow & \cdots & \twoheadleftarrow & P_q \\
\downarrow & \square & \downarrow & \square & & \square & \downarrow \\
P_0'' & \twoheadleftarrow & P_1'' & \twoheadleftarrow & \cdots & \twoheadleftarrow & P_q'' \\
f_0\downarrow & \square & f_1\downarrow & \square & & \square & f_q\downarrow \\
P_0' & \twoheadleftarrow & P_1' & \twoheadleftarrow & \cdots & \twoheadleftarrow & P_q' \\
*\downarrow & & *\downarrow & & & & *\downarrow \\
M_0 & \twoheadleftarrow & M_1 & \twoheadleftarrow & \cdots & \twoheadleftarrow & M_q
\end{array}\right)$$

which gives rise to V'' and V' by composition of the appropriate vertical arrows (and $\operatorname{coker}(f_i) \in \mathscr{H}$ for each i). In order to show that $\mathscr{C}$ is contractible, it will suffice to show that $\mathscr{C}$ is filtering and nonempty.

To see the nonemptiness, we write $M_0 \cong S^{-1}P$, $P \in \mathscr{P}_R$, and split the $M_q \twoheadrightarrow M_0$, allowing us to write $M_i \cong M_0 \oplus S^{-1}P_i$. Then setting $P_i' = P \oplus P_i$ and filling the arrows in an obvious way gives an object of $\mathscr{C}$.

Now we check that $\mathscr{C}$ is filtering. Suppose V' and V'' are any two objects of $\mathscr{C}$, with parts labelled in the obvious way. We search for a third object V with maps to both of these. For some $s \in S$ we get maps f', f'' which make

$$\begin{tikzcd}
P \arrow[r, tail, "s"] \arrow[d, "f'"'] & P \arrow[d, "*"] & P \arrow[l, tail, "s"'] \arrow[d, "f''"] \\
P_0' \arrow[r, "*"] & M_0 & P_0'' \arrow[l, "*"']
\end{tikzcd}$$

commute. Now because S consists of nonzerodivisors and P is projective, the map

$$P \overset{s}{\rightarrowtail} P$$

is an admissible monomorphism, and the map

$$P \xrightarrow{*} M_0$$

is injective. It follows that f' and f'' are injective. From the exact sequence

$$0 \to P \xrightarrow{f'} P_0' \to T' \to 0$$

where P is projective, and P_0' has projective dimension ≤ 1, it follows that T' has projective dimension ≤ 1, so $T' \in \mathscr{H} \subset \mathscr{P}$. Thus f' and f'' are admissible monomorphisms of $\mathscr{P}$. By replacing each of V' and V'' by its pullback to P we reduce to the case where $P_0' = P_0'' = P$. Now we claim that we can pull back further by

$$P \xrightarrow{t} P$$

for some $t \in S$ so that V' and V'' both pull back to the same thing. For this purpose, we state some simple lemmas.

LEMMA 7.2. *Suppose* $0 \to A \to B \to C \to 0$ *is an exact sequence of* R*-modules, and* A *is an* $S^{-1}R$ *module. If two elements of* B *have the same images in* C *and in* $S^{-1}B$ *then they are equal.*

Proof. Their difference is in A, and is S-torsion, thus zero. Q.E.D.

LEMMA 7.3. *Suppose we have two exact sequences of* R*-modules,*

$$E\colon 0 \to A \xrightarrow{j} B \xrightarrow{q} C \to 0 \quad \text{and} \quad E'\colon 0 \to A \xrightarrow{j'} B' \xrightarrow{q'} C \to 0,$$

and suppose A *is an* $S^{-1}R$*-module. Assume we have an isomorphism* f *which makes*

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, equal] & S^{-1}B \arrow[r] \arrow[d, "f", "\sim"'] & S^{-1}C \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & A \arrow[r] & S^{-1}B' \arrow[r] & S^{-1}C \arrow[r] & 0 \end{tikzcd}$$

commute. Then there is a unique isomorphism g *such that* $S^{-1}g = f$ *and which makes*

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "j"] \arrow[d, equal] & B \arrow[r, "q"] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & A \arrow[r, "j'"] & B' \arrow[r, "q'"] & C \arrow[r] & 0 \end{tikzcd}$$

commute.

