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CANONICAL RING OF A CURVE IS KOSZUL: A SIMPLE PROOF

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1. Introduction

In this article we prove, for canonical model of curves, a theorem illustrating the general principle that (to paraphrase Arnold) any homogeneous ring that has a serious reason for being quadratically presented is *Koszul*. In this case we give a new proof, which is both elementary and geometric, of a theorem of Finkelberg and Vishik [VF] (see also [Po]) which says that whenever the canonical ring of a smooth complex projective curve is quadratically presented, it is *Koszul*. Our method is different from [Po]. We use vector bundle technique, building upon the one used in [GL]. We would also like to mention here that our methods fit a more general principle as shown in [GP1], [GP2] and [GP3].

A. The Koszul conditions. Let k be a field. A (commutative) graded k-algebra of the form $R := k \oplus R_1 \oplus \cdots \oplus R_n \cdots$ is said to be *Koszul* if its Koszul complex is exact, or, equivalently, if $k = R/R_{>0}$ has a *linear* minimal resolution over R; namely

$$\cdots \to E_p \to E_{p-1} \to \cdots \to E_2 \to E_1 \to E_0 \to k \to 0$$

with $E_0 = R$ and $E_p = R(-p)^{\oplus r(p)}$ for any $p \geq 1$. Denote the syzygy modules by $R^{(p)} := \ker(E_p \to E_{p-1})$; this means that for any $p \geq 0$ the $R^{(p)}$'s are generated in degree $p+1$ (the minimal degree) as graded R-modules (we refer to the treatment of [BGS] for generalities on Koszul rings, in a much more general context).

When R is a commutative algebra "arising from algebraic geometry", e.g., $R_E = \bigoplus_i H^0(X, E^{\otimes i})$, where X is a projective variety and E some line bundle on X, the Koszul conditions have a convenient interpretation in terms of line bundles due to Lazarsfeld. To see this, it is useful to set the following notation: if F is a sheaf on X, M_F will denote the kernel of the evaluation map $H^0(X, F) \otimes \mathcal{O}_X \to F$. Note that if F is globally generated and locally free on X then M_F is locally free. However, if H is locally free then $H^0(M_F \otimes H)$ is the kernel of the multiplication map $H^0(F) \otimes H^0(H) \to H^0(F \otimes H)$. Therefore, as it is immediate to see, $R_E^{(1)} = \bigoplus_i H^0(X, M_E \otimes E^{\otimes i})$, $R_E^{(2)} = \bigoplus_i H^0(X, M_{M_E \otimes E} \otimes E^{\otimes i})$ and so on. Inductively, let us set $M_E^0 := E$, $M_E^1 := M_E \otimes E$, $M_E^2 := M_{M_E^1} \otimes E, \ldots, M_E^p := M_{M_E^{p-1}} \otimes E$

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for any p. In this setting to be a Koszul algebra means that the multiplication map of global sections

$$H^0(M_E^p) \otimes H^0(E^{\otimes n}) \to H^0(M_E^p \otimes E^{\otimes n}) \tag{1}$$

is surjective for any $p \geq 0$ and $n \geq 1$. We refer for instance to [P] for more details.

B. Primitive pencils. Let us recall the following terminology: a line bundle A on C is said to be *primitive* if both A and $K_C \otimes A^\vee$ are base point free. If moreover $h^0(A) = 2$, A is said to be a *primitive pencil.* It is well known that the existence of certain families of primitive pencils is a meaningful geometric condition. This is also a key point in Finkelberg and Vishik's proof. The following result is well known.

THEOREM 1. *A curve C of genus $g \geq 5$ has a primitive pencil of degree $g - 1$ if and only if it is not hyperelliptic, trigonal or isomorphic to a smooth plane quintic.*

For non bielliptic curves this is generally proved using the Martens-Mumford's Theorem, which ensures that the general element of *every* component of the Brill-Noether variety $W^1_{g-1}(C)$ parametrizes a primitive pencil (see e.g. [ACGH], pp. 372–3). For bielliptic curves there is *one* component of $W^1_{g-1}(C)$ parametrizing primitive pencils (see e.g. [S], [W] and [CS]). The "only if" part of the theorem can be found in [ACGH].

We would like to remark at this point that the statement in [VF] leaves open the case of bielliptic curves. However it is easy to see that the arguments, presented here and in [VF], also work for bielliptic curves.

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2. Some filtrations

In this section we will prove a generalization of a result of [GL] which will be the main technical tool used in the proof.

Let A be a primitive pencil of degree $g - 1$. Hence $K_C \otimes A^\vee$ is a primitive pencil too. Clearly $M_A = A^\vee$ and $M_{K_C \otimes A^\vee} = K_C^\vee \otimes A$. Moreover let $D = p_1 + \cdots + p_d$ be a general divisor in the linear system $|A|$. Since we are over the complex field we can assume that the points p_i are distinct. It is also clear that for every effective divisor D^1 strictly contained in D we have $h^0(\mathcal{O}(D^1)) = 1$ since otherwise A would have base points. Therefore, by Riemann-Roch, $h^0(K_C(-D^1)) = g - \deg D^1$; i.e., any proper effective subdivisor of D imposes independent conditions to the canonical system $H^0(K_C)$. Let us write $D = D^1 + D^2$ and, for any two points $p, q \in D^2$, let $D^3 = D^2 - p - q$.

