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ON EXTENSION OF CR FUNCTIONS FROM PIECEWISE SMOOTH MANIFOLDS INTO A WEDGE

ANDREI E. VITYAEV

1. Introduction

In this paper we study a holomorphic extendibility of CR functions on manifolds in $\mathbf{C}^N$, i.e., functions which are annihilated by all tangential Cauchy-Riemann differential operators. The following classical result says roughly that every CR function on an "angle" formed by manifolds extends analytically into a wedge if the manifolds intersect in a certain "generic" way. The latter condition is formulated in terms of defining functions.

Given a function $g \in C^1(\mathbf{C}^N)$, we write $\partial g = \sum \frac{\partial g}{\partial z_j} dz_j$.

THEOREM 1.1 (Airapetyan-Henkin). *Let $\rho_1, \ldots, \rho_l$, $l > 1$, be real valued functions on $\mathbf{C}^N$ of class C^2 such that $\partial\rho_1 \wedge \cdots \wedge \partial\rho_l \neq 0$ and $\rho_1(0) = \cdots = \rho_l(0) = 0$. Suppose that l smooth manifolds with boundary $M_1, \ldots, M_l$ are given by*

$$M_k = \{z \in \mathbf{C}^N : \rho_j(z) = 0, 1 \leq j \leq l, \ \ j \neq k, \rho_k(z) \geq 0\}. \tag{1}$$

If U is a continuous function on $M_1 \cup \cdots \cup M_l$ which satisfies the tangential Cauchy-Riemann equations on each M_j, $j = 1, \ldots, l$ near 0, then U extends holomorphically to a small wedge with edge $M_1 \cap \cdots \cap M_l$ near 0.

The theorem of Airapetyan and Henkin is a generalization of the Edge of the Wedge theorem, which was proved in 1957 in the connection with dispersion relations in quantum mechanics. See [BMP] for the original proof and [Vl] as a general reference. In [Tu2], Tumanov proved a more precise version of Theorem 1.1. Namely, he proved that in Theorem 1.1 we have extendibility to a small wedge which is formed by small pieces of manifolds $M_1, \ldots, M_l$.

Note that the statement of Theorem 1.1 does not provide us with any kind of estimate on the size of the wedge. In general, not much is known about the size of the holomorphic hull of CR manifolds. We can mention [BDN], [BN], [BPP] where some estimates where obtained in the case of one smooth manifold.

In this paper we prove, by a new method, a stronger version of Theorem 1.1. Namely, we obtain estimates on the size of the wedge of extendibility. The estimates

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depend on the size of the second derivatives of (normalized) defining functions of the manifolds forming the angle.

Now we state the main result of this paper. By $C^{2,\alpha}$, $0 < \alpha < 1$ we denote the space of twice differentiable functions whose second derivatives belong to the Holder class C^α. Let $\|g\|_{\alpha,S}$ denote the C^α norm of g over the set S; i.e., let

$$\|g\|_{\alpha,S} = \sup_{x\in S} |g(x)| + \sup_{x,y\in S,\ |x-y|\le 1} \frac{|g(x)-g(y)|}{|x-y|^\alpha}. \tag{2}$$

If g is a vector or a matrix, the absolute value in (2) has to be replaced by the vector or the matrix norm. Let $\mathbf{T}$ be the unit circle. Let $\mathcal{H}$ be the Hilbert transform on $\mathbf{T}$ normalized by $(\mathcal{H}h)(1) = 0$. The C^α-norm of $\mathcal{H}$ will be denoted by $\kappa = \kappa(\alpha)$. For a function $\omega(\xi)$ on $\mathbf{C}^N$ we denote by $\|\omega''(\xi)\|$ the norm of the $2N \times 2N$ matrix formed by the second derivatives of the real and imaginary parts of ω. By $\frac{\partial\rho}{\partial\xi}$ we denote the $l \times N$ matrix of complex derivatives $\frac{\partial\rho_j}{\partial\xi_k}$.

THEOREM 1.2. *Let $\rho_1, \dots, \rho_l \in C^{2,\alpha}$, $l > 1$ be real valued functions on $\mathbf{C}^N$ such that $\rho_1(0) = \cdots = \rho_l(0) = 0$. Suppose there exist constants C_1, C_2, R such that*

$$C_1|t| \ge \left|\left(\frac{\partial\rho(0)}{\partial\xi}\right)^T t\right| \ge |t|, \quad \forall t \in \mathbf{C}^l, \tag{3}$$

$$\|\rho_j''(\xi)\|_{\alpha,S_1} \le C_2, \quad S_1 = \{\xi \in \mathbf{C}^N : |\xi| \le R\}, \quad j = 1, \dots, l. \tag{4}$$

Then if $U(\xi)$ is a continuous CR-function on $(M_1 \cup \cdots \cup M_l) \cap \{\xi \in \mathbf{C}^N : |\xi| \le R\}$ (defined by (1)) then U extends holomorphically to the wedge $\mathcal{W}$ with edge $M = M_1 \cap \cdots \cap M_l$ which is given by

$$\mathcal{W} = \{\xi + \zeta \in \mathbf{C}^N : \xi \in M, |\xi| \le R_1, \ \zeta \in V_\xi\}$$

where

$$V_\xi = \{\zeta \in \mathbf{C}^N : |\zeta| \le R_2, \ \rho_j(\zeta + \xi) \ge \widetilde{C}|\zeta|^2, \ j = 1, \dots, l\}.$$

The constants R_1, R_2, $\widetilde{C}$ are given by

$$R_1 = \frac{1}{192} \min\left\{\frac{1}{4lC_2}, R\right\},$$

$$\widetilde{C} = C_2 2^{17}(C_1+1)^2 l^{12}(1+\kappa)^4(1+2lC_2)^3,$$

$$R_2 = \frac{1}{2^{15}(C_1+1)l^3(1+\kappa)^2} \min\left\{\frac{1}{l^5(1+\kappa)^2 C_2(1+2lC_2)^3}, R\right\}.$$

Note that the right part of inequality (3) simply means that $\partial\rho_1(0)\wedge\cdots\wedge\rho_l(0)\neq 0$.

Also note that when C_2 tends to zero and R tends to infinity, the wedge tends to fill out the set $\rho_j \geq 0$ which proves the so-called folding screen lemma. The constant C_1 tells us how far the intersection of M_j's is from being non-generic. If the intersection of M_j's is close to being non-generic then the defining functions ρ_j's have to be multiplied by a large constant to satisfy normalization condition (3). This makes C_1 (as well as C_2) large. Note that R_1 does not depend on C_1; i.e., the size of the edge of $\mathcal{W}$ does not depend on C_1.

Obtaining an extension to a wedge in Theorem 1.2 requires an estimate of a neighborhood in the approximation theorem by Baouendi-Treves. No such estimate can be found in the original statement of the theorem in [BT]. But it is possible to extract one from the proof. Section 4 contains a proof of the approximation theorem with neighborhood estimates. It is basically a refinement of the proof in [B].

Theorem 1.2 as well as all other results on the extendibility of CR functions (see [A], [Tr1], [Tu1], [Tu2], [Tu3]) is proved by showing that analytic discs attached to manifolds sweep out an open set in $\mathbf{C}^N$. We, however, use a technique, different from [A] and [Tu2], of "attaching discs" which we discuss now briefly.

We denote by H^2_N the Hardy space of functions valued in $\mathbf{C}^N$, analytic in the open unit disc in $\mathbf{C}$ and whose boundary values are in $L^2(\mathbf{T})$. For $0<\alpha<1$ we denote by $C^\alpha_l(\mathbf{T};\mathbf{R})$ the set of l-vector real valued functions on $\mathbf{T}$ with each component belonging to the Holder class C^α. By $C^\alpha_l(\mathbf{T})$ we denote the space of complex valued functions whose real and imaginary part belong to $C^\alpha_l(\mathbf{T};\mathbf{R})$. We say that an analytic disc $f\in H^2_N\cap C_N(\mathbf{T})$ is attached to a manifold if its boundary $f(\mathbf{T})$ lies on the manifold.