Proof. The uniqueness follows from the previous lemma. We define $g(b)$ as follows. Find $b' \in B'$ so $q'(b') = q(b)$; then find $a \in A$ so

$$j'(a) = f\left(\frac{b}{1}\right) - \frac{b'}{1}.$$

Define $g(b) = b' + j'(a)$. Q.E.D.

If E: $0 \to A \to B \to C \to 0$ is an exact sequence of R-modules, we let E_s be the pushout along

$$A \xrightarrow{s} A,$$

and let E^s be the pullback along

$$C \xrightarrow{s} C.$$

In the next lemma, when we speak of an isomorphism $E \cong E'$, we mean one which is the identity on A and C.

LEMMA 7.4. *Suppose*

$$E: 0 \to A \to B \to C \to 0 \quad \textit{and} \quad E': 0 \to A \to B' \to C \to 0$$

are two exact sequences of finitely presented R-modules. If

$$h: S^{-1}E \to S^{-1}E'$$

is an isomorphism, then for some $s \in S$ there is an isomorphism f: $E_s \to E'_s$ such that $S^{-1}f = h$. Moreover, if f, g are isomorphisms $E \to E'$ such that $S^{-1}f = S^{-1}g$, then for some $s \in S$ we have equality $f_s = g_s$: $E_s \to E'_s$. Finally, the previous two statements apply equally well to pullbacks E^s.

Proof. The statement about pushouts follows from the previous lemmas and the usual inductive limit argument. The statement about pullbacks follows from the canonical isomorphism $E_s \cong E^s$. Q.E.D.

Continue the proof of 7.1. Consider the exact sequences

$$E'_i = (0 \to P_i \to P'_i \to P \to 0) \quad \text{and} \quad E''_i = (0 \to P_i \to P''_i \to P \to 0)$$

which can be extracted from V' and V''. By 7.4 we find $t \in S$ so for all i, there is an isomorphism $E'^t_i \cong E''^t_i$ compatible with the isomorphism $S^{-1}E'_i \cong S^{-1}E''_i$ provided by V' and V''. To see whether these isomorphisms assemble into an isomorphism $V'^t \cong V''^t$ of the pullbacks by t amounts to checking commutativity of the squares

$$\begin{array}{ccc} P'_i & \leftarrow & P'_{i+1} \\ \downarrow & & \downarrow \\ P''_i & \leftarrow & P''_{i+1} \end{array}$$

The maps

$$P'_{i+1} \rightrightarrows P''_i$$

(whose equality would mean the square commutes) agree on P_{i+1}, so their difference factors through the quotient P, and is killed by some $u \in S$ (all i). The pullback by

$$P \overset{u}{\rightarrowtail} P$$

kills the difference, and thus we get an isomorphism $V'^{tu} \cong V''^{tu}$.

The final step in the proof that $\mathscr{C}$ is filtering is to take two arrows

$$V'' \underset{g}{\overset{f}{\rightrightarrows}} V'$$

and produce an arrow

$$V^{\#} \xrightarrow{h} V''$$

with $fh = gh$. As before, we may pull V'' back, replacing P_0'' with some projective P. Then $\ker(f_0 - g_0) \subset P_0''$ is S-torsion, but P_0'' is S-torsion free, so $f_0 = g_0$. The differences $f_i - g_i$ $(i > 0)$ are zero on P_i, so factor through P; they are also S-torsion. Thus we can find $u \in S$ with $uf_i = ug_i$, so pulling back along u: $P \to P$ equalizes all the pairs f_i, g_i. Q.E.D.

