HARMONIC FUNCTIONS ON THE UNIT DISC I[1]

BY

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1. Introduction

This report concerns real- or complex-valued harmonic functions defined on discs in the plane. The principal result may be stated as

THEOREM A. *A function f is harmonic on the unit disc if and only if there is a sequence $\{g_n\}$ of continuous functions on the unit circle such that*

$$\lim (n! \, \| g_n \|)^{1/n} = 0 \tag{1.1}$$

and

$$f(r, \theta) = \sum_{n=0}^{\infty} \frac{1}{2\pi} \int_0^{2\pi} P_r^{(n)}(\theta - t) g_n(t) \, dt. \tag{1.2}$$

In (1.1) the norm is defined by $\| g \| = \sup \{ | g(t) | : 0 \leqq t \leqq 2\pi$ and in (1.2)

$$P_r(\theta - t) = \mathcal{R}[(e^{it} + re^{i\theta})/(e^{it} - re^{i\theta})]$$

is the Poisson kernel for the unit disc. $P_r^{(n)}$ is the nth derivative of P_r.

This theorem was reported in [5]. It has been used by Douglas [2] as a global constraint for harmonic continuation in the disc of a function which is approximated at a finite set of points. Saylor, a student of Douglas, has extended these results to the case of solutions of a linear elliptic second order partial differential equation with analytic coefficients on a domain in R^n bounded by a compact analytic boundary [10].

There is a rich boundary-value theory concerning the Poisson and the Poisson-Stieltjes integrals beginning with the work of Fatou. A nice treatment of old and new results in this theory may be found in [4]. In view of the above result it is natural to ask whether there is a boundary function, in some generalized sense, associated with an arbitrary harmonic function by means of a Poisson representation. In fact, denoting $f_r(\theta) = f(r, \theta)$ and using (1.2) we have, formally

$$f_r = \textstyle\sum_{n=0}^{\infty} P_r^{(n)} * g_n = \sum_{n=0}^{\infty} P_r * g_n^{(n)} = P_r * \sum_{n=0}^{\infty} g_n^{(n)} = P_r * g.$$

The appropriate setting in which these calculations have meaning is a generalized function space having analytic test functions. Let $\mathcal{H}$ denote the linear space of analytic complex-valued functions on the unit circle Γ. Köthe [14], [15] introduced a certain locally convex topology for $\mathcal{H}$. The strong dual $\mathcal{H}'$ is the space of generalized functions. The Fantappie indicator of an element $f \in \mathcal{H}'$ is a function holomorphic on $\Omega - \Gamma$ and zero at ∞. Ω denotes

Received August 24, 1966.

[1] This research was supported by a grant of the Air Force Office of Scientific Research

the Riemann sphere. It converges to f at Γ in the topology of $\mathcal{H}'$. Provided with the topology of uniform convergence on compact subsets of $\Omega - \Gamma$, the space of indicators is isomorphic to $\mathcal{H}'$.

In various settings such spaces of analytic functionals have been studied by Grothendieck [13], Gelfand and Silov [12], Sato [18], Roumieu [17], Lions and Magenes [8], and Meĭman [16]. The topology for the test function space usually appears as the inductive limit of a sequence of normed spaces although the choice of the normed spaces differs among the authors.

In this paper I present an independent description of the $\mathcal{H}$ and $\mathcal{H}'$ topologies using the corresponding sequence spaces of Fourier coefficients. It differs from previous presentations in that no intermediate normed spaces appear. A neighborhood is obtained directly from a dominant sequence. Also, the harmonic functions on the unit disc take the place of the holomorphic indicators, a possibility suggested by Sato [18] and used by Lions and Magenes [8]. The properties for $\mathcal{H}$ and $\mathcal{H}'$ are those obtained by Köthe [14], [15] and by Grothendieck [13].

Let $\mathfrak{F}$ denote the space of complex-valued harmonic functions on the unit disc provided with the topology of uniform convergence on compact sets. A very special case of Theorem 9.1 in [8] establishes a linear and topological isomorphism between $\mathfrak{F}$ and $\mathcal{H}'$ in which the element $\tilde{f} \in \mathcal{H}'$ corresponding to $f \in \mathfrak{F}$ is the trace of f on Γ. They consider, much more generally, elliptic systems on a domain in R^n with compact analytic boundary.

Returning to the question of boundary values we show that the isomorphism of Lions and Magenes is realized by the mapping $\tilde{f} \to f$ defined by the Poisson representation $f_r = P_r * \tilde{f}$. Moreover, $f_r \to \tilde{f}$ in $\mathcal{H}'$ as $r \nearrow 1$.

One may replace the sequence of functions $\{g_n\}$ in Theorem A with a sequence $\{\mu_n\}$ of Radon measures on Γ. In this case one has

$$\lim\, (n!\, \| \mu_n \|)^{1/n} = 0$$

and

$$f(r, \theta) = \sum_{n=0}^{\infty} \int_\Gamma P_r^{(n)}(\theta - t)\, d\mu_n(t) \tag{1.3}$$

where $\| \mu_n \| =$ total variation of μ_n on Γ.

If P_r is replaced with the kernel

$$H_r(\theta - t) = (e^{it} + re^{i\theta})/(e^{it} - re^{i\theta})$$

one obtains a representation for the class of analytic functions on the unit disc. Using (1.3) this may be written

$$f(r, \theta) = \sum_{n=0}^{\infty} \int_\Gamma H_r^{(n)}(\theta - t)\, d\mu_n(t). \tag{1.4}$$

This formula has as antecedent a theorem of Leau and Faber in the theory of Taylor series. (See [9, p. 313]). An analytic function $f(z) = \sum_{p=0}^{\infty} a_p z^p$ has a holomorphic continuation on the Riemann sphere with exactly one singularity which is located at $z = 1$ if and only if $a_p = g(p)$, $p = 0, 1, \cdots$, where g is an entire function of exponential type zero. Setting $g(w) = \sum_{n=0}^{\infty} b_n w^n$ the latter condition is equivalent to $\lim (n! \mid b_n \mid)^{1/n} = 0$ and we obtain after substitution and rearrangement

$$f(re^{i\theta}) = \frac{b_0}{2} + \sum_{n=0}^{\infty} \frac{(-i)^n b_n}{2} H_r^{(n)}(\theta).$$

This is a representation of the form (1.4) with point measures at $t = 0$.

If the support of each μ_n in (1.4) is contained in a fixed compact set K then (1.4) defines a holomorphic function on the complement of this set relative to the sphere. The converse question has not been investigated. That is, if f is holomorphic on the complement of a compact proper subset of the unit circle, is there a representation (1.4) for which $\operatorname{supp} \mu_n \subset K$, $n = 0, 1, \cdots$? Such a result would constitute a precise generalization of the Leau-Faber theorem.

Considering generalized functions on the real axis, Roumieu (see [17, p. 80]) has derived a structure theorem which leads to a representation of the form (1.4) in which H_r is replaced by the Cauchy kernel and the sequence $\{\mu_n\}$ is restricted by the same condition. Meĭman (see [16, §4]) has obtained a similar result. Meĭman was led to this class of generalized functions from a problem in scattering theory. Theorem A is a consequence of a structure theorem of Lions and Magenes (see [8, Prop. 1.3]) but a direct proof is presented in this paper. A structure theorem then follows from Theorem A.

We give here a brief outline of the topics to follow. An estimate of the growth of $P_r^{(n)}$ and $H_r^{(n)}$ with n is derived in Section 2. It is used in the proof of the representation theorem in Section 3. Section 4 contains a description of $\mathcal{H}$ and proof that it is a nonmetrizable complete space. Following a description of $\mathcal{H}'$ in Section 5, $\mathcal{H}$ and $\mathcal{H}'$ are shown to be dual Montel spaces. The Poisson representation of $\mathcal{F}$ is given in the last section together with a structure theorem for $\mathcal{H}'$.

2. An estimate for the kernel and its derivatives

The estimate is found as an application of an inequality derived by S. Mandelbrojt in his lectures. If z is the complex variable denote by $\mathfrak{D}$ the operator $z(d/dz)$ and set $\varphi_n(z) = \mathfrak{D}^n(1/(1 - z))$. Then φ_n is holomorphic on the z-sphere less the point $z = 1$.