LEMMA 2. *In the above situation assume that* $0 \leq \deg D^1 \leq g-3$. *Then we have the exact sequences*

$$0 \to A \to M_{K_C(-D^1)} \otimes K_C \to \Lambda \to 0 \tag{2}$$

$$0 \to K_C(-p-q) \to \Lambda \to \bigoplus_{p_i \in D^3} K_C(-p_i) \to 0 \tag{3}$$

Proof. This lemma is proved in [GL] in the case $D^1 = 0$. The present proof is a straightforward generalization of the argument in [GL] and we include it for sake of self-containedness. First of all let us observe that $K_C(-D^1)$ is base point free: since $K_C \otimes A^\vee$ is base point free the only possible base points are the points of D^2 but if this was the case we would have a divisor strictly contained in D not imposing independent conditions to $H^0(K_C)$. We have a commutative exact diagram

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & M_{K_C(-D)} & \to & M_{K_C(-D^1)} & \to & \Sigma_{K_C(-D^1),D^2} & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & H^0(K_C(-D)) \otimes \mathcal{O}_C & \to & H^0(K_C(-D^1)) \otimes \mathcal{O}_C & \to & V_{K_C(-D^1),D^2} \otimes \mathcal{O}_C & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & K_C(-D) & \to & K_C(-D^1) & \to & K_C(-D^1)_{|D^2} & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 &
\end{array} \tag{4}$$

where $V_{K_C(-D^1),D^2} = H^0(K_C(-D^1))/H^0(K_C(-D))$ and $\Sigma_{K_C(-D^1),D^2} = \ker(V_{K_C(-D^1),D^2} \otimes \mathcal{O}_C \to K_C(-D^1)_{|D^2})$. Moreover, $K_C(-D^1+p+q)$ is base point free too (arguing as above) and then there is also a diagram like (4) taking $K_C(-D^1+p+q)$ instead of $K_C(-D^1)$ and D^3 instead of D^2. Therefore we get a commutative exact diagram

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & \mathcal{O}_C(-p-q) & \to & \Sigma_{K_C(-D^1),D^2} & \to & \Sigma_{K_C(-D^1),D^3} & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & V \otimes \mathcal{O}_C & \to & V_{K_C(-D^1),D^2} \otimes \mathcal{O}_C & \to & V_{K_C(-D^1),D^3} \otimes \mathcal{O}_C & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & K_C(-D^1)_{|p+q} & \to & K_C(-D^1)_{|D^2} & \to & K_C(-D^1)_{|D^3} & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 &
\end{array} \tag{5}$$

where:

(a) the middle column is the last column of diagram (4);

(b) the last column is the last column of the above mentioned diagram like (4) with $K_C(-D^1+p+q)$ instead of $K_C(-D^1)$ and D^3 instead of D^2;

(c) the first column is $V \cong H^0(K_C(-D^1 - D^3)/H^0(K_C(-D^1 - D^2)$ and the third vertical arrow is evaluation, which is surjective since a section $s \in H^0(K_C(-D^1 - D^3))$ which does not vanish on $D = D^1 + D^2$ cannot vanish at either of p and q.

Therefore since $\dim V = 1$, the kernel is $\mathcal{O}_C(-p-q)$.

Next, let us observe that $\Sigma_{K_C(-D^1),D^2}$ is isomorphic to $\bigoplus_{p_i \in D^3} \mathcal{O}_C(-p_i)$. Indeed, since $\dim V_{K_C(-D^1),D^3} = \deg D^3 := n$, the evaluation map $V_{K_C(-D^1),D^3} \otimes \mathcal{O}_C \to K_C(-D^1)_{|D^3}$ decomposes in n surjective maps $V_i \otimes \mathcal{O}_C \to K_C(-D^1)_{|p_i}$, whose kernels are $\mathcal{O}(-p_i)$. The lemma follows taking as sequence (2) and (3) the first rows of diagrams (4) and (5) tensored by K_C (recall that $M_{K_C(-D)} = K_C^\vee \otimes A$). □

THEOREM [*VF*]. If C is a non-hyperelliptic, non-trigonal curve which is not a plane quintic then the canonical ring of C is *Koszul*.