The paper [BRT] suggested a new approach to the problem of constructing families of analytic discs attached to a generic manifold $M=\{z\in\mathbf{C}^N:\rho_1(z)=\cdots=\rho_l(z)=0\}$. A disc $f\in H^2_N\cap C^\alpha_N$ is attached to M if it lies in the zero set of the map

$$(5)\quad \mathcal{R}: f(e^{i\theta})\in H^2_N\cap C^\alpha_N(\mathbf{T})\longrightarrow(\rho_1(f(e^{i\theta})),\ldots,\rho_l(f(e^{i\theta})))\in C^\alpha_l(\mathbf{T};\mathbf{R}).$$

First, we try to prove that the set $\mathcal{A}=\{f\in H^2_N\cap C^\alpha_N:\mathcal{R}(f)=0\}$ forms a Banach manifold near a given small disc f_0. Then, in many cases, we can deal with a tangent space to the manifold $\mathcal{A}$ rather than the manifolds itself. This substantially simplifies the problem.

In this paper we prove Theorem 1.2 by employing the Banach space technique sketched above instead of a construction based on Bishop's equation as in [A] and [Tu2].

Sections of the normal bundle along the boundary of an analytic disc. Now we discuss one more question recently studied in the literature ([G], [O], [C],[F]) and which also arises in the proof of Theorem 1.2.

Let $M=\{\xi\in\mathbf{C}^N:\rho_1(\xi)=\cdots=\rho(\xi)=0\}$ be a generic manifold and suppose $f\in H^2_N\cap C^\alpha_N$ is attached to M. Denote by $T_{f(\mathbf{T})}M$ the restriction of the tangent bundle to the boundary of f; i.e., let

$$T_{f(\mathbf{T})}M = \bigcup_{\zeta \in \mathbf{T}} T_{f(\zeta)}M.$$

The space of C^α sections of this bundle can be defined as

$$C^\alpha(T_{f(\mathbf{T})}M) = \left\{ g \in C_N^\alpha(\mathbf{T}) : \mathrm{Re}\left(\frac{\partial \rho(f(\zeta))}{\partial \xi} g(\zeta) \right) = 0,\ \forall \zeta \in \mathbf{T} \right\}.$$

We can define the normal bundle along $f(\mathbf{T})$ as

$$N_{f(\mathbf{T})}M = T_{f(\mathbf{T})}\mathbf{C}^N / T_{f(\mathbf{T})}M.$$

The problem we study is the following.

PROBLEM 1. *Under what conditions on f and M do holomorphic functions span the space of C^α sections of the normal bundle along $f(\mathbf{T})$? In other words, under what conditions on f and M does the following hold*:

$$H_N^2 \cap C_N^\alpha = C^\alpha(N_{f(\mathbf{T})}M). \tag{6}$$

Here we identify an analytic function $h \in H_N^2 \cap C_N^\alpha$ with its equivalence class in $C^\alpha(N_{f(\mathbf{T})}M)$.

If for particular f and M equation (6) holds, then the derivative $\mathcal{R}'(f)$ of the map $\mathcal{R}$ defined by (5) is onto and we can apply the local submersion theorem to show that the set of analytic disc attached to M forms a Banach manifold near f. Thus Problem 1 arises in the Banach space technique of attaching discs. A variation of Problem 1 also arises in an optimization problem in electrical engineering; see [HV1], [HV2] and [Vi].

If $\|f\|_\alpha$ is sufficiently small then it is easy to see that (6) holds. If M is maximally totally real then the answer depends on the so-called factorization indices of $\partial\rho(f)/\partial\xi$. It follows from the results in [Ve]; see also [O],[G].

In this paper we study Problem 1 when we have a union of manifolds $\widetilde{M}_1, \ldots, \widetilde{M}_l$ with a generic intersection instead of a single manifold M.

Consider $\mathbf{C}^N$ with coordinates $(w, z) = (w_1, \ldots, w_l, z_1, \ldots, z_{N-l})$. Suppose we are given CR-manifolds $\widetilde{M}_1, \ldots, \widetilde{M}_l$ and functions $\rho_1, \ldots, \rho_l$ of class C^2 such that

$$\widetilde{M}_k = \{(w, z) \in \mathbf{C}^N : \rho_j(w, z) = 0, 1 \le j \le l, j \ne k\}$$

and such that $\partial_w\rho_1 \wedge \cdots \wedge \partial_w\rho_l \ne 0$ on $\widetilde{M}_1 \cup \cdots \cup \widetilde{M}_l$. Here we used notation $\partial_w p = \sum \frac{\partial p}{\partial w_j} dw_j$. Divide the unit circle $\mathbf{T} = [0, 2\pi)$ into l open intervals $T_j = (2\pi\frac{j-1}{l}, 2\pi\frac{j}{l})$. Consider the disc $f \in H_N^2 \cap C_N^\alpha$ attached to $\widetilde{M}_1, \ldots, \widetilde{M}_l$ in the following manner: $f|_{T_1} \subset \widetilde{M}_1, \ldots, f|_{T_l} \subset \widetilde{M}_l$.

Define the set of C^α sections of the bundle $N_{f(T_1)}M_1 \cup \cdots \cup N_{f(T_l)}M_l$ in the same way as above. Namely it is the set of all functions $g \in C_N^\alpha(\mathbf{T})$ modulo the equivalence relation

$$g_1 \sim g_2 \Leftrightarrow \forall k \quad \operatorname{Re}\left(\frac{\partial \rho_k(f(\zeta))}{\partial \xi} \cdot (g_1(\zeta) - g_2(\zeta))\right) = 0 \text{ for } \zeta \in \mathbf{T} \setminus T_k.$$

PROBLEM 2. *Under what conditions on f and $\widetilde{M}_j$'s does the following hold*

$$H_N^2 \cap C_N^\alpha = C^\alpha(N_{f(T_1)}M_1 \cup \cdots \cup N_{f(T_l)}M_l). \tag{7}$$

Here we identify an analytic function $h \in H_N^2 \cap C_N^\alpha$ with its equivalence class in $C^\alpha(N_{f(T_1)}M_1 \cup \cdots \cup N_{f(T_l)}M_l)$.

One has to solve Problem 2 with $f = 0$ when proving Theorem 1.2. The solution is quite simple in this case.

For an arbitrary f, Problem 2 seems to be quite hard. The Vekua's theory of partial indices can not be applied. We succeeded only in proving a partial result, Theorem 3.1 which says that (7) holds if the functions ρ_j have a triangular dependence on some of the variables.

This paper has the following structure. In Section 2 we prove Theorem 1.2. In Section 3 we study Problem 2. This section is completely independent of the rest of the paper. Section 4 contains the proof of the approximation theorem with concrete neighborhood estimates.

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2. A new proof of the Airapetyan-Henkin theorem

We will need the two elementary lemmas below.

LEMMA 2.1. *Suppose we are given two Banach spaces A and B and a C^2-map $F : A \to B$ such that $F(0) = 0$. Suppose there exist four constants $\delta_1, \delta_2, \delta_3, \chi$ such that*

$$\|F'(0)\| \le \delta_1, \quad \|(F'(0))^{-1}\| \le \delta_2, \quad \|F''(x)\| \le \delta_3 \text{ for } |x| \le \chi$$

where $F''(x) : A \times A \to B$ is the second derivative of the map F at $x \in A$.

Then there exists a continuous map $L : B \to A$ such that

$$F^{-1}(y) = (F'(0))^{-1}(y) + L(y) \text{ for } |y| \le \min\left\{\frac{1}{4\delta_2^2\delta_3}, \frac{\chi}{2\delta_2}\right\} \tag{8}$$

and such that

$$|L(y)| \le 4\delta_2^2\delta_3|y|^2 \text{ for } |y| \le \min\left\{\frac{1}{4\delta_2^2\delta_3}, \frac{\chi}{2\delta_2}\right\}. \tag{9}$$

Also the image of the ball $|x| \le \min\{\frac{1}{2\delta_2\delta_3}, \chi\}$ *under the map* F *covers a ball*

$$|y| \le \min\left\{\frac{1}{4\delta_2^2\delta_3}, \frac{\chi}{2\delta_2}\right\}.$$

In addition, we have $2\delta_2|y| \ge |x|$ *for* $|x| \le \min\{\frac{1}{2\delta_2\delta_3}, \chi\}$.