PROPOSITION 1. *For each $\rho > 0$ there is an $a > 0$ such that*

$$|\varphi_n(z)| \leqq n!\, a^{n+1} \tag{2.1}$$

on $\{|z - 1| \geqq \rho\}$ for $n = 0, 1, \cdots$.

Proof. If $z = e^s$ and $\psi(s) = 1/(1 - e^s)$, then $\varphi_n(z) = \psi^{(n)}(s)$ for $s \neq \nu 2\pi i$. For $0 < \rho_1 < \rho_2 < 1$, consider the domains

$$R_1 = \{|e^s - 1| > \rho_1\} \quad \text{and} \quad R_2 = \{|e^s - 1| > \rho_2\}$$

in the s-plane. The function ψ is holomorphic in R_1 and bounded by $1/\rho_1$.

Thus for $s \in R_2$ and $d = \operatorname{dist}(\partial R_1, \partial R_2)$

$$|\psi^{(n)}(s)| = \left| \frac{n!}{2\pi i} \int_{|\lambda - s| = d} \frac{\psi(\lambda)}{(\lambda - s)^{n+1}} \, d\lambda \right| \leqq \frac{1}{\rho_1} \frac{n!}{d^n}.$$

Setting $a = \max(1/\rho_1, 1/d)$ and transforming back to the z-plane we obtain (2.1).

By comparing the power series expansions on $\{r < 1\}$ one finds that $H_r(\theta) = 2\varphi_0(re^{i\theta}) - 1$ and $H_r^{(n)}(\theta) = 2i^n\varphi_n(re^{i\theta})$ for $n = 1, 2, \cdots$. If $\rho > 0$ choose a as in Proposition 1 so that in addition we have $2|\varphi_0(z)| + 1 \leqq a$ on $\{|z - 1| \geqq \rho\}$. Then

$$(2.2) \qquad |H_r^{(n)}(\theta - t)| \leqq n!\, a^n$$

for $n = 0, 1, \cdots$ and $|re^{i\theta} - e^{it}| \geqq \rho$.

Thus if μ_n is a measure on Γ and $z = re^{i\theta}$, then

$$f_n(z) = \int_\Gamma H_r^{(n)}(\theta - t)\, d\mu_n(t)$$

is analytic on $\{|z| < 1\}$ and $|f_n(z)| \leqq n!\, a^{n+1} \|\mu_n\|$ on $\{|z| \leqq \rho\}$ if $\rho < 1$. For a sequence $\{\mu_n\}$ satisfying $\lim (n! \|\mu_n\|)^{1/n} = 0$ it follows readily that

$$(2.3) \qquad f(z) = \sum_{n=0}^{\infty} f_n(z) = \sum_{n=0}^{\infty} \int_\Gamma H_r^{(n)}(\theta - t)\, d\mu_n(t)$$

is an analytic function on $\{|z| < 1\}$ where the series converges uniformly on compact sets. We will use the convenient terminology of G. MacLane in describing this as subuniform convergence.

Our object is to show that any analytic function has such a representation. The next estimate for $H_r^{(n)}$ leads to a clue for the proof of this fact. Using a power series expansion on $\{r < 1\}$ one obtains

$$(2.4) \qquad |H_r^{(n)}(\theta - t)| \leqq B_n/(1 - r)^{n+1}.$$

Thus $|f_n(z)| \leqq B_n \|\mu_n\|/(1 - r)^{n+1}$. Writing $f_n(z) = \sum_{k=0}^{\infty} a_k z^k$ and using Cauchy's estimate for the coefficients we find

$$(2.5) \qquad |a_k| = O(k^{n+1})$$

An arbitrary analytic function on $\{|z| < 1\}$ has power series coefficients $\{a_k\}$ satisfying $\limsup |a_k|^{1/k} \leqq 1$. If success is to be achieved, there must be a decomposition of this sequence into a sum of sequences which grow as in (2.5). Such a decomposition is the result in Lemma 1.

Evans observed that there are functions which are not of the type f_n; that is, representable as a Stieltjes integral. (See [3, p. 59]). In his example $a_k = a^{\sqrt{k}}$, $a > 1$.

3. The representation theorem

According to the remarks of the last section we seek a decomposition $a_k = \sum_n a_{k,n}$ so that for each n the sequence $\{a_{k,n}\}$ grows as a power of k. Further, some uniformity must prevail for the constants involved as one varies n. The precise manner is expressed in

LEMMA 1. *Let* $\{a_k\}$ *be a sequence satisfying* $\limsup |a_k|^{1/k} \leqq 1$. *Then there are sequences* $\{a_{k,n}\}$ *and a finite-valued function* B *such that*

$$a_k = \sum_{n=0}^{k} a_{k,n}$$

and

$$|a_{k,n}| \leqq \frac{B(\varepsilon)\varepsilon^{n+2}}{(n+2)!} k^n$$

for $0 \leqq n \leqq k$, $k = 1, 2, \cdots$, *and for all* $\varepsilon > 0$.

Proof. Let $d_\nu = \sup_{k \geqq \nu} |a_k|^{1/k}$, $\nu = 1, 2, \cdots$. Then the sequence $\{d_\nu\}$ decreases to a limit no greater than 1. This may be expressed by writing $d_\nu \leqq 1 + \varepsilon_\nu$ where $\varepsilon_\nu \searrow 0$. Then for $k \geqq 1$

$$|a_k| \leqq d_k^k \leqq (1 + \varepsilon_k)^k = 1 + \binom{k}{1}\varepsilon_k + \cdots + \binom{k}{n}\varepsilon_k^n + \cdots + \varepsilon_k^k$$

and it is possible to write $a_k = a_{k,0} + \cdots + a_{k,n} + \cdots + a_{k,k}$ so that

$$|a_{k,n}| \leqq \binom{k}{n}\varepsilon_k^n$$

for $n = 0, 1, \cdots, k$. Setting $a_0 = a_{0,0}$ we have a decomposition of the desired form and it remains to show that the growth condition is valid.

$$|a_{k,n}| \leqq \binom{k}{n}\varepsilon_k^n \leqq \frac{k^n}{n!}\varepsilon_k^n.$$

If $\varepsilon > 0$, choose $k(\varepsilon)$ such that $\varepsilon_k/\varepsilon < e^{-1}$ for $k \geqq k(\varepsilon)$. Using the inequality $(n+1)(n+2)e^{-n} < 3$ we have $(n+1)(n+2)(\varepsilon_k/\varepsilon)^n < 3$ for $k \geqq k(\varepsilon)$. There are only finitely many pairs k, n such that $0 \leqq n \leqq k < k(\varepsilon)$ so that

$$B(\varepsilon) = \sup_{\substack{0 \leqq n \leqq k \\ k=1,2,\cdots}} \frac{(n+1)(n+2)}{\varepsilon^2}\left(\frac{\varepsilon_k}{\varepsilon}\right)^n < \infty$$

and

$$|a_{k,n}| \leqq \frac{B(\varepsilon)\varepsilon^{n+2}}{(n+2)!} k^n.$$

This completes the proof and we are now in a position to prove the representation theorem.

There are several forms in which the representation may be written corresponding to the various classes of functions which have been studied in connection with the Poisson integral. There are for example the Hardy H^p

classes or the positive harmonic functions. All these forms are obtained on starting with the class A of functions continuous on $\{|z| \leqq 1\}$ and analytic on $\{|z| < 1\}$. Provided with the norm

$$\| g \| = \sup \{ | g(e^{it}) | : 0 \leqq t < 2\pi \},$$

A is a Banach space. (See [4, Chapter 6]).

THEOREM 1. *A function f is analytic on $\{|z| < 1\}$ if and only if there is a sequence $\{g_n\} \subset A$ such that*

$$\lim (n! \| g_n \|)^{1/n} = 0 \tag{3.1}$$

and

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{2\pi} \int_0^{2\pi} P_r^{(n)}(\theta - t) g_n(e^{it}) \, dt \tag{3.2}$$

where $z = re^{i\theta}$.

