ILLINOIS JOURNAL OF MATHEMATICS
Volume 44, Number 3, 2000

BETTI NUMBERS OF ALMOST COMPLETE INTERSECTIONS

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ABSTRACT. We investigate the minimal free resolutions of cyclic modules R/I, where I is an almost complete intersection in the local ring R. Our results concern various binomial lower bounds for the Betti numbers of the resolution. For example, we show that the sum of the Betti numbers is at least 2^d where d is the dimension of R.

Introduction

Throughout this paper (R, m, k) will denote a local ring with maximal ideal m and residue field k. If M is a finitely-generated module over R, then one can consider the minimal free resolution of M,

$$\cdots \to R^{b_2} \to R^{b_1} \to R^{b_0} \to M \to 0.$$

The ranks of the modules appearing in this resolution are called the *Betti numbers* of M; the ith Betti number is b_i. It is easy to see that one has the formula $b_i = \dim_k Tor_i(M, k)$.

The question to be considered in this paper, attributed to Horrocks, is the following (see [Ha]):

When R is a regular local ring of dimension n and M is a module of finite length, is it necessarily the case that $b_i(M) \geq \binom{n}{i}$ for all values of i?

If we let $r_i(M)$ denote the rank of the ith *map* in a minimal free resolution of M, a stronger question would ask whether $r_i(M) \geq \binom{n-1}{i-1}$. This is stronger because in any free resolution one must have $r_i + r_{i+1} = b_i$, thanks to the Buchsbaum–Eisenbud acyclicity criterion [BE4]. On the other hand, a weaker question asks whether $\sum_{i=0}^{n} b_i(M) \geq 2^n$. These are all known (and relatively easy) when $n \leq 4$, when $i \in \{0, 1, n-1, n\}$, as well as in several special cases (cf. [BE1], [Ch], [EG1], [EG2], [Sa], for instance).

Now we will restrict our attention in particular to the case where M is a cyclic module R/I and I is m-primary. Note that this implies $\mu(I) \geq \operatorname{ht} I = \operatorname{ht} m = \dim R = n$, where $\mu(I)$ is the minimal number of generators of I. If $\mu(I)$ is actually *equal* to n then I is generated by a regular sequence (i.e., I is a complete intersection), and the conjectured lower bounds are actually obtained because R/I is resolved by the Koszul complex.

Received December 21, 1998; received in final form December 15, 1999.
1991 Mathematics Subject Classification. Primary 13D02; Secondary 13D25.

In this paper we begin an investigation of the case where $\mu(I) = n + 1$, which is when I is a so-called *almost complete intersection.* Our attention will not be limited only to regular rings, however. We show in the first section that the conjectured lower bounds hold for at least half of the betti numbers and at least half of the ranks, and we establish the conjectured bound on the *sum* of the betti numbers. This is all done by exploiting a kind of 'almost duality' which is enjoyed by almost complete intersections.

In the second section we introduce some new combinatorial arguments which rule out the existence of certain resolutions. These techniques are very elementary, depending only on general position arguments and counting. We do have to make use of a little linkage theory, but this is not at all difficult. In the last section we list some open problems.

This work was done several years ago, while the author was at the University of Michigan. I would like to thank Mel Hochster for introducing me to the problem, as well as for much guidance and encouragement during the actual research.

Preliminaries

If I is a proper ideal, $\operatorname{depth}_I R$ denotes the length of a maximal regular sequence contained in I. The ideal is said to be *perfect* if $\operatorname{pd} R/I = \operatorname{depth}_I R$, in which case the *type* of I is defined to be value of the last (nonzero) betti number of R/I. Perfect ideals of type 1 are called *Gorenstein. Complete intersections* are ideals for which $\mu(I) = \operatorname{depth}_I R$ (which are necessarily perfect, being resolved by the Koszul complex) and *almost complete intersections* are perfect ideals for which $\mu(I) = \operatorname{depth}_I R + 1$. Be advised that some authors use slightly different definitions. It should be remembered that almost complete intersections are *never* Gorenstein—this is proven in both [BE2] and [Ku] for the case in which the ring is regular, but that assumption can easily be eliminated.

Also note that if $f\colon R^n \to R^m$ then $I_t(f)$ denotes the ideal generated by the size t minors of any matrix for f, and that the *rank* of f is the largest t for which $I_t(f) \neq 0$. If $x_1, \dots, x_n \in R$ then $K_\bullet(x_1, \dots, x_n; R)$ denotes the Koszul complex on the x's.