LEMMA 2.2. *Suppose we are given a real vector valued function* $\rho = (\rho_1, \dots, \rho_l)$ *of class* $C^{2,\alpha}$ *on* $\mathbf{C}^N$. *Suppose there exist three constants* $\gamma_1, \gamma_2, \sigma$ *such that the following two conditions are satisfied*:

$$(10) \qquad \gamma_1|t| \ge \left|\left(\frac{\partial\rho(p_0)}{\partial\xi}\right)^T t\right| \ge |t|, \quad \forall t \in \mathbf{C}^l,$$

$$(11) \qquad \|\rho''(\xi)\|_{\alpha,S_2} \le \gamma_2, \quad S_2 = \{\xi : |\xi - p_0| \le \sigma\}.$$

Then there exists a $\mathbf{C}$*-affine change of coordinates* $\xi \to \xi' = (x + iy, u + iv)$ *and a set of functions* $h = (h_1, \dots h_l)$ *such that* $p_0 \to 0$ *and such that for every* $|\xi'| \le \min\{\frac{1}{4l\gamma_2}, \sigma\}$ *the following five conditions hold*

$$\rho_j(\xi') = \rho_j(x + iy, u + iv) = 0 \iff y_j = h_j(x, u, v), \quad 1 \le j \le l,$$

where $h(0) = 0$, $\nabla h(0) = 0$,

$$(12) \quad |\nabla h(x, u, v)|_{\mathbf{R}^l \times \mathbf{R}^{2N-l}} \le \frac{1}{2} \ \text{for} \ |(x, u, v)|_{\mathbf{R}^{2N-l}} \le \frac{1}{2}\min\left\{\frac{1}{4l\gamma_2}, \sigma\right\},$$

$$(13) \quad \|h_j''\|_{\alpha,S_3} \le 8\gamma_2 l(1 + l\gamma_2)^3, \quad S_3 = \{|(x, u, v)|_{\mathbf{R}^{2N-l}} \le \frac{1}{2}\min\left\{\frac{1}{4l\gamma_2}, \sigma\right\}\},$$

$$(14) \qquad (2\gamma_1 + 1)|\xi - p_0| \ge |\xi'| \ge |\xi - p_0|.$$

If for some $c > 0$ *we have* $\rho_j(\xi - p_0) \ge c|\xi - p_0|^2$ *for* $|\xi - p_0| \le \sigma$ *then*

$$(15) \qquad \operatorname{Im}\xi_j' \ge \left(\frac{c}{(2\gamma_1 + 1)^2} - \frac{\gamma_2}{2}\right)|\xi'|^2, \quad \text{for } |\xi'| \le \sigma.$$

The proofs of Lemmas 2.1 and 2.2 are very elementary, though somewhat lengthy. We do not include the proofs in this paper and refer the reader to [AMR] as a general reference.

Denote by M the intersection of M_j's; i.e., $M = \{\xi \in \mathbf{C}^N : \rho_j(\xi) = 0,\ j = 1, \dots, l\}$. Divide the unit disc $\mathbf{T} = [0, 2\pi)$ into l intervals given by $T_j = (2\pi\frac{j-1}{l}, 2\pi\frac{j}{l})$ for $1 \le j \le l$. The following theorem is the main step toward proving Theorem 1.2.

PROPOSITION 2.3. *Suppose we are given l manifolds with boundary*

$$M_k = \{\xi \in \mathbf{C}^N : \rho_j(\xi) = 0, j \neq k, \rho_k(\xi) \geq 0\}, \quad 1 \leq k \leq l$$

and suppose that for a given point $p_0 \in M$ inequalities (10) *and* (11) *hold.*
Then the set

$$(16) \quad \{f(0) : f \in H_N^2 \cap C_N^\alpha, \|f - p_0\|_\infty \leq \frac{\sigma}{4}, f|_{T_1} \subset M_1, \dots, f|_{T_l} \subset M_l\}$$

contains a set V_{p_0} given by

$$V_{p_0} = \{\zeta \in \mathbf{C}^N : |\zeta - p_0| \leq \widetilde{\sigma},\ \rho_j(\zeta) \geq \widetilde{\gamma}|\zeta - p_0|^2,\ j = 1, \dots, l\}$$

where

$$\tilde{\sigma} = \frac{1}{(2\gamma_1 + 1)32l^3(1+\kappa)^2} \min\left\{\frac{1}{32l^5(1+\kappa)^2(1+l\gamma_2)^3\gamma_2}, \sigma\right\},$$

$$\tilde{\gamma} = \gamma_2 2^{13}(2\gamma_1 + 1)^2 l^{12}(1+\kappa)^4(1+l\gamma_2)^3.$$

Proof. Denote by $\widetilde{M}_j$ a continuation of M_j near 0:

$$\widetilde{M}_j = \{z \in \mathbf{C}^N : \rho_k(z) = 0,\ k \neq j\}.$$

We start with a change of coordinates. Let us write

$$\gamma_1^* = 2\gamma_1 + 1,\quad \gamma_2^* = 8l\gamma_2(1+l\gamma_2)^3,\quad \sigma^* = \frac{1}{2}\min\left\{\frac{1}{4l\gamma_2}, \sigma\right\}.$$

It follows from Lemma 2.2 that there exists a new set of coordinates $\xi' = (w_1, \dots w_l, z_1, \dots, z_{N-l}) = (x+iy, u+iv)$ such that $\widetilde{M}_j = \{(w,z) \in \mathbf{C}^N : \operatorname{Im} w = h(\operatorname{Re} w, z)\}$ for $|(w,z)| \leq 2\sigma^*$. We will adopt notation $\tilde{\rho}_j(w,z) = \operatorname{Im} w_j - h_j(\operatorname{Re} w, z)$ for $j = 1, \dots, l$ in the sequel. Lemma 2.2 also implies that

$$\|\tilde{\rho}_j''\|_{\alpha, S_4} \leq \gamma_2^*,\quad S_4 = \{|\xi'| \leq \sigma^*\}.$$

We consider the set of discs attached to the manifolds $\widetilde{M}_j$ in the following manner: $f|_{T_1} \subset \widetilde{M}_1, \dots, f|_{T_l} \subset \widetilde{M}_l$. Our next objective is to show that the set of such analytic discs forms a Banach manifold. To do so we introduce the map

$$(17) \quad \begin{aligned} &\mathcal{R} : H_N^2 \cap C_N^\alpha \to C^\alpha(\mathbf{T} \setminus T_1; \mathbf{R}) \times \cdots \times C^\alpha(\mathbf{T} \setminus T_l; \mathbf{R}),\\ &\mathcal{R}(f)(\zeta) = (\tilde{\rho}_1(f(\zeta)|_{\mathbf{T}\setminus T_1}), \dots, \tilde{\rho}_l(f(\zeta)|_{\mathbf{T}\setminus T_l})). \end{aligned}$$

Then the set of attached discs $\mathcal{A}$ is equal to $\{f \in H_N^2 \cap C_N^\alpha : \mathcal{R}(f) = 0\}$.

We are going to apply the submersion theorem to $\mathcal{R}$ to show that in a certain neighborhood of zero disc the set of attached discs forms a Banach manifold. In order to do so we need to show that the derivative of $\mathcal{R}$ is onto and its kernel splits, i.e., has a closed complement. Consider the derivative map

$$\mathcal{R}'(f) : H_N^2 \cap C_N^\alpha \to C^\alpha(\mathbf{T} \setminus T_1; \mathbf{R}) \times \cdots \times C^\alpha(\mathbf{T} \setminus T_l; \mathbf{R}),$$

$$(18) \qquad \begin{aligned} \mathcal{R}'(f)g = 2\,\mathrm{Re}(Ag), \; A_{jk} &= \tfrac{\partial \tilde{\rho}_j(f)}{\partial w_k}\Big|_{\mathbf{T}\setminus T_j}, \; 1 \le k \le l, \\ A_{jk} &= \tfrac{\partial \tilde{\rho}_j(f)}{\partial z_{k-l}}\Big|_{\mathbf{T}\setminus T_j}, \; l < k \le N. \end{aligned}$$

For an analytic disc f, also consider the map

$$\mathcal{S}(f) : \{g \in H_l^2 \cap C_l^\alpha : \mathrm{Re}\, g(1) = 0\} \longrightarrow C_l^\alpha(\mathbf{T}; \mathbf{R}), \;\; \mathcal{S}(f)(g) = 2\,\mathrm{Re}\left(\frac{\partial \tilde{\rho}}{\partial w} g\right)$$

where $\frac{\partial \tilde{\rho}}{\partial w}$ stands for the $l \times l$ matrix of w-derivatives of $\tilde{\rho} = (\tilde{\rho}_1, \ldots, \tilde{\rho}_l)$.