Proof. The sufficiency of (3.1) and (3.2) is proved using (2.2). Conversely, if $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is analytic then $\lim \sup | a_k |^{1/k} \leqq 1$. Let $\{a_k\}$ be decomposed into the sequences $\{a_{k,n}\}$ of Lemma 1. For notational convenience suppose $a_0 = a_{0,0} = 0$. Except for a trivial modification when $n = 2$,

$$\textstyle\sum_{k=n-2}^{\infty} | a_{k,n}/k^n | \leqq (B(\varepsilon)\varepsilon^n/n!) \sum_{k=n-2}^{\infty} 1/k^2 < \infty \tag{3.3}$$

which implies that

$$g_n(z) = (-i)^n \sum_{k=n-2}^{\infty} \frac{a_{k,n-2}}{k^n} z^k = \frac{1}{2\pi} \int_0^{2\pi} P_r(\theta - t) g_n(e^{it}) \, dt$$

is an element of A. Differentiating n times with respect to θ

$$\sum_{k=n-2}^{\infty} a_{k,n-2} z^k = \frac{1}{2\pi} \int_0^{2\pi} P_r^{(n)}(\theta - t) g_n(e^{it}) \, dt.$$

Denoting this function $f_n(z)$ we have

$$f(z) = \sum_{n=2}^{\infty} f_n(z) = \sum_{n=2}^{\infty} \frac{1}{2\pi} \int_0^{2\pi} P_r^{(n)}(\theta - t) g_n(e^{it}) \, dt$$

provided (3.1) is satisfied. But (3.3) implies

$$\| g_n \| \leqq 2B(\varepsilon)\varepsilon^n/n!$$

for each $\varepsilon > 0$ and (3.1) is an immediate consequence.

If $a_0 \neq 0$ set $g_0(z) = a_0$ and include the integral for $n = 0$. This completes the proof of the theorem.

It is obvious from the process of decomposition that the representation is not unique. More directly this is evident from the possibility of integration by parts. If α is a function of bounded variation on the interval $[0, 2\pi]$

with $\alpha(0) = \alpha(2\pi)$, then

$$\int_0^{2\pi} P_r^{(n)}(\theta - t)\, d\alpha(t) = \int_0^{2\pi} P_r^{(n+1)}(\theta - t)\alpha(t)\, dt.$$

Thus it may be possible to replace an integral with one of higher order or one of lower order. It is even possible to write a Poisson integral as the sum of a nonterminating series of integrals.

The possibility of a canonical representation will be discussed in a following paper.

4. The space $\mathfrak{H}$

The distributions on the unit circle Γ are the continuous linear functionals on the space $\mathfrak{D}$ of infinitely differentiable functions on Γ provided with the topology of Schwartz [11]. In this topology a sequence $\varphi_\nu \to 0$ if and only if $\varphi_\nu^{(n)} \to 0$ uniformly on Γ for each n. A functional g is a distribution, i.e. $g \in \mathfrak{D}'$, if and only if it has a Fourier series $\sum_{0 \leq |k|} a_k e^{ik\theta}$ such that $|a_k| \leq B(|k|+ 1)^n$ for some constants B and n.

A harmonic function has a trace on Γ which is a distribution if and only if it has a terminating representation of the form

$$f(r, \theta) = \sum_{n=0}^{N} \frac{1}{2\pi} \int_0^{2\pi} P_r^{(n)}(\theta - t) g_n(t)\, dt. \tag{4.1}$$

In fact f has such a representation if and only if

$$f(r, \theta) = \sum\nolimits_{0 \leq |k|} a_k r^{|k|} e^{ik\theta}$$

with coefficient growth as above. (See (2.5) and (3.3)). Moreover

$$f_r \to \tilde{f} = \sum\nolimits_{0 \leq |k|} a_k e^{ik\theta}$$

in the topology of $\mathfrak{D}'$ as $r \nearrow 1$. The mapping $f \to \tilde{f}$ thus establishes a one to one correspondence between the two spaces. Using the convolution product in $\mathfrak{D}'$ this correspondence may be written as the Poisson representation

$$f_r = P_r * \tilde{f}$$

since $P_r(\theta) = \sum_{0 \leq |k|} r^{|k|} e^{ik\theta}$.

If one replaces $\mathfrak{D}$ by $\mathfrak{H}$ and $\mathfrak{D}'$ by $\mathfrak{H}'$ then the structure described above extends to the class $\mathfrak{F}$ of all harmonic functions. The topology of $\mathfrak{H}$ will be introduced in this section.

In terms of Fourier series developments the class $\mathfrak{H}$ may be defined by

$$\mathfrak{H} = \{\varphi : \varphi(\theta) = \sum\nolimits_{0 \leq |k|} c_k e^{ik\theta}, \limsup_{|k| \to \infty} |c_k|^{1/|k|} < 1\}. \tag{4.2}$$

$\mathfrak{H}$ is dense in $\mathfrak{D}$ and its topology will be finer than the relative topology.

A description of the topology will make use of a certain class of sequences.

$\mathfrak{A}$ will denote the class of sequences $\alpha = \{\alpha_\nu\}_{0 \leqq \nu}$ which satisfy

$$\alpha_\nu \geqq \alpha_{\nu+1} > 0 \quad \text{for all } \nu \tag{4.3}$$
$$\lim_{\nu \to \infty} \frac{\alpha_{\nu+1}}{\alpha_\nu} = 1.$$

The collection of sets

$$V(\alpha) = \{\varphi \in \mathfrak{K} : |c_k| \leqq \alpha_{|k|} \quad \text{for all} \quad k\}$$

for all $\alpha \in \mathfrak{A}$ is a base for the neighborhood system at the origin for the desired topology on $\mathfrak{K}$.

To facilitate a proof that $\mathfrak{K}$ is a linear topological space with this topology and the derivation of other properties we first obtain three facts concerning the sequences in $\mathfrak{A}$.

PROPOSITION 2. *If $\alpha \in \mathfrak{A}$, then for each $a > 0$ and $\rho < 1$ there exists ν_0 such that*

$$a\rho^\nu \leqq \alpha_\nu \quad \textit{for} \quad \nu \geqq \nu_0 . \tag{4.4}$$

If $\alpha \in \mathfrak{A}$ and $\rho < 1$, then there is an $a > 0$ such that

$$a\rho^\nu \leqq \alpha_\nu \quad \textit{for all} \quad \nu. \tag{4.5}$$

If $\{\gamma_\nu\}_{0 \leqq \nu}$ is a sequence with the property that for each $\alpha \in \mathfrak{A}$ there is a $b > 0$ such that $\gamma_\nu \leqq b\alpha_\nu$ for all ν, then there exists $a > 0$ and $\rho < 1$ such that

$$\gamma_\nu \leqq a\rho^\nu \quad \textit{for all} \quad \nu. \tag{4.6}$$

Proof. To prove (4.4) choose $\eta > 1$ such that $\rho\eta < 1$ and choose ν_1 such that $\alpha_{\nu+1}/\alpha_\nu \geqq \rho\eta$ *for* $\nu \geqq \nu_1$. Then

$$\alpha_\nu \geqq (\rho\eta)^{\nu - \nu_1} \alpha_{\nu_1} \quad \text{for} \quad \nu \geqq \nu_1$$

and

$$a\rho^\nu \leqq \alpha_\nu \frac{a(\rho\eta)^{\nu_1}}{\eta^\nu \alpha_{\nu_1}} .$$

One obtains (4.4) on choosing $\nu_0 \geqq \nu_1$ so that the multiplier of α_ν is smaller than 1. The property (4.5) is an immediate consequence of (4.4).

The validity of (4.6) is established indirectly. Suppose to the contrary that for each $a > 0$ and $\rho < 1$ there is a ν such that $\gamma_\nu > a\rho^\nu$. If this is so, then ν may be chosen as large as one wishes. For if $\gamma_\nu \leqq a\rho^\nu$ for $\nu \geqq \nu_0$, we may choose $c > 0$ such that $\gamma_\nu \leqq c\rho^\nu$ for $0 \leqq \nu < \nu_0$, and then $\gamma_\nu \leqq \max(a, c)\rho^\nu$ for all ν.