Section 1

Suppose that $\cdots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0$ is a free resolution of a finitely generated module M (but not necessarily a *minimal* resolution!) We want to explain how to recover the betti numbers of M from the information in this complex. It is well known, of course, that $F_\bullet$ decomposes into $G_\bullet \oplus Q_\bullet$, where $G_\bullet$ is a minimal resolution of M and $Q_\bullet$ is a split-exact resolution of zero. Thus, $Q_\bullet$ remains exact when we tensor with the residue field k. So if q_i denotes the ith map in $Q_\bullet$, we have $\operatorname{rank} Q_i = \operatorname{rank}(q_i \otimes \mathrm{id}_k) + \operatorname{rank}(q_{i+1} \otimes \mathrm{id}_k)$.

To simplify things, let us adopt the notation $\operatorname{rank}_k h$ for $\operatorname{rank}(h \otimes \mathrm{id}_k)$, and refer to it as the *k-rank* of the map h. (Notice that this is also equal to the largest value of t for which $I_t(h) = R$). We have shown, then, that

$$\operatorname{rank} F_i = \operatorname{rank} G_i + \operatorname{rank} Q_i = b_i(M) + \operatorname{rank}_k q_i + \operatorname{rank}_k q_{i+1}.$$

However, it follows from the nature of our decomposition of $F_\bullet$, and in particular from the fact that $G_\bullet$ is a *minimal* resolution, that $\operatorname{rank}_k q_i = \operatorname{rank}_k f_i$; thus we may write

$$b_i(M) = \operatorname{rank} F_i - (\operatorname{rank}_k f_i + \operatorname{rank}_k f_{i+1}).$$

This is the relation we were after, and it will be an important tool in the proof of our main result:

(1.1) THEOREM. *Let I be an almost complete intersection of depth d. Then there exist nonnegative integers Γ_j, $0 \le j \le d+1$, such that*

(*a*) $\max\{0, b_j - \binom{d+1}{j}\} \le \Gamma_j \le b_j$,
(*b*) $b_j + b_{d-j+1} = \Gamma_j + \Gamma_{j+1} + \binom{d+1}{j}$,
(*c*) $r_j + r_{d-j+2} = \Gamma_j + \binom{d}{j-1}$,
(*d*) $\Gamma_j = \Gamma_{d-j+2}$.

Proof. Choose a minimal set of generators $x_1, \ldots, x_{d+1}$ for I such that $x_1, \ldots, x_d$ is a regular sequence, and write $K_\bullet$ instead of $K_\bullet(x_1, \ldots, x_{d+1}; R)$. It follows from the depth-sensitivity of the Koszul complex that $K_\bullet$ has homology only in degrees 0 and 1. We further note that

$$\begin{aligned} H_1(x_1, \ldots, x_{d+1}; R) \cong H_1(x_{d+1}; R/(x_1, \ldots, x_d)) &\cong \frac{(x_1, \ldots, x_d) : I}{(x_1, \ldots, x_d)} \\ &\cong \operatorname{Hom}(R/I, R/(x_1, \ldots, x_d)) \\ &\cong \operatorname{Ext}^d(R/I, R). \end{aligned}$$

(The first and last isomorphisms arise from homological shifting.) Now if $F_\bullet \to R/I \to 0$ is the minimal resolution of R/I, then it is a consequence of the perfection of I that $F_\bullet^*$ (the dual of $F_\bullet$) is the minimal resolution for $\operatorname{Ext}^d(R/I, R) \cong H_1(\underline{x}; R)$. An augmentation $F_d^* \to H_1(\underline{x}; R)$ induces in the natural way a map $F_d^* \xrightarrow{\psi} Z_1$, where Z_1 is the module of one-cycles in the Koszul complex, and it follows that ψ lifts to a comparison map of complexes:

$$\begin{tikzcd} 0 \arrow[r] & K_{d+1} \arrow[r] & K_d \arrow[r] & K_{d-1} \arrow[r] & \cdots \arrow[r] & K_2 \arrow[r] & K_1 \arrow[r] & K_0 \arrow[r] & R/I \\ & & & & & & & & \\ 0 \arrow[r] & F_0 \arrow[r] \arrow[uu, "\psi_0"] & F_1 \arrow[r] \arrow[uu, "\psi_1"] & F_2 \arrow[r] \arrow[uu, "\psi_2"] & \cdots \arrow[r] & F_{d-1} \arrow[r] \arrow[uu, "\psi_{d-1}"] & F_d. \arrow[uu, "\psi_d=\psi"] & & \end{tikzcd}$$