3. The proof

We keep the notation of the previous sections. The strategy will be to prove the theorem of Finkelberg and Vishik by verifying conditions (1) for $E = K_C$ and in order to do that one repeatedly uses Lemma 2. To this purpose let us introduce the following slight variation on the notation of Section 1.A: if E is a sheaf on C we let $\tilde{M}_E^0 := E$, $\tilde{M}_E^1 := M_{\tilde{M}_E^0} \otimes K_C$ and inductively define $\tilde{M}_E^j := M_{\tilde{M}_E^{j-1}} \otimes K_C$ for any j. For C, A and D as in the previous sections we will prove:

PROPOSITION 3. *Let D^1 be any effective or zero divisor contained in D such that $0 \leq \deg D^1 \leq 2$. Then the map $H^0(\tilde{M}^j_{K_C(-D^1)}) \otimes H^0(K_C^{\otimes n}) \to H^0(\tilde{M}^j_{K_C(-D^1)} \otimes K_C^{\otimes n})$ is surjective for any $j \geq 0$.*

In view of Section 1.A, the case $D^1 = 0$ of the proposition is the theorem (since $\tilde{M}^j_{K_C} = M^j_{K_C}$). To prove Proposition 3 it is convenient to use the following ad hoc terminology:

Definitions. Given three vector bundles E, E_1 and E_2 on C we will say that E is *cohomologically the direct sum of E_1 and E_2*, and we will write $E \equiv E_1 \oplus E_2$, if there is an extension $0 \to E_i \to E \to E_j \to 0$, exact on global sections, with $1 \leq i, j \leq 2$, $i \neq j$. Inductively, we will say that $E \equiv \bigoplus_{i=1}^m E_i$ if $E \equiv F \oplus G$ and $F \equiv \bigoplus_{i \in X_1} E_i$ and $G \equiv \bigoplus_{i \in X_2} E_i$ with $X_1 \amalg X_2 = \{1, \dots, m\}$. In this case we will also say that E is *cohomologically a direct sum of copies of certain bundles* $F_1, \dots, F_k$ if every E_i is isomorphic to some F_j.

The proof of the following lemma is by induction on m and left to the reader:

LEMMA 4. *Suppose that E_i are globally generated sheaves for $i = 1, \dots, m$ and that $E \equiv \bigoplus_{i=1}^m E_i$. Moreover let K be a locally free sheaf on C and assume that the* *multiplication maps* $H^0(E_i)\otimes H^0(K) \to H^0(E_i\otimes K)$ *are surjective. Then* $M_E\otimes K \equiv \bigoplus_{i=1}^m M_{E_i} \otimes K$ *and the multiplication map* $H^0(E) \otimes H^0(K) \to H^0(E \otimes K)$ *is surjective.*

We are now ready to prove Proposition 3. To simplify the notation we will prove the statement only for $n = 1$, since the general case is similar but easier. The key point is the following:

LEMMA 5. *Under the hypotheses of Proposition* 3, *for any* $j \geq 1$, $\tilde{M}^j_{K_C(-D^1)}$ *is cohomologically a direct sum of copies of* A, $K_C \otimes A^\vee$, *and line bundles of the form* $K_C(-D^1)$, *with* D^1 *again as in the statement of Proposition* 3 (*i.e.*, D^1 *contained in* D *and* $0 \leq \deg D^1 \leq 2$).

Proof of Lemma 5. Induction on j: the case $j = 1$ follows from Lemma 2. The only thing to show is that sequences (2) and (3) are exact at the global sections level, and this holds since on the one hand $h^0(M_{K_C(-D^1)}\otimes K_C) \leq h^0(A)+h^0(\Lambda) = 2+(g-3-\deg D^1)h^0(K_C(-p_i))+h^0(K_C(-p-q)) = g^2-(g-1)\deg D^1-3g+3$ (we have $h^0(K_C(-p_i)) = g-1$ and $h^0(K_C(-p-q)) = g-2$ since C is not hyperelliptic), and on the other hand $h^0(M_{K_C(-D^1)} \otimes K_C) \geq g^2-(g-1)\deg D^1-3g+3$ since it is the dimension of the kernel of the multiplication map $H^0(K_C(-D^1)) \otimes H^0(K_C) \to H^0(K_C^{\otimes 2}(-D^1)$. This also proves that such multiplication maps are surjective, a well known and easy fact. If the statement is true at $j-1$ then it is true at j. This follows applying Lemma 4 to $M_{\tilde{M}^{j-1}_{K_C(-D^1)}} \otimes K_C := \tilde{M}^j_{K_C(-D^1)}$. In fact all of A, $K_C \otimes A^\vee$ and line bundles of type $K_C(-D^1)$ as above are globally generated, and, moreover, the multiplication maps $H^0(A) \otimes H^0(K_C) \to H^0(K_C \otimes A)$, $H^0(K_C \otimes A^\vee) \otimes H^0(K_C) \to H^0(K_C^{\otimes 2} \otimes A^\vee)$ are obviously surjective, while the multiplication maps $H^0(K_C(-D^1)) \otimes H^0(K_C) \to H^0(K_C^{\otimes 2}(-D^1))$ are surjective by the previous step. Then, by Lemma 4, $\tilde{M}^j_{K_C(-D^1)}$ is cohomologically a direct sum of copies of A, $K_C\otimes A^\vee$ and of bundles of type $\tilde{M}_{K_C(-D^1)}$, again with $0 \leq \deg D^1 \leq 2$. The statement at j then follows since, by the initial step, the bundles $\tilde{M}_{K_C(-D^1)}$ with $0 \leq \deg D^1 \leq 2$ are in turn cohomologically direct sum of copies of A, $K_C \otimes A^\vee$ and line bundles of type $K_C(-D^1)$ as above. This proves Lemma 5. □

Finally, Lemma 5 and the last part of the statement of Lemma 4 prove the Theorem. □

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