In particular let $\rho_n \nearrow 1$, and choose ν_1 for which $\gamma_{\nu_1} > \rho_1^{\nu_1}$. Now define a sequence $\{\nu_n\}$ inductively as follows. If $\nu_1, \cdots, \nu_{n-1}$ have been selected, choose ν_n to satisfy

$$\nu_n > \frac{n}{n-1} \nu_{n-1} \tag{4.7}$$

$$\gamma_{\nu_n} > n\rho_n^{\nu_n}. \tag{4.8}$$

Now let $\rho_n = \rho^{1/n}$ where $\rho < 1$ and define the sequence $\alpha = \{\alpha_\nu\}_{0 \leqq \nu}$ by

$$\begin{aligned} \alpha_\nu &= \rho^\nu, & 0 \leqq \nu < \nu_1, \\ &= \rho^* \text{ where } * = (\nu - \nu_n)\frac{\dfrac{\nu_{n+1}}{n+1} - \dfrac{\nu_n}{n}}{\nu_{n+1} - \nu_n} + \frac{\nu_n}{n}, & \nu_n \leqq \nu < \nu_{n+1}. \end{aligned} \tag{4.9}$$

Because of (4.7) the sequence is decreasing and $\alpha_{\nu+1}/\alpha_\nu > \rho^{1/n}$ for $\nu_n \leqq \nu < \nu_{n+1}$ which implies $\lim (\alpha_{\nu+1}/\alpha_\nu) = 1$.

The sequence $\alpha \in \mathfrak{A}$ and by hypothesis there is a $b > 0$ such that $\gamma_\nu \leqq b\alpha_\nu$ for all ν. But this is contradictory for we should have according to (4.8) and (4.9)

$$n\rho^{\nu_n/n} < \gamma_{\nu_n} \leqq b\alpha_{\nu_n} = b\rho^{\nu_n/n}$$

or $n < b$. The alternative is that there does exist $a > 0$ and $\rho < 1$ satisfying (4.6). This completes the proof of the proposition.

We are now in a position to prove the assertions of the following four propositions.

PROPOSITION 3. *The collection of sets $V(\alpha)$, $\alpha \in \mathfrak{A}$, determines a topology on $\mathfrak{H}$ with which $\mathfrak{H}$ is locally convex.*

Proof. The collection $V(\alpha)$ is a local base. For if $\alpha, \beta \in \mathfrak{A}$, there is a $\gamma \in \mathfrak{A}$ such that $V(\gamma) \subset V(\alpha) \cap V(\beta)$. In fact, the sequence $\gamma = \{\min (\alpha_\nu, \beta_\nu)\} \in \mathfrak{A}$ and $V(\gamma) = V(\alpha) \cap V(\beta)$.

To prove that a topology is determined which is compatible with the linear structure it is sufficient to show that each $V(\alpha)$ is circled and radial and that the collection forms a uniform structure.

Since $|\lambda c_k| \leqq |\lambda| \alpha_{|k|} \leqq \alpha_{|k|}$ for $|\lambda| \leqq 1$, $V(\alpha)$ is circled. If $\varphi \in \mathfrak{H}$, then $\limsup |c_k|^{1/|k|} < 1$. There is an $a > 0$ and a $\rho < 1$ such that $|c_k| \leqq a\rho^{|k|}$ for all k. Using (4.5) there is a $\lambda > 0$ such that $\lambda a \rho^{|k|} \leqq \alpha_{|k|}$ for all k. It follows that $\lambda\varphi \in V(\alpha)$ which means that $V(\alpha)$ absorbs φ. Therefore $V(\alpha)$ is radial. If $\alpha \in \mathfrak{A}$ and $\beta = \{\alpha_\nu/2\}$ then $V(\beta) + V(\beta) \subset V(\alpha)$.

The proof is completed on observing that each $V(\alpha)$ is convex.

PROPOSITION 4. *$\mathfrak{H}$ is complete.*

Proof. Let $\{\varphi_\mu\}$ be a Cauchy net in $\mathfrak{H}$. Then $\varphi_\mu - \varphi_\nu \in V(\alpha)$ for $\mu, \nu \geqq \mu(\alpha)$. Thus $|c_k^\mu - c_k^\nu| \leqq \alpha_{|k|}$ for $\mu, \nu \geqq \mu(\alpha)$ and for all k. This implies that $\{c_k^\mu\}$ is a Cauchy net having a limit c_k for each k.

We have $|c_k^\mu - c_k| \leqq \alpha_{|k|}$ for $\mu \geqq \mu(\alpha)$. For a particular μ the condition $\limsup |c_k^\mu|^{1/|k|} < 1$ implies the existence of $a > 0$ and $\rho < 1$ such that $|c_k^\mu| \leqq a\rho^{|k|}$ for all k. By (4.5) there is a $b > 0$ such that $a\rho^{|k|} \leqq b\alpha_{|k|}$ for all

k. Therefore

$$|c_k| \leqq |c_k^\mu| + \alpha_{|k|} \leqq (b+1)\alpha_{|k|}$$

for all k. But this happens for each $\alpha \in \mathfrak{A}$. Now applying (4.6), there is an $a > 0$ and a $\rho < 1$ such that $|c_k| \leqq a\rho^{|k|}$ for all k. Hence

$$\varphi(\theta) = \sum\nolimits_{0 \leqq |k|} c_k e^{ik\theta} \in \mathfrak{H}.$$

Since $\varphi_\mu - \varphi \in V(\alpha)$ for $\mu \geqq \mu(\alpha)$, the net converges to φ in $\mathfrak{H}$ proving the proposition.

PROPOSITION 5. *The collection of sets*

$$E(a, \rho) = \{\varphi \in \mathfrak{H} : |c_k| \leqq a\rho^{|k|} \quad \textit{for all} \quad k\}$$

for all $a > 0$ *and* $\rho < 1$ *is a fundamental system of bounded sets for* $\mathfrak{H}$. *Each* $E(a, \rho)$ *is compact.*

Proof. If $\alpha \in \mathfrak{A}$ choose $b > 0$ such that $a\rho^{|k|} \leqq b\alpha_{|k|}$ for all k. Then $(1/b)E(a, \rho) \subset V(\alpha)$. This proves that $E(a, \rho)$ is bounded.

Conversely, if A is a bounded set, then $\lambda A \subset V(\alpha)$, $\lambda = \lambda(\alpha) > 0$, for each $\alpha \in \mathfrak{A}$. We have

$$\gamma_k = \sup\{|c_k| : \varphi \in A\} \leqq (1/\lambda(\alpha))\alpha_{|k|}$$

for all k and each $\alpha \in \mathfrak{A}$. According to (4.6) there is a pair $a > 0$, $\rho < 1$ such that $\gamma_k \leqq a\rho^{|k|}$ for all k. Consequently $A \subset E(a, \rho)$.

The Tychonoff product theorem may be used to prove that $E(a, \rho)$ is compact. For each k, $D_k = \{|z| \leqq a\rho^{|k|}\}$ is a compact disc in the complex plane and $E(a, \rho)$ may be identified with the cartesian product $X_{0 \leqq |k|} D_k$. If the product topology is finer than the relative topology on $E(a, \rho)$ then $E(a, \rho)$ is compact.

Let $\varphi \in E(a, \rho)$ have Fourier coefficient sequence $\{c_k\}$ and consider its neighborhood

$$(\varphi + V(\alpha)) \cap E(a, \rho)$$
$$= \{\psi \in \mathfrak{H} : |d_k - c_k| \leqq \alpha_{|k|} \quad \text{and} \quad |d_k| \leqq a\rho^{|k|} \quad \text{for all} \quad k\}$$

where $\{d_k\}$ is the sequence of coefficients of ψ. Assuming only that $\varphi, \psi \in E(a, \rho)$ we have $|d_k - c_k| \leqq 2a\rho^{|k|}$ and using (4.4) there is a ν_0 such that $|d_k - c_k| \leqq \alpha_{|k|}$ for $|k| \geqq \nu_0$. Therefore

$$(\varphi + V(\alpha)) \cap E(a, \rho)$$
$$= \{\psi \in \mathfrak{H} : |d_k - c_k| \leqq \alpha_{|k|} \quad \text{for} \quad |k| < \nu_0, \ |d_k| \leqq a\rho^{|k|} \quad \text{for all} \quad k\}.$$

This however is a neighborhood of φ in the product topology proving that the product topology is finer. This completes the proof. It is easy to show in fact that the two topologies are the same.