If $C_\bullet$ denotes the mapping cone, then the long exact sequence for a mapping cone shows readily that $C_\bullet$ is a resolution of R/I (almost all the terms in the sequence are zero). Moreover, since the boundary maps for $K_\bullet$ and $F_\bullet$ all have k-rank zero (i.e., the matrices for the maps have entries in the maximal ideal) it's easy to see that the k-rank of the ith map in the mapping cone is equal to the k-rank of ψ_{d+2-i}. The remark preceding the proposition now shows that $b_i = \operatorname{rank} C_i - (\operatorname{rank}_k \psi_{d+2-i} + \operatorname{rank}_k \psi_{d+1-i})$. So if we set $\Gamma_i = \operatorname{rank} F_i - \operatorname{rank}_k \psi_i$ and use the fact that $\operatorname{rank} C_i = \operatorname{rank} K_i + \operatorname{rank} F_{d+2-i}$, we obtain

$$\begin{aligned} b_i &= \operatorname{rank} K_i + \operatorname{rank} F_{d+2-i} + (\Gamma_{d+2-i} - \operatorname{rank} F_{d+2-i}) + (\Gamma_{d+1-i} - \operatorname{rank} F_{d+1-i}) \\ &= \tbinom{d+1}{i} - b_{d+1-i} + \Gamma_{d+2-i} + \Gamma_{d+1-i}. \end{aligned}$$

And now by interchanging i and $d+1-i$ this can be rewritten in the slightly simpler form

$$b_i + b_{d+1-i} = \tbinom{d+1}{i} + \Gamma_i + \Gamma_{i+1}.$$

Substituting this into the identity $r_i = b_i - b_{i+1} + \cdots + (-1)^{d-i} b_d$ yields $r_i + r_{d+2-i} = \Gamma_i + \binom{d}{i-1}$; an immediate consequence is that $\Gamma_i = \Gamma_{d+2-i}$. □

(1.2) *Remark.* An alternative (dual) approach to proving the result is as follows: Let Z_1 and B_1 be the 1-cycles and 1-boundaries in the Koszul complex on $x_1, \ldots, x_{d+1}$. Then $0 \to K_{n+1} \to \cdots \to K_2$ is the minimal resolution of B_1, and $0 \to F_n \to \cdots \to F_2$ is the minimal resolution of Z_1. The inclusion map $B_1 \hookrightarrow Z_1$ lifts to a comparison map of complexes, the mapping cone of which then gives a free-resolution of $H_1(\underline{x}; R)$. But $F_\bullet^*$ is the *minimal* resolution of $H_1(\underline{x}; R)$, so it splits off from this mapping cone.

Said differently, if $\gamma\colon K_\bullet \to F_\bullet$ is a map of complexes such that $\gamma_0 = \gamma_1 = \mathrm{id}_R$, then the mapping cone of the following diagram is a resolution for $H_1(\underline{x}; R)$:

$$\begin{tikzcd} & 0 \arrow[r] & F_d \arrow[r] & F_{d-1} \arrow[r] & \cdots \arrow[r] & F_3 \arrow[r] & F_2 \\ 0 \arrow[r] & K_{d+1} \arrow[r] \arrow[u] & K_d \arrow[r] \arrow[u, "\gamma_d"'] & K_{d-1} \arrow[r] \arrow[u, "\gamma_{d-1}"'] & \cdots \arrow[r] & K_3 \arrow[r] \arrow[u, "\gamma_3"'] & K_2. \arrow[u, "\gamma_2"'] \end{tikzcd}$$

Thus, the mapping cone for the *dual* of this diagram is a resolution for R/I, and the Γ_i's appearing in the proposition are simply the *k-nullities* of the γ_i^*'s. This is often the best way to 'remember' the result, and it connects strongly to ideas from linkage theory which will be discussed in the next section. Notice that part (c) now states that the k-nullities of γ_i^* and γ_{d-i+2}^* are equal, for all i—it's natural to wonder whether there's a more direct way to see this.