Obviously, if $\mathcal{S}(f)$ is onto then the derivative map $\mathcal{R}'(f)$ is onto. Note that $\frac{\partial \tilde{\rho}(0)}{\partial w}$ is equal to $-0.5i$ times the identity matrix. Therefore $\|\mathcal{S}(0)\| = 1$, $\|\mathcal{S}^{-1}(0)\| = 1 + \kappa$, κ is the C^α norm of the Hilbert transform. Consider the derivative of $\mathcal{S}$ at f:

$$\mathcal{S}'(f) : \{g \in H_l^2 \cap C_l^\alpha : \mathrm{Re}\, g(1) = 0\} \times \{g \in H_l^2 \cap C_l^\alpha : \mathrm{Re}\, g(1) = 0\} \longrightarrow C_l^\alpha(\mathbf{T}; \mathbf{R}).$$

Its k-th component is given by

$$(19) \qquad \begin{aligned} \mathcal{S}_k'(f)(q, g) &= 2 \sum_{j,m} \frac{\partial \tilde{\rho}_k(f)}{\partial w_j \partial \overline{w}_m} q_j \overline{g}_m + \sum_{j,m} \frac{\partial \tilde{\rho}_k(f)}{\partial w_j \partial w_m} q_j g_m \\ &\quad + \sum_{j,m} \frac{\partial \tilde{\rho}_k(f)}{\partial \overline{w}_j \partial \overline{w}_m} \overline{q}_j \overline{g}_m. \end{aligned}$$

Therefore $\mathcal{S}'(f)$ has a norm not exceeding $l\gamma_2^*$ for f such that $\|f\|_\alpha \le \sigma^*$.

Now we want to estimate the size of the neighborhood of 0 in $\{g \in H_l^2 \cap C_l^\alpha : \mathrm{Re}\, g(1) = 0\}$ in which $\mathcal{S}(f)$ is invertible. To do so we write

$$\mathcal{S}(f) = \mathcal{S}(0)[I + \mathcal{S}^{-1}(0)(\mathcal{S}(f) - \mathcal{S}(0))]$$

which implies that $\mathcal{S}(f)$ is invertible if

$$\|\mathcal{S}^{-1}(0)\| \cdot \|\mathcal{S}(f) - \mathcal{S}(0)\| < 1.$$

In the view of the fact that $\|\mathcal{S}'(f)\| < l\gamma_2^*$ for $\|f\|_\alpha \le \sigma^*$, the map $\mathcal{S}(f)$ is invertible for

$$\|f\|_\alpha \le \frac{1}{2}\min\left\{\frac{1}{l\gamma_2^*(1+\kappa)}, \sigma^*\right\} \tag{20}$$

and so is $\mathcal{R}'(f)$.

In order to show that the kernel of (18) splits for f such that $\mathcal{R}'(f)$ is onto, we only need to show that the kernel of the projection

$$\text{(21) } \mathcal{P} : C^\alpha(\mathbf{T};\mathbf{R}) \times \cdots \times C^\alpha(\mathbf{T};\mathbf{R}) \to C^\alpha(\mathbf{T}\setminus T_1;\mathbf{R}) \times \cdots \times C^\alpha(\mathbf{T}\setminus T_l;\mathbf{R})$$

splits. Denote by $C_0^\alpha([t_1,t_2];\mathbf{R})$ the set $\{g \in C^\alpha([t_1,t_2]) : g(t_1) = g(t_2) = 0\}$. Then we have $C^\alpha(\mathbf{T};\mathbf{R}) = C_0^\alpha(T_1;\mathbf{R}) \oplus C_0^\alpha(\mathbf{T}\setminus T_1;\mathbf{R}) \oplus \mathbf{R}^2$, where $\mathbf{R}^2$ is identified with the space $\{c_1 d_1(\zeta) + c_2 d_2(\zeta)\}$ for fixed functions d_1, d_2 satisfying $d_1(0) = 1, d_1(2\pi/l) = 0, d_2(0) = 0, d_2(2\pi/l) = 1$. Therefore $C^\alpha(\mathbf{T};\mathbf{R}) \times \cdots \times C^\alpha(\mathbf{T};\mathbf{R}) = \ker\mathcal{P} \oplus B$, where

$$B = (C_0^\alpha(\mathbf{T}\setminus T_1;\mathbf{R}) \oplus \mathbf{R}^2) \times \cdots \times (C_0^\alpha(\mathbf{T}\setminus T_l;\mathbf{R}) \oplus \mathbf{R}^2). \tag{22}$$

Thus we have just shown that $\mathcal{A}$ is a Banach manifold for f such that (20) holds. The tangent space to $\mathcal{A}$ at the zero disc is given by

$$T_0\mathcal{A} = \{f \in H_N^2 \cap C_N^\alpha : \operatorname{Im} f_1|_{\mathbf{T}\setminus T_1} = 0, \ldots, \operatorname{Im} f_l|_{\mathbf{T}\setminus T_l} = 0\}.$$

Here $f_1, \ldots, f_N$ are the components of the analytic disc f written in (w, z) coordinates. In other words, the tangent space is the space of discs attached to the tangent planes $T_0M_1, \ldots, T_0M_l$.

Now we are going to consider the set of discs attached to the angle $M_1 \cup \cdots \cup M_l$, rather than to the union of manifolds $\widetilde{M}_1 \cup \cdots \cup \widetilde{M}_l$. Denote by $C_0^\alpha(T_j;\mathbf{R})$ the set of functions in $C^\alpha(\overline{T}_j;\mathbf{R})$ which vanish at the end points of the interval T_j. Let Λ be a Banach space

$$\Lambda = C_0^\alpha(T_1;\mathbf{R}) \times \cdots \times C_0^\alpha(T_l;\mathbf{R}) \times \mathbf{R}^l \times (H_{N-l}^2 \cap C_{N-l}^\alpha)$$

endowed with a norm

$$\begin{aligned}\|(\psi_1,\ldots,\psi_l,a,\varphi_1,\ldots,\varphi_{N-l})\|_\Lambda^2 &= \|\psi_1\|_\alpha^2 + \cdots + \|\psi_l\|_\alpha^2 + |a|^2 \\ &\quad + \|\varphi_1\|_\alpha^2 + \cdots + \|\varphi_{N-l}\|_\alpha^2.\end{aligned}$$

If we introduce a map $\mathcal{F} : \mathcal{A} \to \Lambda$ defined by

$$\begin{aligned}\text{(23) } \mathcal{F}(f) &= \mathcal{F}(f_1,\ldots,f_N) \\ &= (\tilde{\rho}_1(f|_{T_1}),\ldots,\tilde{\rho}_l(f|_{T_l}), \operatorname{Re} f_1(1),\ldots,\operatorname{Re} f_l(1), f_{l+1},\ldots,f_N)\end{aligned}$$

then its derivative at zero is given by

$$\mathcal{F}'(0)g = (\operatorname{Im} g_1|_{T_1},\ldots,\operatorname{Im} g_l|_{T_l}, \operatorname{Re} g_1(1),\ldots,\operatorname{Re} g_l(1), g_{l+1},\ldots,g_N).$$

The operator $\mathcal{F}'(0) : T_0\mathcal{A} \to \Lambda$ is invertible and its inverse K is given by the formula

$$K(\psi_1, \dots, \psi_l, a, \varphi_1, \dots, \varphi_{N-l}) = (a_1 + i\widetilde{\psi}_1 - \mathcal{H}\widetilde{\psi}_1, \dots, a_l + i\widetilde{\psi}_l - \mathcal{H}\widetilde{\psi}_l, \varphi_1, \dots, \varphi_{N-l})$$

where we denote by $\widetilde{\psi}_j$ the extension of $\psi_j \in C_0^\alpha(T_j)$ to $\mathbf{T}$ by zero. Note also that $\|K\| = 1 + \kappa$. Therefore the map $\mathcal{F}$ is a local isomorphism.

In order to estimate the size of the neighborhood of 0 on which $\mathcal{F}$ is an isomorphism, we need to estimate the norm of the second derivative of $\mathcal{F}$. Since each component of $\mathcal{F}''$ is given by a formula similar to (20), we have $\|\mathcal{F}''\| \le l\gamma_2^*$ for f satisfying (20). By Lemma 2.1 we can write

$$\mathcal{F}^{-1}(\lambda) = K(\lambda) + L(\lambda), \tag{24}$$

$$\|L(\lambda)\|_\alpha \le 4(1+\kappa)^2 l\gamma_2^* \|\lambda\|_\Lambda^2 \text{ for } \|\lambda\|_\Lambda \le \frac{1}{4(1+\kappa)} \min\left\{\frac{1}{(1+\kappa)l\gamma_2^*}, \sigma^*\right\}. \tag{25}$$

Note that for $\lambda = (\psi_1, \dots, \psi_l, a, \varphi_1, \dots, \varphi_{N-l})$ the analytic disc $\mathcal{F}^{-1}(\lambda)$ is attached to the angle $M_1 \cup \dots \cup M_l$ if and only if $\psi_j \ge 0$, $j = 1, \dots, l$.