PROPOSITION 6. *$\mathcal{H}$ is not metrizable.*

Proof. If $\mathcal{H}$ were metrizable there would be a countable collection $V(\alpha^n)$ constituting a local base at the origin. We will describe a sequence $\alpha \in \mathfrak{A}$ such that no $V(\alpha^n)$ is contained in $V(\alpha)$.

It is clear that $V(\alpha^n) \subset V(\alpha)$ if and only if $\alpha_\nu^n \leqq \alpha_\nu$ for all ν. Thus we seek α such that for each n, $\alpha_\nu < \alpha_\nu^n$ for some ν. This occurs if there is a sequence $\{\nu_n\}$ such that $\alpha_{\nu_n} < \alpha_{\nu_n}^n$.

Choose α_1 such that $0 < \alpha_1 < \alpha_1^1$ and $\nu_1 = 1$. Applying (4.4) to the sequences $\{(\frac{1}{2})^{\nu-\nu_1}\alpha_{\nu_1}\}$ and $\{\alpha_\nu^2\}$ choose ν_2 such that

$$(\tfrac{1}{2})^{\nu_2-\nu_1}\alpha_{\nu_1} < \alpha_{\nu_2}^2 .$$

Define $\alpha_\nu = (\frac{1}{2})^{\nu-\nu_1}\alpha_{\nu_1}$ for $\nu_1 < \nu \leqq \nu_2$. If $\nu_1, \cdots, \nu_{n-1}$ have been defined choose ν_n such that

$$\left(\frac{n-1}{n}\right)^{\nu_n-\nu_{n-1}} \alpha_{\nu_{n-1}} < \alpha_{\nu_n}^n$$

and define

$$\alpha_\nu = \left(\frac{n-1}{n}\right)^{\nu-\nu_{n-1}} \alpha_{\nu_{n-1}}$$

for $\nu_{n-1} < \nu \leqq \nu_n$. Then the sequence $\alpha = \{\alpha_\nu\}$ is decreasing and

$$\alpha_{\nu+1}/\alpha_\nu = (n-1)/n$$

for $\nu_{n-1} \leqq \nu < \nu_n$ so that $\lim (\alpha_{\nu+1}/\alpha_\nu) = 1$ and $\alpha \in \mathfrak{A}$.

Since $\alpha_{\nu_n} < \alpha_{\nu_n}^n$ we obtain the desired contradiction. Therefore a base at the origin is necessarily uncountable and $\mathcal{H}$ is not metrizable.

Before considering the dual space, let us note that the topology of $\mathcal{H}$ is finer than the $\mathfrak{D}$ topology on $\mathcal{H}$. The topology of $\mathfrak{D}$ is described by the increasing sequence of norms

$$p_n(\varphi) = \sup_{0 \leqq |k|} (|k| + 1)^n |c_k|, \qquad n = 0, 1, \cdots .$$

Observing that $\alpha = \{\varepsilon/(\nu+1)^n\} \in \mathfrak{A}$ we have $V(\alpha) = \{\varphi \in \mathcal{H} : p_n(\varphi) \leqq \varepsilon\}$ so that each $\mathfrak{D}$ neighborhood is also a neighborhood in $\mathcal{H}$.

One fact about $\mathcal{H}$ remains to be proved. The space is barrelled. We defer the verification until the next section where some properties of the dual space may be used. Since, according to Proposition 5, closed bounded subsets of $\mathcal{H}$ are compact, $\mathcal{H}$ is a Montel space. (See [1, p. 89]).

5. The Space $\mathcal{H}'$

We turn now to a study of the dual space $\mathcal{H}'$ of continuous linear functionals on $\mathcal{H}$. The duality $\langle\varphi, f\rangle$ on $\mathcal{H} \times \mathcal{H}'$ will be chosen to be linear on $\mathcal{H}$ and antilinear (complex conjugate of linear) on $\mathcal{H}'$. Thus

$$\langle \varphi, f_1 + f_2 \rangle = \langle \varphi, f_1 \rangle + \langle \varphi, f_2 \rangle \quad \text{and} \quad \langle \varphi, cf \rangle = \bar{c}\langle \varphi, f \rangle.$$

The Fourier coefficients of $f \in \mathcal{H}'$ are defined by $a_k = \overline{\langle e^{ik\theta}, f \rangle}$.

Our immediate object is to show that $\mathcal{H}'$may be identified with the space of sequences $\{a_k\}$ satisfying $\limsup_{|k|\to\infty} |a_k|^{1/|k|} \leqq 1$. In the correspondence $\{a_k\}$ is the sequence of Fourier coefficients of an element of $\mathcal{H}'$. A sequence satisfies the above condition if and only if there is finite-valued function B such that $|a_k| \leqq B(\eta)\eta^{|k|}$ for all k and all $\eta > 1$. However, for certain calculations we need two other characterizations of such sequences.

Let us first observe that one may limit attention to those functions B which are convex and decreasing (not necessarily strictly decreasing). For if

$$B_1(\eta) = \sup_{0\leqq|k|} |a_k| \eta^{-|k|} \leqq B(\eta) < \infty$$

for each $\eta > 1$, then B_1 has these properties and $|a_k| \leqq B_1(\eta)\eta^{|k|}$.

We define now a class $\mathcal{B}$ of sequences $\beta = \{\beta_\nu\}_{0\leqq\nu}$ which satisfy

$$(5.1) \qquad \begin{aligned} 0 < \beta_\nu \leqq \beta_{\nu+1} \quad &\text{for all} \quad \nu \\ \lim_{\nu\to\infty} \beta_{\nu+1}/\beta_\nu &= 1. \end{aligned}$$

It may be noted that $\mathcal{B}$ is the dual of the class $\mathcal{A}$ defined by (4.3) in the sense that $\{\beta_\nu\} \in \mathcal{B}$ if and only if $\{1/\beta_\nu\} \in \mathcal{A}$. As may be guessed, the class $\mathcal{B}$ plays a role in describing the bounded subsets of $\mathcal{H}'$ analogous to that of $\mathcal{A}$ in determining a base for the topology of $\mathcal{H}$.

PROPOSITION 7. *If $\beta \in \mathcal{B}$ there is a convex decreasing function B such that*

$$(5.2) \qquad \beta_\nu \leqq \inf_{\eta>1} B(\eta)\eta^\nu$$

for all $\nu \geqq 0$. If B is a convex decreasing function there is a $\beta \in \mathcal{B}$ such that

$$(5.3) \qquad \inf_{\eta>1} B(\eta)\eta^\nu \leqq \beta_\nu$$

for all $\nu \geqq 0$.

Proof. The first assertion can be proved readily but the second requires somewhat more effort. If $\beta \in \mathcal{B}$ and $\eta > 1$ there is, using (4.4), a ν_0 such that $\eta^{-\nu} \leqq 1/\beta_\nu$ for $\nu \geqq \nu_0$. This implies that $B(\eta) = \sup_{0\leqq\nu} \beta_\nu \eta^{-\nu}$ is finite. The function B satisfies (5.2).

Now let B be a strictly positive convex decreasing function on the interval $(1, \infty)$. Define B'_+ to be the derivative of B from the right at each point of $(1, \infty)$ and denote $f(\eta) = -\eta B'_+(\eta)/B(\eta)$. Then f is nonnegative, subuniformly bounded on $(1, \infty)$, continuous from the right on $(1, \infty)$, and continuous except at a countable set of points. If $1 < t_0 < \infty$ then

$$B(\eta) = B(t_0) \exp \int_\eta^{t_0} \frac{f(t)}{t}\, dt.$$

It will be helpful to replace B by a larger function B_1 for which the function f_1 corresponding to f is decreasing. The properties of f make it possible to choose an f_1 which is strictly decreasing, continuous, and greater than f on the interval $(1,\ t_0]$. We may as well assume that $\lim_{\eta \downarrow 1} f_1(\eta) = \infty$. Take f_1 to be zero on $(t_0,\ \infty)$. The function B_1 defined by

$$B_1(\eta) = B(t_0) \exp \int_\eta^{t_0} \frac{f_1(t)}{t}\, dt$$

dominates B and is convex decreasing on $(1,\ \infty)$ with minimum value $B_1(t_0)$.