The following easy corollary gives the promised lower bounds for the betti numbers:

(1.3) COROLLARY. *Let I and d be as above, and write $b_i = b_i(R/I)$, $r_i = r_i(R/I)$.*

(a) *If $0 \le k \le d$ then either $b_k \ge \binom{d}{k}$ or $b_{d+1-k} \ge \binom{d}{d+1-k}$.*
(b) *If $1 \le k \le d$ then either $r_k \ge \binom{d-1}{k-1}$ or $r_{d+2-k} \ge \binom{d-1}{d+1-k}$.*
(c) *When $d = 2m+1$, $b_{m+1} \ge \binom{d}{m+1}$; when $d = 2m$, $r_{m+1} \ge \binom{d-1}{m}$.*
(d) $\sum_{i=0}^{d} b_i \ge 2^d$.

Proof. Using the positivity of the coefficients Γ_i, parts (b) and (c) of (1.1) yield immediately that $b_k + b_{d-k+1} \ge \binom{d+1}{k}$ and $r_k + r_{d-k+2} \ge \binom{d}{k-1}$ whenever $0 \le k \le d+1$. This gives (a) and (b), from which (c) follows immediately. If the first inequality is summed over all k, one obtains

$$2\sum_{k=0}^{d+1} b_k \ge \sum_{k=0}^{d+1} \binom{d+1}{k} = 2^{d+1}.$$

After remembering that $b_{d+1} = 0$, we obtain (d). □

Section 2

If I is an almost complete intersection of depth 5, then the Horrocks bounds can now be shown to hold for all of the betti numbers of R/I except for b_2 (the bound for b_3 is obtained from (1.3), and the others are trivial). In this section that bound will be established as well in the case where type$(I) \le 5$. We do this by some elementary counting arguments, making use of general position and linkage theory.

The following lemma recalls some of the basic facts from the theory of linkage (to be found in [HU], for instance). We go ahead and include the proofs because our situation is slightly different from that usually found in the literature, and because the same ideas will shortly be used elsewhere. Note that when I and J are ideals of R then $I : J$ denotes $\{r \in R : rJ \subseteq I\}$.

(2.1) LEMMA. *Let I be a perfect ideal of depth d and let $x_1, \ldots, x_d$ be a (maximal) regular sequence contained in I such that $(x_1, \ldots, x_d) \ne I$. Set $J = (\underline{x}) : I$. Then:*

(a) *J is perfect, of depth d.*
(b) $\mathrm{Ext}^d(R/J, R) \cong I/(\underline{x})$.
(c) $(\underline{x}) : J = I$.
(d) Type$(J) \le \mu(I)$. *Moreover, if the x's are part of a minimal set of generators for I, then* type$(J) = \mu(I) - d$ (*= the so-called* deviation *of I*).

Proof. (a) Let $0 \to F_d \to \cdots \to F_0 \to R/I$ be the minimal resolution of R/I, and write $K_\bullet$ for $K_\bullet(x_1, \ldots, x_d;\, R)$ (which is the minimal resolution for $R/(\underline{x})$). The projection $R/(\underline{x}) \to R/I$ lifts to a map of complexes $\pi\colon K_\bullet \to F_\bullet$. It is well known that a resolution for R/J can be obtained from the following diagram by dualizing and then taking the mapping cone (see [PS], for instance):

$$\begin{tikzcd} 0 \arrow[r] & F_d \arrow[r] & F_{d-1} \arrow[r] & \cdots \arrow[r] & F_2 \arrow[r] & F_1 \\ 0 \arrow[r] & K_d \arrow[r] \arrow[u, "\pi_d"] & K_{d-1} \arrow[r] \arrow[u, "\pi_{d-1}"] & \cdots \arrow[r] & K_2 \arrow[r] \arrow[u, "\pi_2"] & K_1 . \arrow[u, "\pi_1"] \end{tikzcd}$$

It is then immediate that $\operatorname{pd} R/J \le d$. However it is always true that $pd\, R/J \ge \operatorname{depth}_J R$, and the fact that $x_1, \ldots, x_d$ is a regular sequence contained in J shows that $\operatorname{depth}_J R \ge d$. Thus $\operatorname{pd} R/J = \operatorname{depth}_J R = d$.

(b) Since taking the mapping cone after dualizing gives a resolution of R/J which has minimal length (namely d), taking the mapping cone *before* dualizing must give a resolution of $\operatorname{Ext}^d(R/J, R)$ (this uses the fact that J is perfect). Yet the top row of the above diagram is a resolution of I and the bottow row is a resolution of $(\underline{x})$, so that the mapping cone is a resolution of $I/(\underline{x})$.