Let $\phi_1, \dots, \phi_l$ be functions in $C^\alpha(\mathbf{T}; \mathbf{R})$ which satisfy the conditions

$$\phi_j\big|_{T_j} > 0, \quad \phi_j\big|_{\mathbf{T}\setminus T_j} = 0, \quad \tilde{\phi}_j(0) = 1, \quad \|\phi_j\|_\alpha \le 2l^2 \tag{26}$$

where $\tilde{\phi}_j$ denotes the harmonic extension of ϕ_j to the unit disc in $\mathbf{C}$. In fact, we can find piecewise linear functions ϕ_j's which satisfy (26).

Define the evaluation map $E : \mathcal{A} \to \mathbf{C}^N$ by $E(f) = f(0)$. Consider the map

$$\mathcal{G} : \{(t\phi, a, b) : t, a \in \mathbf{R}^l, b \in \mathbf{C}^{N-l}\} \subset \Lambda \to \mathbf{C}^N$$

which is given by taking the restriction of $E \circ \mathcal{F}^{-1}$. Then we have $\|(\mathcal{G}'(0))^{-1}\| \le 2l^3(1+\kappa)$ (since $(\mathcal{G}'(0))^{-1}(a + it, b) = (t\phi, a + t\mathcal{H}\tilde{\phi}(0), b)$). And we have $\|\mathcal{G}''\| \le 8(1+\kappa)^2 l\gamma_2^*$ for $\lambda = (t\phi, a, b)$ satisfying (25).

Therefore we can apply Lemma 2.1 with

$$\delta_2 = 2l^3(1+\kappa),\ \delta_3 = 8(1+\kappa)^2 l\gamma_2^*,\ \chi = \frac{1}{4(1+\kappa)} \min\left\{\frac{1}{(1+\kappa)l\gamma_2^*}, \sigma^*\right\}$$

and conclude that for every $\zeta \in \mathbf{C}^N$ with

$$|\zeta| \le \min\left\{\frac{1}{2^7 l^7 (1+\kappa)^4 \gamma_2^*}, \frac{\sigma^*}{16(1+\kappa)^2 l^3}\right\}, \ \operatorname{Im}\zeta_j \ge 2^9 l^{11}(1+\kappa)^4\gamma_2^* |\zeta|^2 \tag{27}$$

there exists (t, a, b) such that

$$\begin{aligned} \zeta &= E \circ \mathcal{F}^{-1}(t_1\phi_1, \dots, t_l\phi_l, a, b) \\ &= (a_1 + it_1, \dots, a_l + it_l, b) + E \circ L(t\phi, a, b). \end{aligned} \tag{28}$$

In other words, the center of analytic disc $\mathcal{F}^{-1}(t\phi, a, b)$ is precisely ζ. In order to show that this disc is attached to the angle, we have to prove that $t_j \geq 0,\ j = 1, \dots, l$. Equation (29) implies that

$$|\zeta - (a + it, b)| \leq 16l^5(1+\kappa)^2\gamma_2^*|(a+it, b)|^2.$$

Lemma 2.1 implies that

$$|(a+it, b)| \leq \|(t\phi, a, b)\|_\Lambda \leq 4l^3(1+\kappa)|\zeta|$$

and therefore we have

$$\begin{aligned} t_j &\geq \operatorname{Im}\zeta_j - |\operatorname{Im}\zeta_j - t_j| \geq 2^9 l^{11}(1+\kappa)^4\gamma_2^*|\zeta|^2 - 16l^5(1+\kappa)^2\gamma_2^*|(a+it, b)|^2 \\ &\geq 2^9 l^{11}(1+\kappa)^4\gamma_2^*|\zeta|^2 - 2^8 l^{11}(1+\kappa)^4\gamma_2^*|\zeta|^2 \geq 0 \end{aligned}$$

and thus any ζ satisfying (27) can be covered by a center of an analytic disc attached to the angle (since $t_j \geq 0$).

The inequalities in (27) are written in (w, z) coordinates. We can use (14) and (15) to convert them to the original ξ coordinates and come up with $\tilde{\sigma}$ and $\tilde{\gamma}$ defined in the statement of Theorem 2.3.

To finish the proof we just have to notice that we used only the discs f satisfying (20) when written in (w, z) coordinates. Then the definition of σ^* together with (14) imply that we used discs f with $\|f - p_0\|_\infty \leq \sigma/4$ when written in the original ξ coordinates. □

Proof of Theorem 1.2. Let us apply Proposition 2.3 for a point $p_0 \in M$ such that

$$|p_0| \leq \beta = \frac{1}{192}\min\left\{\frac{1}{4lC_2}, R\right\}.$$

Inequalities (3) and (4) imply that

$$\left|\left(\frac{\partial\rho(p_0)}{\partial\xi}\right)^T t\right| \leq (C_1 + \beta l C_2)|t|.$$

At the same time, since $1 - \beta l C_2 \geq 1/2$ we have

$$\left|\left(\frac{\partial\rho(p_0)}{\partial\xi}\right)^T t\right| \geq \left|\left(\frac{\partial\rho(0)}{\partial\xi}\right)^T t\right| - \beta l C_2|t| \geq \frac{|t|}{2}.$$

Therefore the vector of defining functions ρ has to be multiplied by 2 to satisfy inequality (10) . After this normalization of ρ we can apply Theorem 2.3 with

$$\gamma_1 = 2(C_1 + \beta l C_2),\ \gamma_2 = 2C_2,\ \sigma = \beta$$

to fill out the set V_{p_0} using discs f satisfying

$$\|f\|_\infty \le \|f - p_0\|_\infty + \beta \le \frac{1}{96}\min\left\{\frac{1}{4lC_2}, R\right\}.$$

This allows us to apply Theorem 4.4 to get extension of CR functions into the set V_{p_0}. □

3. Sections of the normal bundle along the boundary of an analytic disc

As we mentioned in the introduction this section is completely independent of the rest of the paper. Let us restate the assumptions associated with Problem 2 stated in the introduction.

ASSUMPTIONS. *Consider $\mathbf{C}^N$ with coordinates $(\underset{\sim}{w}, z) = (w_1, \dots, w_l, z_1, \dots, z_{N-l})$. Suppose we are given CR-manifolds $\widetilde{M}_1, \dots, \widetilde{M}_l$ and functions $\rho_1, \dots, \rho_l$ of class C^2 such that*

$$\widetilde{M}_k = \{(w, z) \in \mathbf{C}^N : \rho_j(w, z) = 0, 1 \le j \le l, j \ne k\}$$

and such that $\partial_w \rho_1 \wedge \cdots \wedge \partial_w \rho_l \ne 0$ on $\widetilde{M}_1 \cup \cdots \cup \widetilde{M}_l$.

Here we use notation $\partial_w p = \sum \frac{\partial p}{\partial w_j} dw_j$.

Now we state the main result of this section which says that (7) holds for every f if the functions ρ_j have a triangular dependence on some of the variables.

THEOREM 3.1. *Suppose the assumptions above hold, and suppose, in addition, that the following hold:*

$$\begin{array}{ll} \rho_2(w, z) & \text{does not depend on } w_1 \\ \rho_3(w, z) & \text{does not depend on } w_1, w_2 \\ \vdots & \dots \\ \rho_l(w, z) & \text{does not depend on } w_1, \dots, w_{l-1} \end{array}$$

Then $\underset{\sim}{\text{equation}}$ (7) holds for $\underset{\sim}{\text{every}}$ $f \in H^2_N \cap C^\alpha_N$ attached to $\widetilde{M}_j$'s, i.e., for F such that $f|_{T_1} \subset \widetilde{M}_1, \dots, f|_{T_l} \subset \widetilde{M}_l$.

The following lemma deals with derivative map (18) in general.