The function B_1 also has the property that $B_1(\eta)\eta^\nu$ is convex for each $\nu > 0$ which will make it easy to find its minimum value. For $\nu > 0$

$$\begin{aligned}(B_1(\eta)\eta^\nu)' &= (-f_1(\eta) + \nu)\eta^{\nu-1}B_1(\eta), \quad 1 < \eta < t_0, \\ &= \nu\eta^{\nu-1}B_1(t_0), \qquad\qquad t_0 < \eta.\end{aligned}$$

If we let $\nu_0 = \min\{\nu : f_1(t_0) < \nu\}$ and define η_ν by $f_1(\eta_\nu) = \nu$ for $\nu \geqq \nu_0$, then $\eta_\nu \searrow 1$ and

$$\begin{aligned}\inf_{\eta>1} B_1(\eta)\eta^\nu &= B_1(t_0)t_0^\nu, \quad 0 \leqq \nu < \nu_0, \\ &= B_1(\eta_\nu)\eta_\nu^\nu, \quad \nu_0 \leqq \nu.\end{aligned}$$

A suitable estimate for $B_1(\eta_\nu)\eta_\nu^\nu$ will provide us with the desired sequence $\beta \in \mathcal{B}$. On the interval $[\eta_{\nu_0},\ t_0]$, $f_1(t) \leqq f_1(\eta_{\nu_0}) = \nu_0$ so that

$$B_1(\eta_{\nu_0}) = B_1(t_0) \exp \int_{\eta_{\nu_0}}^{t_0} \frac{f_1(t)}{t}\, dt \leqq B_1(t_0) \left(\frac{t_0}{\eta_{\nu_0}}\right)^{\nu_0}$$

or $B_1(\eta_{\nu_0})\eta_{\nu_0}^{\nu_0} \leqq B_1(t_0)t_0^{\nu_0}$. For each $\nu > \nu_0$, $f_1(t) \leqq f_1(\eta_\nu) = \nu$ on the interval $[\eta_\nu,\ \eta_{\nu-1}]$ implying

$$\begin{aligned}B_1(\eta_\nu) &= B_1(\eta_{\nu_0}) \exp \sum_{j=\nu_0+1}^{\nu} \int_{\eta_j}^{\eta_{j-1}} \frac{f_1(t)}{t}\, dt \\ &\leqq B_1(t_0)(t_0/\eta_{\nu_0})^{\nu_0} \textstyle\prod_{j=\nu_0+1}^{\nu} (\eta_{j-1}/\eta_j)^j\end{aligned}$$

Thus $B_1(\eta_\nu)\eta_\nu^\nu \leqq B_1(t_0)t_0^{\nu_0}\eta_{\nu_0} \cdot \cdots \cdot \eta_{\nu-1}$ for $\nu > \nu_0$.

The sequence β defined by

$$\beta_\nu = \begin{cases} B_1(t_0)t_0^\nu, & 0 \leqq \nu \leqq \nu_0, \\ B_1(t_0)t_0^{\nu_0}\eta_{\nu_0} \cdot \cdots \cdot \eta_{\nu-1}, & \nu_0 < \nu, \end{cases}$$

is in the class $\mathcal{B}$ and satisfies (5.3). This completes the proof of the proposition.

PROPOSITION 8. *The following conditions for a sequence* $\{a_k\}$ *are equivalent.*

(5.4) $\sum_{0 \leqq |k|} |c_k \bar{a}_k| < \infty$ *for all sequences* $\{c_k\}$ *such that*

$$\limsup_{|k| \to \infty} |c_k|^{1/|k|} < 1.$$

(5.5) $\sum_{0 \leqq |k|} c_k \bar{a}_k$ *is convergent for all sequences* $\{c_k\}$ *such that*

$$\limsup_{|k| \to \infty} |c_k|^{1/|k|} < 1.$$

(5.6) $\limsup_{|k| \to \infty} |a_k|^{1/|k|} \leqq 1.$

(5.7) *There is a finite valued function* B *such that* $|a_k| \leqq B(\eta)\eta^{|k|}$ *for all* k *and all* $\eta > 1$.

(5.8) *There is a sequence* $\beta \in \mathcal{B}$ *such that* $|a_k| \leqq \beta_{|k|}$ *for all* k.

Proof. The implications (5.4) $\Rightarrow$ (5.5) $\Rightarrow$ (5.6) $\Rightarrow$ (5.4) are either trivial or straightforward. The equivalence (5.6) $\Leftrightarrow$ (5.7) is immediate from the definitions and the equivalence (5.7) $\Leftrightarrow$ (5.8) is a consequence of Proposition 7.

As a side remark we mention that the equivalence of (5.8) with (5.6) leads to an apparently new elementary condition, in the form of a ratio test, for determining the radius of convergence for power series.

With these preliminary results in hand it is possible to establish the identification of $\mathcal{H}'$ with the space of sequences $\{a_k\}$ satisfying the conditions of Proposition 8 and at the same time to obtain the appropriate Parseval relation for the duality $\langle \varphi, f \rangle$.

THEOREM 2. *If* $f \in \mathcal{H}'$ *and* $\{a_k\}$ *is the sequence of Fourier coefficients of* f, *then* $\langle \varphi, f \rangle = \sum_{0 \leqq |k|} c_k \bar{a}_k$ *for all* $\varphi \in \mathcal{H}$ *where* $\{c_k\}$ *is the sequence of Fourier coefficients of* φ. *The series is absolutely convergent. Conversely, any sequence* $\{a_k\}$ *such that the series is convergent for all* $\varphi \in \mathcal{H}$ *is the sequence of Fourier coefficients of an* $f \in \mathcal{H}'$.

Proof. Consider first the case $f \in \mathcal{H}'$. If $\varphi \in \mathcal{H}$ there is an $a > 0$ and a $\rho < 1$ such that $|c_k| \leqq a\rho^{|k|}$ for all k. Set $\varphi_\nu = \sum_{0 \leqq |k| \leqq \nu} c_k e^{ik\theta}$. If $\alpha \in \mathcal{A}$ there is, according to (4.4), a ν_0 such that $a\rho^\nu \leqq \alpha_\nu$ for $\nu \geqq \nu_0$. This implies

$$\varphi - \varphi_\nu = \sum_{\nu < |k|} c_k e^{ik\theta} \in V(\alpha)$$

for $\nu \geqq \nu_0$. Consequently $\varphi_\nu \to \varphi$ in $\mathcal{H}$ and

$$\sum_{0 \leqq |k| \leqq \nu} c_k \bar{a}_k = \langle \varphi_\nu, f \rangle \to \langle \varphi, f \rangle.$$

For the converse define $\langle \varphi, f \rangle = \sum_{0 \leqq |k|} c_k \bar{a}_k$ for all $\varphi \in \mathcal{H}$. Then f is linear and condition (5.8) yields a $\beta \in \mathcal{B}$ such that $|a_k| \leqq \beta_{|k|}$ for all k. The sequence $\alpha = \{\varepsilon/(5\nu^2\beta_\nu)\} \in \mathcal{A}$ and $\varphi \in V(\alpha)$ implies

$$|\langle \varphi, f \rangle| \leqq \sum_{0 \leqq |k|} |c_k| \, |\bar{a}_k| \leqq \sum_{0 \leqq |k|} \alpha_{|k|} \beta_{|k|} \leqq \varepsilon.$$

Therefore f is continuous and the proof is completed on noting that $\overline{\langle e^{ik\theta}, f \rangle} = a_k$ for all k.

Having characterized the elements of $\mathcal{H}'$ we wish to describe convergence in this space. $\mathcal{H}'$ is to be provided with the strong topology. This is the topology of uniform convergence on the bounded subsets of $\mathcal{H}$. The first result below gives a simple description of a base for the topology of $\mathcal{H}'$ and the second result concerns the bounded subsets of $\mathcal{H}'$.

PROPOSITION 9. *The sets*

$$V'(\varepsilon, \eta) = \{f \in \mathcal{H}' : |a_k| \leqq \varepsilon\eta^{|k|} \quad \textit{for all} \quad k\}$$

for all $\varepsilon > 0$ *and* $\eta > 1$ *form a base for the neighborhood system at the origin.*

Proof. By the definition of the strong topology *a* base is formed by the sets

$$W'(a, \rho) = \{f \in \mathcal{H}' : |\langle\varphi, f\rangle| \leqq 1 \quad \text{for all} \quad \varphi \in E(a, \rho)\}$$

for all $a > 0$ and $\rho < 1$. That is, the polars of a fundamental system of bounded sets in $\mathcal{H}$. It will be shown that these bases are equivalent.