(c) Let $Q = (\underline{x}) : J$. By (a) we know that Q is perfect of depth d, and from (b) it follows that

$$\operatorname{Ext}^d(R/Q, R) \cong \frac{J}{(\underline{x})} = \frac{(\underline{x}) : I}{(\underline{x})} \cong \operatorname{Hom}(R/I, R/(\underline{x})) \cong \operatorname{Ext}^d(R/I, R).$$

But since both Q and I are perfect we have $R/Q \cong \operatorname{Ext}^d(\operatorname{Ext}^d(R/Q, R), R)$ and $R/I \cong \operatorname{Ext}^d(\operatorname{Ext}^d(R/I, R), R)$, so $R/Q \cong R/I$; hence $Q = I$.

(d) In analogy to the argument given in (1.1) we see that

$$\operatorname{type}(J) = b_d(R/J) = \operatorname{rank} F_1 - \operatorname{rank}_k \pi_1 = \mu(I) - \operatorname{rank}_k \pi_1 \le \mu(I).$$

If the x's actually form part of a minimal set of generators for I, then this means that π_1 is split; that is, $\operatorname{rank}_k \pi_1 = d$, and so $\operatorname{type}(J) = \mu(I) - d$. □

(2.2) *Remarks.* (i) When the x's form part of a minimal set of generators for I, parts (a) and (d) of the lemma show that if I is an almost complete intersection then $(\underline{x}) : I$ is Gorenstein. Conversely, it's not hard to show using the same ideas that if I is Gorenstein then $(\underline{x}) : I$ is an almost complete intersection.

(ii) Two perfect ideals I and J are said to be *linked* if there is a maximal regular sequence $x_1, \ldots, x_d$ in $I \cap J$ such that $J = (\underline{x}) : I$—part (c) of the lemma guarantees the symmetry of this condition. The following proposition makes use of the fact that the property of being 'linked to a complete intersection' is particularly nice. While the result is possibly known to the experts, we do not know of a proof appearing in the literature and so one is included here.

(2.3) PROPOSITION. If I is a perfect ideal of depth d which is linked to a complete intersection, then $b_i(R/I) \geq \binom{d}{i} + \binom{d-1}{i-1}$.

Proof. The assumption on I is that it contains a regular seqence $x_1, \ldots, x_d$ such that $(x_1, \ldots, x_d) : I$ is a complete intersection—that is, it is generated by some regular sequence $y_1, \ldots, y_d$. We'll write $K_\bullet$ and $\mathcal{K}_\bullet$ for $K_\bullet(\underline{y}; R)$ and $K_\bullet(\underline{x}; R)$, respectively. We can lift the identity on R to a map ψ: $K_1 \to \mathcal{K}_1$, and then this extends in a natural way to a comparison map of complexes as follows:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathcal{K}_d & \longrightarrow & \mathcal{K}_{d-1} & \longrightarrow & \cdots & \longrightarrow & \mathcal{K}_2 & \longrightarrow & \mathcal{K}_1 \\
 & & \uparrow{\scriptstyle \bigwedge^d \psi} & & \uparrow{\scriptstyle \bigwedge^{d-1} \psi} & & & & \uparrow{\scriptstyle \bigwedge^2 \psi} & & \uparrow{\scriptstyle \psi} \\
0 & \longrightarrow & K_d & \longrightarrow & K_{d-1} & \longrightarrow & \cdots & \longrightarrow & K_2 & \longrightarrow & K_1
\end{array}$$

(we are now thinking of the Koszul complex in terms of the exterior algebra model). The lemma shows that, after dualizing, the mapping cone of this diagram gives a resolution of $R/((\underline{x}) : (\underline{y}))$, which is precisely R/I by the symmetry of linkage. Arguing again as in (1.1), we discover that

$$b_i(R/I) = \operatorname{rank} \mathcal{K}_{d-i+1} + \operatorname{rank} K_{d-i} - \operatorname{rank}_k \bigwedge^{d-i+1} \psi - \operatorname{rank}_k \bigwedge^{d-i} \psi.$$