LEMMA 3.2. *Suppose the assumptions of Theorem 3.1 hold. Given an analytic disc $f \in H^2_N \cap C^\alpha_N(\mathbf{T})$ such that $f|_{T_1} \subset \widetilde{M}_1, \dots, f|_{T_l} \subset \widetilde{M}_l$, consider the matrix of w-derivatives*

$$\begin{pmatrix} \frac{\partial \rho_1(f)}{\partial w_1}|_{\mathbf{T}\setminus T_1} & \cdots & \frac{\partial \rho_1(f)}{\partial w_l}|_{\mathbf{T}\setminus T_1} \\ \cdots & \cdots & \cdots \\ \frac{\partial \rho_l(f)}{\partial w_1}|_{\mathbf{T}\setminus T_l} & \cdots & \frac{\partial \rho_l(f)}{\partial w_l}|_{\mathbf{T}\setminus T_l} \end{pmatrix}. \tag{29}$$

Suppose we can extend the elements of this matrix, i.e., find C^α-functions a_{jk} satisfying

$$a_{jk}|_{\mathbf{T}\setminus T_j} = \frac{\partial \rho_j(f)}{\partial w_k}|_{\mathbf{T}\setminus T_j} \tag{30}$$

so that the Riemann-Hilbert problem

$$2\,\mathrm{Re}\left[\begin{pmatrix} a_{11} & \dots & a_{1l} \\ \dots & \dots & \dots \\ a_{l1} & \dots & a_{ll} \end{pmatrix}\begin{pmatrix} h_1 \\ \dots \\ h_l \end{pmatrix}\right] = \begin{pmatrix} \widetilde{\psi}_1 \\ \dots \\ \widetilde{\psi}_l \end{pmatrix} \tag{31}$$

is solvable for any right hand side $\widetilde{\psi} \in C^\alpha_l(\mathbf{T};\mathbf{R})$ and the matrix (a_{jk}) is invertible. Then equation (7) holds.

Proof. We want to show that derivative map (18) is onto. This leads to the Riemann-Hilbert problem

$$2\,\mathrm{Re}\left[\begin{pmatrix} b_{11} & \dots & b_{1N} \\ \dots & \dots & \dots \\ b_{l1} & \dots & b_{lN} \end{pmatrix}\begin{pmatrix} u_1 \\ \dots \\ u_N \end{pmatrix}\right] = \begin{pmatrix} \psi_1 \cdot 1_{\mathbf{T}\setminus T_1} \\ \dots \\ \psi_l \cdot 1_{\mathbf{T}\setminus T_l} \end{pmatrix}. \tag{32}$$

Here b's are given by the formulas

$$\begin{aligned} b_{jk} &= \tfrac{\partial \rho_j(f)}{\partial w_k}|_{\mathbf{T}\setminus T_j},\ k = 1,\dots,l, \\ b_{jk} &= \tfrac{\partial \rho_j(f)}{\partial z_{k-l}}|_{\mathbf{T}\setminus T_j},\ k = l+1,\dots,N. \end{aligned} \tag{33}$$

For map (18) to be onto it is necessary and sufficient that (32) have a holomorphic solution $u \in C^\alpha$ for every collection $\psi_j \in C^\alpha(\mathbf{T}\setminus T_j;\mathbf{R})$, $j = 1,\dots,l$. Let $\widetilde{\psi}_j \in C^\alpha$ be arbitrary extensions of ψ_j to the whole $\mathbf{T}$. Then if problem (31) is solvable for any right hand side part then so is (32). □

Proof of Theorem 3.1. First note that $\partial_w\rho_1 \wedge \dots \wedge \partial_w\rho_l \neq 0$ and the fact that matrix (29) is triangular imply that $\frac{\partial \rho_j(f)}{\partial w_j}|_{\mathbf{T}\setminus T_j} \neq 0$, $j = 1,\dots,l$. Let c_j be C^α-extensions of $\frac{\partial \rho_j(f)}{\partial w_j}$ to the whole $\mathbf{T}$ such that the winding numbers of c_j's around 0 are all equal to zero. Extend other entries of the matrix arbitrary. We need to show the solvability of the following problem:

$$\mathrm{Re}\left[\begin{pmatrix} c_1 & * & \dots & * \\ 0 & c_2 & \dots & * \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & c_l \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ \dots \\ u_l \end{pmatrix}\right] = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \dots \\ \psi_l \end{pmatrix}. \tag{34}$$

For each c_j there exists r_j, a real valued function, non-zero everywhere on $\mathbf{T}$, such that $r_j c_j = h_j$ where h_j is holomorphic and with the winding number equal to zero. The existence of r_j can be shown by considering $\log c_j$. Multiply both sides of (34) on the left by the diagonal matrix with entries $r_1, \dots, r_l$. Then make a change of variables in (34): let $u' = Hu$ where H is the diagonal matrix with entries $h_1, \dots, h_l$. Then (34) becomes

$$(35) \qquad \operatorname{Re}\left[\begin{pmatrix} 1 & \widetilde{c}_{12} & \dots & \widetilde{c}_{1l} \\ 0 & 1 & \dots & \widetilde{c}_{2l} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{pmatrix}\begin{pmatrix} u'_1 \\ u'_2 \\ \dots \\ u'_l \end{pmatrix}\right] = \begin{pmatrix} \psi'_1 \\ \psi'_2 \\ \dots \\ \psi'_l \end{pmatrix}.$$

This problem can be always solved by a version of Gaussian elimination. We set $u'_l = \psi'_l + i\mathcal{H}\psi'_l$ where $\mathcal{H}$ denotes the Hilbert transform. Then (35) reduces to the $l-1$ dimensional problem

$$(36) \operatorname{Re}\left[\begin{pmatrix} 1 & \widetilde{c}_{12} & \dots & \widetilde{c}_{1,l-1} \\ 0 & 1 & \dots & \widetilde{c}_{2,l-1} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{pmatrix}\begin{pmatrix} u'_1 \\ u'_2 \\ \dots \\ u'_{l-1} \end{pmatrix}\right] = \begin{pmatrix} \psi'_1 - \operatorname{Re}\widetilde{c}_{1l}u'_l \\ \psi'_2 - \operatorname{Re}\widetilde{c}_{2l}u'_l \\ \dots \\ \psi'_{l-1} - \operatorname{Re}\widetilde{c}_{l-1,l}u'_l \end{pmatrix}.$$

Therefore the derivative map (18) is onto and thus (7) holds.

4. The proof of the approximation theorem with neighborhood estimates

This section contains a proof of the approximation theorem of Baouendi-Treves with explicit estimates on the size of the neighborhood in which holomorphic polynomials converge uniformly to a given CR-function.

These estimates are new. They are obtained as a modification of the proof of the theorem in [B].

THEOREM 4.1. *Suppose we are given a generic manifold*

$$M = \{\xi \in \mathbf{C}^N : \rho_1(\xi) = 0, \dots \rho_l(\xi) = 0\}$$

where $\rho_1, \dots, \rho_l \in C^2$ *are normalized so that*

$$(37) \qquad \left|\left(\frac{\partial \rho(0)}{\partial \xi}\right)^T t\right| \geq |t|, \quad \forall t \in \mathbf{C}^l.$$

Suppose there exists a constant $C > 0$ *such that*

$$(38) \qquad \|\rho''_j(\xi)\| \leq C \text{ for } |\xi| \leq R.$$

Then for any continuous CR function U *on* $M \cap \{\xi \in \mathbf{C}^N : |\xi| \leq R\}$ *there exists a sequence of holomorphic polynomials* p_j *in* $\mathbf{C}^N$ *such that* $p_j \to U$ *uniformly on* $M \cap \{\xi : |\xi| \leq R^*\}$ *where*

$$R^* = \frac{1}{96}\min\left\{\frac{1}{4lC}, R\right\}.$$

Proof. We start with a change of coordinates and defining functions given by Lemma 2.2. For $\xi, \zeta \in \mathbf{C}^N$ let us define

$$\xi \cdot \zeta = \xi_1 \cdot \zeta_1 + \cdots + \xi_N \cdot \zeta_N, \quad [\zeta]^2 = \zeta \cdot \zeta.$$

We will make use of the notation

$$\widetilde{R} = \frac{1}{2\sqrt{2}} \min\left\{\frac{1}{4lC}, R\right\}.$$

Define a function H by

$$H(t, v) = (h(x, u + iv), v), \quad t = (x, u). \tag{39}$$

For every $v \in \mathbf{R}^{N-l}$, $|v| \leq \widetilde{R}$, define a submanifold of M by

$$M_v = \{t + iH(t, v); |t| < \widetilde{R}\}.$$

Let us define a function $g(t) \in \mathbf{C}^\infty(\mathbf{R}^N)$ to be 1 for $|t| \leq \frac{2}{3}\widetilde{R}$ and to have support strictly inside of $|t| \leq \widetilde{R}$. The lemma below is the first step toward proving Theorem 4.1. We will write $\operatorname{Exp}(x)$ instead of e^x. □

LEMMA 4.2. *Let g be defined as above. Let U be a continuous function on M. Suppose $|v| \leq \widetilde{R}$. Then for $\zeta \in \{t + iH(t, v) : |t| \leq \frac{1}{2}\widetilde{R}\} \subset M_v$ we have*

$$U(\zeta) = \lim_{\epsilon \to 0} \pi^{-N/2} \epsilon^{-N} \int_{\zeta' \in M_v} g(\operatorname{Re} \zeta') U(\zeta') \operatorname{Exp}(-\epsilon^{-2}[\zeta - \zeta']^2) d\zeta'. \tag{40}$$

Moreover this limit is uniform for $|v| \leq \widetilde{R}$, $\zeta \in \{t + iH(t, v) : |t| \leq \frac{1}{2}\widetilde{R}\} \subset M_v$.