Given $a > 0$ and $\rho < 1$ choose $\eta > 1$ such that $\rho\eta < 1$ and $\varepsilon > 0$ such that $\varepsilon \leqq (1 + \rho\eta)/[a(1 - \rho\eta)]$. Then $f \in V'(\varepsilon, \eta)$ and $\varphi \in E(a, \rho)$ imply

$$|\langle\varphi, f\rangle| \leqq \textstyle\sum_{0 \leqq |k|} |c_k| \, |\bar{a}_k| \leqq a\varepsilon \sum_{0 \leqq |k|} (\rho\eta)^{|k|} \leqq 1.$$

Therefore $V'(\varepsilon, \eta) \subset W'(a, \rho)$.

On the other hand if $f \in W'(a, \rho)$ then $|\langle\varphi, f\rangle| \leqq 1$ for all $\varphi \in \mathcal{H}$ such that $|c_k| \leqq a\rho^{|k|}$. In particular if $c_{k_0} = a\rho^{|k_0|}$ and $c_k = 0$ for $k \neq k_0$ then $|a\rho^{|k_0|}\bar{a}_{k_0}| \leqq 1$. Therefore $|a_k| \leqq (1/a)(1/\rho)^{|k|}$ for all k. Therefore given $\varepsilon > 0$ and $\eta > 1$ choose $a = 1/\varepsilon$ and $\rho = 1/\eta$ to obtain $W'(a, \rho) \subset V'(\varepsilon, \eta)$. This proves the proposition.

PROPOSITION 10. *Each of the collections*

$$E'(B) = \{f \in \mathcal{H}' : |a_k| \leqq B(\eta)\eta^{|k|} \quad \text{for all} \quad k \quad \text{and all} \quad \eta > 1\}$$

for all convex decreasing functions B and

$$F'(\beta) = \{f \in \mathcal{H}' : |a_k| \leqq \beta_{|k|} \quad \text{for all} \quad k\}$$

for all $\beta \in \mathcal{B}$ *is a fundamental system of bounded sets for* $\mathcal{H}'$.

Proof. These two collections are equivalent as a consequence of Proposition 7 in the sense that each $E'(B)$ contains an $F'(\beta)$ and conversely. Therefore we need only prove that the collection of set $E'(B)$ is a fundamental system.

For each $\varepsilon > 0$ and $\eta > 1$, $(\varepsilon/B(\eta))E'(B) \subset V'(\varepsilon, \eta)$. This proves that $E'(B)$ is bounded. Conversely, if A is bounded then $\lambda A \subset V'(\varepsilon, \eta)$, $\lambda = \lambda(\varepsilon, \eta)$ for all $\varepsilon > 0$ and $\eta > 1$. In particular $|\lambda(1, \eta)a_k| \leqq \eta^{|k|}$ for all k and all $\eta > 1$ if $f \in A$. Therefore

$$B(\eta) = \sup\{|a_k|\eta^{-|k|} : \quad \text{for all} \quad f \in A \quad \text{and all} \quad k\} \leqq 1/\lambda(1, \eta) < \infty$$

The function B is convex decreasing and $A \subset E'(B)$. This establishes the proposition.

We may return now to the question left open at the end of Section 4. The space $\mathcal{H}$ is barrelled if each barrel in $\mathcal{H}$ is a neighborhood of the origin. A subset V of $\mathcal{H}$ is a barrel if it is closed, convex, circled and radial. Since V is closed, convex, and contains the origin it is the polar of its polar, $V = V^{00}$. The radial property for V is equivalent to the boundedness of V^0 relative to the weak topology on $\mathcal{H}'$ (pointwise boundedness).

But since $\mathcal{H}$ is complete the weakly bounded sets are also strongly bounded. (See [1, p. 86]). Therefore each barrel in $\mathcal{H}$ is the polar of a bounded subset of $\mathcal{H}'$. We need only show that the polar of a bounded subset of $\mathcal{H}'$ is a neighborhood of the origin in $\mathcal{H}$. It is sufficient to consider the fundamental sets $F'(\beta)$.

If $V = \{\varphi \in \mathcal{H} : |\langle \varphi, f\rangle| \leqq 1$ for all $f \in F'(\beta)\}$ for some $\beta \in \mathcal{B}$ define $\alpha = \{1/(5\nu^2\beta_\nu)\} \in \mathcal{A}$. Then $\varphi \in V(\alpha)$ and $f \in F'(\beta)$ implies

$$|\langle \varphi, f\rangle| \leqq \sum\nolimits_{0\leqq|k|} |c_k|\,|\bar{a}_k| \leqq \sum\nolimits_{0\leqq|k|} \alpha_{|k|}\,\beta_{|k|} \leqq 1.$$

Therefore $V(\alpha) \subset V$ and V is a neighborhood of zero.

We have therefore proved with the help of Proposition 5 that $\mathcal{H}$ is a Montel space. Each Montel space is reflexive, the strong dual of the strong dual is the original space, and its dual is a Montel space. Thus

THEOREM 3. *The spaces $\mathcal{H}$ and $\mathcal{H}'$ are Montel spaces. $\mathcal{H}$ is the strong dual of $\mathcal{H}'$.*

A number of properties of $\mathcal{H}'$ follow readily from the results already established. Since $\{e^{ik\theta}\}$ is total in $\mathcal{H}$ we conclude from the Banach-Steinhaus theorem that boundedness in $\mathcal{H}'$ and convergence of the Fourier coefficients implies strong convergence. In particular the Fourier series of each $f \in \mathcal{H}'$ converges to f in $\mathcal{H}'$. As remarked by Köthe, after Schwartz, this result suppresses further the distinction between Fourier series and trigonometric series which are not Fourier series.

$\mathcal{H}'$ is a Frechet space and contains $\mathcal{D}'$ as a dense subspace. The topology of $\mathcal{D}'$ is finer than the $\mathcal{H}'$ topology on $\mathcal{D}'$. Convolution may be defined by

$$f * g = \sum\nolimits_{0\leqq|k|} a_k\, b_k\, e^{ik\theta}$$

where $f = \sum_{0\leqq|k|} a_k\, e^{ik\theta}$ and $g = \sum_{0\leqq|k|} b_k\, e^{ik\theta}$. The mapping $f \rightarrow f * g$ is continuous for each $g \in \mathcal{H}'$.

Differentiation may be defined as in $\mathcal{D}'$ and it commutes with convolution. Antidifferentiation is possible if and only if $a_0 = \overline{\langle 1, f\rangle} = 0$. In the closed hyperplane $\mathcal{H}_0' = \{f \in \mathcal{H}' : \langle 1, f\rangle = 0\}$ antidifferentiation is always possible and is unique. The element $F \in \mathcal{H}_0'$ such that $F^{(n)} = f$ will be called the n-th integral of f.

The real elements of $\mathcal{H}'$ are those with Fourier coefficients satisfying $a_{-k} = \bar{a}_k$ and each positive f is a positive measure.

6. The Poisson Integral and a Structure Theorem

The generalized functions serve as a boundary space for the harmonic functions. A generalized function is called analytic if it has a Fourier series of the form $\sum_{0 \leqq k} a_k e^{ik\theta}$. The closed subspace of analytic generalized functions serve as a boundary space for the holomorphic functions. These correspondences are realized by means of the Poisson integral.

The Poisson kernel is defined for $0 \leqq r \leqq 1$ by $P_r = \sum_{0 \leqq |k|} r^{|k|} e^{ik\theta}$. In particular P_1 is the Dirac delta function. If $\beta_\nu = 1$ for $\nu \geqq 0$ then $\beta = \{\beta_\nu\} \in \mathcal{B}$ and $P_r \in F'(\beta)$. Since the coefficients are continuous functions of r and the family $\{P_r\}$ is bounded, P_r is a continuous mapping of [0, 1] into $\mathcal{H}'$. Hence $P_r * g$ is a continuous mapping for each $g \in \mathcal{H}'$.