But if we set $p = \operatorname{rank}_k \psi$ then ψ has a matrix of the form $\begin{pmatrix} I_p & \mathcal{O} \\ \mathcal{O} & M \end{pmatrix}$ where I_p denotes the $p \times p$ identity matrix and M is a matrix with entries in the maximal ideal of R. It is then easy to see that, for any choice of t, *rank*$_k \bigwedge^t \psi = \binom{p}{t}$. Using this to simplify the above equation, we find that

$$b_i(R/I) = \binom{d}{d-i+1} + \binom{d}{d-i} - \binom{p}{d-i+1} - \binom{p}{d-i} = \binom{d+1}{i} - \binom{p+1}{d-i+1}.$$

In particular note that $d + 1 = \mu(I) = b_1(R/I) = d + 1 - \binom{p+1}{d}$, which implies that $\binom{p+1}{d} = 0$. Hence $d > p + 1$, and it then follows that

$$b_i(R/I) \geq \binom{d+1}{i} - \binom{d-1}{d-i+1} = \binom{d}{i} + \binom{d}{i-1} - \binom{d-1}{d-i+1} = \binom{d}{i} + \binom{d-1}{i-1}. \qquad \square$$

(2.4) *Discussion.* We now return our attention to the betti numbers of R/I, where I is an almost complete intersection of depth d. We may restrict to the case where I is *not* linked to a complete intersection, since otherwise the desired lower bounds are provided by (2.3). Perhaps the first question to settle is whether this case even occurs! From (2.1d) we see that if I were linked to a complete intersection then we must have type$(I) \leq d$, and so it will suffice to find almost complete intersections with type *larger* than d. But note that if we had a Gorenstein ideal J with $\mu(J) = d + t$ and we chose a regular sequence $x_1, \dots, x_d$ which is part of a minimal set of generators for J, then by Remark (2.2i) the ideal $(\underline{x}) : I$ would be an almost complete intersection of type t. So we have reduced the problem to that of finding Gorenstein ideals requiring more that $2d$ generators. However, in [BE1], Gorenstein ideals requiring arbitrarily large numbers of generators are constructed in rings of dimension 3, and of course we can get examples in larger dimension rings simply by deforming (that is, by adding indeterminates).

Having come to the conclusion that this case really does need to be considered, let us proceed. The first step is to choose a minimal set of generators $f_1, \dots, f_{d+1}$ for I which are in general position—i.e., such that any d of them form a regular sequence (Lemma 8.2 of [BE3] gives a proof that this is always possible). It follows as in (1.1) that $H_1(f_1, \dots, f_{d+1}; R) \cong \mathrm{Ext}^d(R/I, R)$, and the minimal number of generators for the latter module is precisely the *type* of I—call this number t. We may then pick relations $v_i = (u_{i1}, \dots, u_{i,d+1})$, $1 \le i \le t$, so that the homology classes represented by $v_1, \dots, v_t$ form a minimal set of generators for $H_1(\underline{f}; R)$. It follows that the v_i, together with the Koszul relations on the f's, generate *all* of the relations on the f's, and it's not hard to see that the v_i's must be *minimal* among these relations. Thus, there exist Koszul relations $\rho_1, \dots, \rho_w$ such that the v_i's together with the ρ_j's form a minimal basis for the entire module of relations on the f's. (Of course this implies that $b_2(R/I) = t + w$). Said differently, if we let

$$U = \begin{pmatrix} u_{11} & \cdots & u_{1t} \\ \vdots & \cdots & \vdots \\ u_{d+1,1} & \cdots & u_{d+1,t} \end{pmatrix}$$

then there is a $(d+1) \times (b_2 - t)$ matrix V, all of whose columns are Koszul relations on the f's, such that R/I has a minimal resolution beginning as follows:

$$\cdots \to R^{b_2} \xrightarrow[(U|V)]{} R^{d+1} \xrightarrow[(f_1, \dots, f_{d+1})]{} R \to R/I \to 0$$

(matrices act on the left).

Now it is apparent that the elements in the jth row of $(U \mid V)$ generate the ideal $(f_1, \dots, \hat{f_j}, \dots, f_{d+1}) : I$, which we will denote by I_j (note that $\hat{f_j}$ means f_j is omitted). Since the f's were chosen to be in general position, $f_1, \dots, \hat{f_j}, \dots, f_{d+1}$ is a regular sequence (for any choice of j). Thus, by the first remark of (2.2), each I_j is Gorenstein, of depth d. However, it was assumed from the start that I is not linked to a complete intersection, and therefore no I_j is generated by a regular sequence. Moreover, we will never have $\mu(I_j) = d + 1$ —otherwise I_j would be an almost complete intersection, and therefore could not be Gorenstein. Hence $\mu(I_j) \ge d + 2$ for all j, and so each row of the matrix $(U \mid V)$ has at least $d + 2$ nonzero entries.