Proof. If we consider a map $\zeta(t, v) = t + iH(t, v)$ then for fixed $|v| \leq \widetilde{R}$, the map $\zeta(\cdot, v)$ parametrizes M_v. This allows us to change the variable of integration:

$$\begin{aligned} U(\zeta(t, v)) = {} & \lim_{\epsilon \to 0} \pi^{-N/2} \epsilon^{-N} \int_{t' \in \mathbf{R}^N} g(t') U(\zeta(t', v)) \\ & \times \operatorname{Exp}(-\epsilon^{-2}[\zeta(t, v) - \zeta(t', v)]^2) \det\left(\frac{\partial \zeta}{\partial t'}(t', v)\right) dt'. \end{aligned}$$

Substituting $t' = t - \epsilon s$ yields

$$\begin{aligned} & \pi^{-N/2} \int_{s \in \mathbf{R}^N} g(t - \epsilon s) U(\zeta(t - \epsilon s, v)) \operatorname{Exp}(-\epsilon^{-2}[\zeta(ts, v) - \zeta(t - \epsilon s, v)]^2) \\ & \det\left(\frac{\partial \zeta}{\partial t}(t - \epsilon s, v)\right) ds. \end{aligned}$$

If $|t| \leq \frac{1}{2}\widetilde{R}$ then $g(t-\epsilon s) \to 1$ as $\epsilon \to 1$. Moreover

$$\zeta(t,v) - \zeta(t-\epsilon s, v) = -\frac{\partial \zeta}{\partial t}(t,v)\cdot(\epsilon s) + \mathcal{O}(\epsilon^2).$$

Therefore we have

$$\epsilon^{-2}\left[\zeta(t,v) - \zeta(t-\epsilon s, v)\right]^2 = \left[\frac{\partial \zeta}{\partial t}(t,v)\cdot s\right]^2 + \mathcal{O}(\epsilon^2).$$

This means that pointwise in s, but uniformly in $|t| \leq \frac{1}{2}\widetilde{R}$, $|v| \leq \widetilde{R}$, the integrand converges, as $\epsilon \to 0$, to

$$U(\zeta(t,v))\,\mathrm{Exp}\left(-\left[\frac{\partial \zeta}{\partial t}(t,v)\cdot s\right]^2\right)\det\left(\frac{\partial \zeta}{\partial t}(t,v)\right).$$

To show that the integral converges, we must uniformly in ϵ dominate the integrand by an integrable function of $s \in \mathbf{R}^N$. Certainly $f \cdot \det(\partial\zeta/\partial t)$ is globally bounded. So it suffices to dominate the exponential term by an integrable function. We have

$$\begin{aligned} g(t-\epsilon s)&\left|\mathrm{Exp}\left(-\epsilon^{-2}\left[\zeta(t,v)-\zeta(t-\epsilon s,v)\right]^2\right)\right| \\ &= g(t-\epsilon s)\,\mathrm{Exp}\left(-\epsilon^{-2}\,\mathrm{Re}\left(\left[\zeta(t,v)-\zeta(t-\epsilon s,v)\right]^2\right)\right). \end{aligned}$$

Since $\zeta(t,v) = t + iH(t,v)$, we obtain

$$-\epsilon^{-2}\,\mathrm{Re}\left(\left[\zeta(t,v)-\zeta(t-\epsilon s,v)\right]^2\right) = -|s|^2 + \epsilon^{-2}\left[H(t,v) - H(t-\epsilon s,v)\right]^2.$$

Inequality (12) implies that $|\partial H(\tau,v)/\partial t| < \frac{1}{2}$ for $\tau \in \operatorname{supp} g$ and $|v| \leq \widetilde{R}$. Therefore we have

$$\left[H(t,v) - H(t-\epsilon s,v)\right]^2 \leq \left(\frac{1}{2}\epsilon|s|\right)^2 = \frac{1}{4}\epsilon^2|s|^2 \text{ for } |t| \leq \frac{\widetilde{R}}{2},\ (t-\epsilon s) \in \operatorname{supp} g.$$

Combining the last two inequalities yields

$$g(t-\epsilon s)\,\mathrm{Exp}\left(-\epsilon^{-2}\,\mathrm{Re}\left(\left[\zeta(t,v)-\zeta(t-\epsilon s,v)\right]^2\right)\right) \leq g(t-\epsilon s)\,\mathrm{Exp}\left(-\frac{3|s|^2}{4}\right) \quad \text{for } |t| \leq \frac{\widetilde{R}}{2}.$$

The right hand side is an integrable function.

Thus the dominated convergence theorem implies that for $|t| \leq \frac{1}{2}\widetilde{R}$, $|v| \leq \widetilde{R}$, the integral converges to

$$U(\zeta(t,v))\pi^{-N/2}\int_{s\in\mathbf{R}^N} \mathrm{Exp}\left(-\left[\frac{\partial \zeta}{\partial t}(t,v)\cdot s\right]^2\right)\det\left(\frac{\partial \zeta}{\partial t}(t,v)\right)ds$$

and the limit is uniform for $|t| \le \frac{1}{2}\widetilde{R}$, $|v| \le \widetilde{R}$. Note that $\frac{\partial \zeta}{\partial t}(t, v) = \frac{\partial}{\partial t}(t + iH(t, v)) = I + i\partial H/\partial t$ Moreover $|\partial H(t, v)/\partial t| \le 1/2$ for $|t| \le \frac{1}{2}\widetilde{R}$, $|v| \le \frac{1}{2}\widetilde{R}$. Therefore the proof of the lemma will be completed by the following proposition:

PROPOSITION. *Suppose A is an $N \times N$ complex matrix such that $\|\operatorname{Im} A\| \le \frac{1}{2}\|\operatorname{Re} A\|$ and such that $\operatorname{Re} A$ is non-singular, then*

$$\pi^{-N/2} \int_{s \in \mathbf{R}^N} \operatorname{Exp}(-[A \cdot s]^2) \det A \, ds = 1.$$

The proof of this proposition is elementary and we refer the reader to the proof in [B] for details. □

Lemma 4.2 would provide us with a sequence of entire functions if the integral in (40) did not depend on the choice of M_v. The following lemma shows that if f is CR then it is enough to integrate over M_0 only.

LEMMA 4.3. *If U is a smooth CR function in a neighborhood of $M \cap \{\xi' : |\xi'| \le \sqrt{2}\widetilde{R}\}$ then*

$$U(\zeta) = \lim_{\epsilon \to 0} \pi^{-N/2} \epsilon^{-N} \int_{\zeta' \in M_0} g(\operatorname{Re} \zeta') U(\zeta') \operatorname{Exp}(-\epsilon^{-2}[\zeta - \zeta']^2) d\zeta' \tag{41}$$

uniformly for $|v| \le \widetilde{R}/(24\sqrt{2})$, $|t| \le \frac{1}{2}\widetilde{R}$.