8. New definitions for K-theory

Let $\mathscr{M}$ be an exact category. We say that a sequence

$$E: 0 \to M_1 \to \cdots \to M_n \to 0$$

in $\mathscr{M}$ is *exact* if it can be obtained by splicing together short exact sequences of $\mathscr{M}$. Let $\mathscr{E}_n$ be the category of exact sequences of length n (like E) in $\mathscr{M}$. Define a short exact sequence of $\mathscr{E}_n$ to be one which is exact in each degree.

LEMMA 8.1. *$\mathscr{E}_n$ is an exact category.*

Proof. Let $\mathscr{M}$ be a full exact subcategory of an abelian category $\mathscr{A}$, closed under extension and exact sequence. Let $\mathscr{C}$ be the category of complexes in $\mathscr{A}$; it is an abelian category, and $\mathscr{E}_n$ is a full subcategory of $\mathscr{A}$. One checks easily that $\mathscr{E}_n$ is closed under extension in $\mathscr{C}$, so is an exact category. Q.E.D.

Letting $\mathscr{M}^n$ denote the n-fold cartesian product of $\mathscr{M}$, we consider the evident exact functor F_n: $\mathscr{E}_n \to \mathscr{M}^n$ defined by

$$F_n(E) = (M_1, \ldots, M_n).$$

For any $(M_1, \ldots, M_n) \in \mathcal{M}^n$ we see that

$$(M_1, \ldots, M_n) \oplus (0, M_1, \ldots, M_{n-3}, M_{n-2} \oplus M_n, M_{n-1}) \in \operatorname{im} F_n,$$

so im F_n is cofinal in obj $\mathcal{M}^n$, and F_n is dominant (2.3).

THEOREM 8.2. *There is a natural homotopy equivalence $G\mathcal{M} \to 0|SF_n$, and thus $K_i\mathcal{M} = \pi_i(0|SF_n)$.*

Proof. Any object E of $\mathcal{E}_n$ fits into a short exact sequence of $\mathcal{E}_n$:

$$\begin{array}{ccccccccccc} 0 & \to M_1 & \to & \cdots & \to M_{n-2} & \to M' & \to & 0 & \to 0 \\ & \downarrow & & & \downarrow & \downarrow & & \downarrow & \\ 0 & \to M_1 & \to & \cdots & \to M_{n-2} & \to M_{n-1} & \to & M_n & \to 0 \\ & \twoheadrightarrow & & & \twoheadrightarrow & \twoheadrightarrow & & \twoheadrightarrow & \\ 0 & \to 0 & \to & \cdots & \to 0 & \to M_n & \to & M_n & \to 0. \end{array}$$

From Waldhausen's version of the additivity theorem [8, p. 183] we deduce a natural fibration sequence $S\mathcal{E}_{n-1} \to S\mathcal{E}_n \to S\mathcal{M}$ for $n \geq 3$, and thus a homotopy cartesian square

$$\begin{array}{ccc} S\mathcal{E}_{n-1} & \to & S\mathcal{E}_n \\ \downarrow & & \downarrow \\ S\mathcal{M}^{n-1} & \to & S\mathcal{M}^n \end{array}.$$

Therefore the map $0|SF_{n-1} \to 0|SF_n$ is a homotopy equivalence, by Theorem C′.

As for $0|SF_2$, observe that $\mathcal{E}_2$ is equivalent to $\mathcal{M}$, and $0|SF_2$ is equivalent to $G\mathcal{M}$. Alternatively, one can see directly that the homotopy fiber of

$$S\mathcal{M} \xrightarrow{\Delta} S\mathcal{M}^2$$

is homotopy equivalent to the loop space on $S\mathcal{M}$, and thus arrive at a second proof that $|G\mathcal{M}| \sim \Omega|S\mathcal{M}|$ [5]. Q.E.D.

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