If we now suppose further that $t < d + 2$, then each row of V must have at least $d + 2 - t$ nonzero entries. Since V has $d + 1$ rows, there must be at least $(d + 1)(d + 2 - t)$ nonzero entries in the matrix. And yet each column of V is a Koszul relation on the f's, which will have precisely 2 nonzero entries, and so it follows that V must have at least $\frac{1}{2}(d + 1)(d + 2 - t)$ columns. We now recall that the number of columns of V is $b_2 - t$, and thus obtain the inequality

$$b_2 \geq t + \frac{(d+1)(d+2-t)}{2} = \frac{1}{2}(d^2 + 3d - (d-1)t + 2).$$

We have therefore proven:

(2.5) PROPOSITION. *Let I be an almost complete intersection of depth d, let t be the type of I, and assume that I is* not *linked to a complete intersection. Then if $t \leq d + 2$, $b_2 \geq \frac{1}{2}[d^2 + 3d - (d - 1)t] + 1$. In particular, if* $\operatorname{type}(I) \leq 4$ *and* $\operatorname{depth}_I R \geq 2$ *then* $b_2(R/I) \geq \binom{d}{2} + 3$.

(2.6) *Remark.* The above proposition can be used to answer a question raised by Avramov and Buchweitz [AB]. In a footnote at the end of their paper, they point out that the smallest possible sequence of Betti numbers they cannot rule out has the form $(b_0, b_1, \ldots, b_5) = (1, 6, 9, 10, 8, 2)$ over a dimension 5 ring. Note that a module with such a resolution must be of the form R/I, where I is an almost complete intersection of type 2. Clearly I cannot be linked to a complete intersection, or else the betti numbers would have to be much higher by (2.3). The above proposition now tells us that b_2 must be at least 13, and so no such resolution can exist.

(2.7) *Discussion.* Now suppose that I is an almost complete intersection of depth 5—we are only lacking the lower bound for b_2, and the above proposition gives it to us when $\operatorname{type}(I) \leq 5$. Moreover, the line of reasoning given in (2.4) shows that $b_2 \geq \operatorname{type}(I)$ always, and so we obtain the bound on b_2 when $\operatorname{type}(I) \geq 10$. The Buchsbaum-Eisenbud acyclicity criterion tells us that a resolution of R/I must have the following form (the numbers below the maps indicate their ranks):

$$0 \to R^t \xrightarrow[t]{} R^{t+a} \xrightarrow[a]{} R^{a+b} \xrightarrow[b]{} R^{5+b} \xrightarrow[5]{\psi_2} R^6 \xrightarrow[1]{\psi_1} R \to R/I.$$

The Syzygy Theorem, in the cases where it is known, immediately gives $b \geq 3$; however this can also be obtained in general using other methods.

Note additionally that (1.3) says $a + b \geq 10$, and that (2.5) gives $b \geq 4$ in the case $t = 6$. Based on this, the 'simplest' possible counterexample to Horrocks' question which we cannot rule out at the moment would have a resolution of the form

$$0 \to R^6 \to R^{a+6} \to R^{a+4} \to R^9 \to R^6 \to R \qquad \text{where } a \geq 6.$$

Section 3

We now indicate some open problems. Unless otherwise stated, R is simply an arbitrary local ring; however, the main case of interest is always when R is regular.

(1) There seem to be no known examples of almost complete intersections for which the strong bounds given in (2.3) do not hold.

(2) If I is an ideal which is linked to a complete intersection, then it's easy to see that $\operatorname{type}(R/I) \leq \operatorname{depth}_I R$. It would be useful to have a collection of counterexamples to the converse, if they exist (especially when I is an almost complete intersection).

(3) Questions abound even for almost complete intersections of depth 4. The Buchsbaum-Eisenbud acyclicity criterion, together with (1.1), shows that a resolution must have the form

$$0 \to R^a \to R^{a+b} \to R^{b+4} \to R^5 \to R$$

where $a \geq 2$, $b \geq 3$, and $a - 4 \leq b \leq a + 6$. It seems very likely, however, that there should be much stronger relations between a and b. Although a can probably take on any value in the specified range, it's doubtful whether the *pair* (a, b) can take on all of these values. What additional restrictions can be put on a and b?

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