Proof. For fixed $|v| \le \widetilde{R}$ we consider

$$\widetilde{M}_v = \{\zeta(t', \lambda v) = t' + iH(t', \lambda v) \in M : |t'| \le \widetilde{R}, 0 \le \lambda \le 1\}.$$

Then $\widetilde{M}_v$ is an $(N+1)$-(real)-dimensional submanifold of M and its boundary is the union of M_0, M_v and the set

$$\{\zeta(t', \lambda v) : |t'| = \widetilde{R}, 0 \le \lambda \le 1\}. \tag{42}$$

We apply Stokes' theorem to (40) to conclude that for $|t| \le \frac{1}{2}\widetilde{R}$, $|v| \le \widetilde{R}$ we have

$$\begin{aligned} U(\zeta) = {} & \lim_{\epsilon \to 0} \pi^{-N/2} \epsilon^{-N} \int_{\zeta' \in M_0} g(\operatorname{Re} \zeta') U(\zeta') \operatorname{Exp}(-\epsilon^{-2}[\zeta - \zeta']^2) \, d\zeta' \\ & + \lim_{\epsilon \to 0} \pi^{-N/2} \epsilon^{-N} \int_{\zeta' \in \widetilde{M}_v} d_{\zeta'} \left\{ g(\operatorname{Re} \zeta') U(\zeta') \operatorname{Exp}(-\epsilon^{-2}[\zeta - \zeta']^2) \right\} d\zeta'. \end{aligned}$$

Note that since the support of $g(\operatorname{Re} \zeta)$ does not intersect the set (42) the corresponding term drops out. The second integral involves the exterior derivative $d_{\zeta'} = \partial_{\zeta'} +$

$\overline{\partial}_{\zeta'}$ but due to the presence of $d_{\zeta'}$ the only contributing term comes from $\overline{\partial}_{\zeta'}$. By taking the almost holomorphic extension of U from M to $\mathbf{C}^N$, we may assume that $\overline{\partial}_{\zeta'}U(\zeta(t', v)) = 0$ for $|t'| \leq \widetilde{R}$, $|v| \leq \widetilde{R}$. Therefore we have

$$U(\zeta) = \lim_{\epsilon\to 0} \pi^{-N/2}\epsilon^{-N} \int\limits_{\zeta'\in M_0} g(\operatorname{Re}\zeta')U(\zeta')\operatorname{Exp}(-\epsilon^{-2}[\zeta - \zeta']^2)d\zeta' +$$

$$+ \lim_{\epsilon\to 0} \pi^{-N/2}\epsilon^{-N} \int\limits_{\zeta'\in\widetilde{M}_v} \left(\overline{\partial}_{\zeta'}g(\operatorname{Re}\zeta')\right) U(\zeta')\operatorname{Exp}(-\epsilon^{-2}[\zeta - \zeta']^2)d\zeta'. \tag{43}$$

We want to show that the second limit is zero. To estimate $|\operatorname{Exp}(-\epsilon^{-2}[\zeta - \zeta']^2)|$ we need to estimate the real part of the exponent:

$$\operatorname{Re}\{[\zeta - \zeta']^2\} = |t - t'|^2 - |H(t, v) - H(t', \lambda v)|^2.$$

Equation (39) and estimate (12) imply that $|\partial H/\partial t| < 1/2$, $|\partial H/\partial v| \leq 2$ for $|v| \leq \widetilde{R}$ and for $t \in \operatorname{supp} g$. Therefore we have

$$\begin{aligned} |H(t, v) - H(t', \lambda v)| &\leq |H(t, v) - H(t', v)| + |H(t', v) - H(t', \lambda v)| \\ &\leq 1/2|t - t'| + 2|v - \lambda v|. \end{aligned}$$

Squaring the last inequality and making use of the inequality $2ab \leq \frac{1}{4}a^2 + 4b^2$ gives us

$$|H(t, v) - H(t', \lambda v)|^2 \leq \frac{1}{2}|t - t'|^2 + 8|v - \lambda v|^2.$$

The inequality $|v - \lambda v| < |v|$ (which follows from $0 \leq \lambda \leq 1$) implies that

$$\operatorname{Re}\{[\zeta - \zeta']^2\} \geq \frac{1}{2}|t - t'|^2 - 8|v|^2.$$

Now observe that since $g(\operatorname{Re}\zeta') \equiv 1$ for $|\operatorname{Re}\zeta'| = |t'| \leq \frac{2}{3}\widetilde{R}$, we have $\overline{\partial}_{\zeta'}g = 0$ for $|\operatorname{Re}\zeta'| = |t'| \leq \frac{2}{3}\widetilde{R}$ and therefore

$$\frac{1}{2}|t - t'|^2 \geq \frac{1}{2}\left(\frac{\widetilde{R}}{6}\right)^2 \quad \text{for } t' \in \operatorname{supp}\overline{\partial}_{\zeta'}g, \text{ and } |t| \leq \frac{\widetilde{R}}{2}.$$

At the same time we have

$$8|v|^2 \leq \frac{\widetilde{R}^2}{12^2}, \text{ for } |v| \leq \frac{\widetilde{R}}{24\sqrt{2}}.$$

Therefore we have

$$\operatorname{Re}\{[\zeta - \zeta']^2\} \geq \frac{\widetilde{R}^2}{12^2}.$$

for t', v as above. The last inequality allows us to estimate the second integral in (43):

$$\left| \epsilon^{-N} \int\limits_{\zeta' \in \widetilde{M}_v} \left(\overline{\partial}_{\zeta'} g(\operatorname{Re} \zeta')\right) U(\zeta') \operatorname{Exp}(-\epsilon^{-2}[\zeta - \zeta']^2) d\zeta' \right| \leq \epsilon^{-N} \operatorname{Exp}\left(-\epsilon^{-2} \frac{\widetilde{R}}{12^2}\right) K.$$

where K is a constant determined by the size of M_v and the sup-norm of $(\overline{\partial}_{\zeta'} g(\operatorname{Re} \zeta')) f(\zeta')$.

This inequality shows that the second limit in (43) is zero as $\epsilon \to 0$ uniformly for $|t| < \frac{1}{2}\widetilde{R}$ and $|v| \leq \widetilde{R}/(24\sqrt{2})$. □

End of proof of Theorem 4.1. The function on the right side of (41) is an entire function of ζ. Therefore f can be approximated by holomorphic polynomials uniformly on M for $|\xi'| \leq \widetilde{R}/(24\sqrt{2})$ or (in old coordinates) for $|\xi| \leq \widetilde{R}/(24\sqrt{2})$.

THEOREM 4.4. *Suppose we are given l generic manifolds with boundary*

$$M_k = \{\xi \in \mathbf{C}^N : \rho_j(\xi) = 0, 1 \leq j \leq l, j \neq k, \rho_k(\xi) \geq 0\}$$

where $\rho_1, \ldots, \rho_l \in C^2$ are normalized so that

$$\left| \left(\frac{\partial \rho(0)}{\partial \xi}\right)^T t \right| \geq |t|, \quad \forall t \in \mathbf{C}^l. \tag{44}$$

Suppose there exists a constant $C > 0$ such that

$$\|\rho_j''(\xi)\| \leq C, \; 1 \leq j \leq l \text{ for } |\xi| \leq R. \tag{45}$$

Suppose U is a continuous function on $M_1 \cup \cdots \cup M_l$ which is CR on each of $M_j \cap \{\xi \in \mathbf{C}^N : |\xi| \leq R\}$. Then there exists a sequence of holomorphic polynomials p_j in $\mathbf{C}^N$ such that $p_j \to U$ uniformly on

$$(M_1 \cup \cdots \cup M_l) \cap \{\xi \in \mathbf{C}^N : |\xi| \leq R^*\}$$

where

$$R^* = \frac{1}{96} \min\left\{\frac{1}{4lC}, R\right\}.$$

Proof. The proof is just a minor adaptation of the proof of Theorem 4.1. By Lemma 2.2 there exists a new set of coordinates $\xi' = (x + iy, u + iv)$ and a function $h = (h_1, \ldots, h_l)$, satisfying (12), so that

$$M_k = \{\xi \in \mathbf{C}^N : y_j - h_j(x, u, v) = 0, 1 \leq j \leq l, j \neq k, y_k - h_k(x, u, v) \geq 0\}.$$

Let $M_0 = \{\xi' = (x+iy, u+iv) \in \mathbf{C}^N : y = h(x,u,0)\}$ for $|\xi'| < \sqrt{2}\widetilde{R}$. Then M_0 is a maximally totally real submanifold of the $(M_1 \cup \cdots \cup M_l)$. At the same time it is a maximally totally real submanifold of each of the M_j's individually.

Applying Lemma 4.3 with $M = M_j$ for $j = 1, \ldots, l$ implies that

$$U(\zeta) = \lim_{\epsilon \to 0} \pi^{-N/2} \epsilon^{-N} \int_{\zeta' \in M_0} g(\operatorname{Re} \zeta') U(\zeta') \operatorname{Exp}(-\epsilon^{-2}[\zeta - \zeta']^2) d\zeta' \tag{46}$$

uniformly for $|\zeta| \leq \widetilde{R}/(24\sqrt{2})$, $\zeta \in M_j$. The integral in the right hand side of (46) is an entire function which proves Theorem 4.4. □

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Department of Mathematics, University of California at San Diego.
Current address: Department of Mathematics, University of Washington, Seattle, WA 98195-4350
vityaev@math.washington.edu