THEOREM 4. *A function f is harmonic on $\{r < \rho\}$ if and only if there is a generalized function $\tilde{f}$ such that*

$$f_r = P_{r/\rho} * \tilde{f} \tag{6.1}$$

for each $r < \rho$. Moreover $f_r \to \tilde{f}$ in $\mathcal{H}'$ as $r \nearrow \rho$ and consequently $\tilde{f}$ is uniquely determined.

Proof. A function f is harmonic on $\{r < \rho\}$ if and only if

$$f(r, \theta) = \sum_{0 \leqq |k|} b_k r^{|k|} e^{ik\theta}$$

where $\limsup_{|k| \to \infty} | b_k |^{1/|k|} \leqq 1/\rho$. Denoting $a_k = b_k \rho^{|k|}$ and $\tilde{f} = \sum_{0 \leqq |k|} a_k e^{ik\theta}$ we have $\limsup_{|k| \to \infty} | a_k |^{1/|k|} \leqq 1$ and

$$f_r(\theta) = \sum_{0 \leqq |k|} a_k (r/\rho)^{|k|} e^{ik\theta}$$

for $r < \rho$. Therefore $\tilde{f} \in \mathcal{H}'$ and $f_r = P_{r/\rho} * \tilde{f}$.

The convergence $f_r \to P_1 * \tilde{f}$ as $r \nearrow \rho$ was established in the remarks preceding the theorem and $P_1 * \tilde{f} = \tilde{f}$. This completes the proof.

COROLLARY. *A function f is holomorphic on $\{r < \rho\}$ if and only if there is an analytic generalized function f such that* (6.1) *holds.*

At this point we may confirm that the topological spaces $\mathcal{H}$ and $\mathcal{H}'$ are the same as those defined by Lions and Magenes [8]. We have remarked earlier that $\mathcal{H}$ was chosen the same in both instances. Theorem 4 above and their Theorem 9.1 both establish linear isomorphisms of $\mathcal{H}'$ with $\mathfrak{F}$. Since the strong dual topology is used also by them for $\mathcal{H}'$ is sufficient to equate the topologies for $\mathcal{H}$. In their construction $\mathcal{H}$ is the limit of an increasing sequence of Banach spaces. The inductive limit topology is used and it is bornological (see [7, p. 406]). But this is the Mackey topology for the duality $\langle \mathcal{H}, \mathcal{H}' \rangle$. The same is true in the present paper because $\mathcal{H}$ is barrelled.

Denote by P the one to one linear mapping $\tilde{f} \to f$ of $\mathcal{H}'$ onto the space $\mathfrak{F}$ of harmonic functions on $\{r < 1\}$ defined by $f_r = P_r * \tilde{f}$. The isomorphism of $\mathcal{H}'$ and $\mathfrak{F}$ established by Lions and Magenes is realized by the mapping P.

THEOREM 5. *If $\mathfrak{F}$ is provided with the topology of subuniform convergence, then P is a topological isomorphism of $\mathcal{H}'$ onto $\mathfrak{F}$.*

Proof. Recall that a base at the origin for $\mathcal{H}'$ is formed by the sets

$$V'(\varepsilon, \eta) = \{\tilde{f} \in \mathcal{H}' : |a_k| \leqq \varepsilon\eta^{|k|} \quad \text{for all} \quad k\}$$

for all $\varepsilon > 0$ and $\eta > 1$. A base at the origin for $\mathfrak{F}$ is given by the sets

$$U(a, \rho) = \{f \subset \mathfrak{F} : |f(r, \theta)| \leqq a \quad \text{for} \quad r \leqq \rho\}$$

for all $a > 0$ and $\rho < 1$.

If $a > 0$ and $\rho < 1$ we have

$$V(\varepsilon, \eta) \subset P^{-1}U(a, \rho)$$

for $\varepsilon = a(1 - \rho)/(3 + \rho)$ and $\eta = (1 + \rho)/2\rho$. This proves that P is continuous. If $\varepsilon > 0$ and $\eta > 1$ set $a = \varepsilon$ and $\rho = 1/\eta$. Then $U(a, \rho) \subset PV'(\varepsilon, \eta)$ proving that P^{-1} is continuuos and completing the proof of the theorem.

We may remark that the bounded subsets of $\mathfrak{F}$ are the families of harmonic functions which are subuniformly bounded on $\{r < 1\}$. It is interesting to compare this description with that of the bounded subsets $F'(\beta)$ of $\mathcal{H}'$ in Proposition 10.

Theorem 1 leads to a structure theorem similar to Proposition 1.3 in [8] for generalized functions. To instroduce this idea we recall the structure theorem for distributions. If $\tilde{f} \in \mathfrak{D}'$ there is a finite sequence $\{g_0, \cdots, g_N\}$ of continuous functions on Γ such that $\tilde{f} = \sum_{n=0}^{N} g_n^{(n)}$. Using Theorem 4 this yields the representation (4.1) for those harmonic functions having a boundary function which is a distribution.

THEOREM 6. *If f is a generalized function there is a sequence $\{g_n\}_{n=0}^{\infty}$ of continuous functions on Γ such that*

$$\lim_{n\to\infty} (n! \, \| g_n \|)^{1/n} = 0 \tag{6.2}$$

$$f = \sum_{n=0}^{\infty} g_n^{(n)}. \tag{6.3}$$

Conversely, if $\{g_n\}$ is a sequence of continuous functions satisfying (6.2) *then* (6.3) *defines a generalized function f.*

Proof. It is convenient to prove first that (6.2) implies the convergence of $\sum_{0\leqq n} g_n^{(n)}$ in $\mathcal{H}'$. Condition (6.2) is equivalent the existence of a convex decreasing function B such that $n! \, \| g_n \| \leqq B(\varepsilon)\varepsilon^n$ for all $n \geqq 0$ and all $\varepsilon > 0$. Each $g_n^{(n)}$ is a generalized function and has a Fourier series. For $n > 0$, $g_n^{(n)} \in \mathcal{H}_0'$. Set $g_0 = \sum_{0\leqq|k|} a_k^0 e^{ik\theta}$ and $g_n^{(n)} = \sum_{0\leqq|k|} a_k^n e^{ik\theta}$. For $n > 0$ the

nth integral of $g_n^{(n)}$ will differ from g_n by a constant

$$a_0^n = (1/2\pi) \int_0^{2\pi} g_n(\theta)\, d\theta$$

which may as well be assumed to be zero. Then $g_n = \sum_{0<|k|} (ik)^{-n} a_k^n e^{ik\theta}$. Thus for all k and n and $\varepsilon > 0$

$$| (ik)^{-n} a_k^n | = \left| \frac{1}{2\pi} \int_0^{2\pi} e^{-ik\theta} g_n(\theta)\, d\theta \right| \leqq \|g_n\| \leqq \frac{B(\varepsilon)\varepsilon^n}{n!}$$

or $| a_k^n | \leqq B(\varepsilon)\varepsilon^n | k |^n / n!$.

Denoting $b_k^N = a_k^0 + \cdots + a_k^N$, $\eta = e^\varepsilon$, and $B_1(\eta) = B(\log \eta)$ we have $| b_k^N | \leqq B(\varepsilon)[1 + \varepsilon | k | + \cdots + (\varepsilon | k |)^N / N!] \leqq B(\varepsilon) e^{\varepsilon |k|} = B_1(\eta)\eta^{|k|}$ for all k and N and all $\eta > 1$. For each N

$$h_N = \sum_{0 \leqq n \leqq N} g_n^{(n)} = \sum_{0 \leqq |k|} b_k^N e^{ik\theta}$$

is an element of the bounded set $E'(B_1)$. For each k, the sequence $\{b_k^N\}$ has a limit b_k since it is the sequence of partial sums of a dominated series.

Therefore the sequence $\{h_n\}$ is bounded and has convergent Fourier coefficients. It converges in $\mathcal{H}'$ to a generalized function

$$f = \sum_{0 \leqq n} g_n^{(n)} = \sum_{0 \leqq |k|} b_k e^{ik\theta}.$$

Now let $\tilde{f} \in \mathcal{H}'$ and define $f_r = P_r * \tilde{f}$. By Theorem 1 there is a sequence $\{g_n\}$ of continuous functions satisfying (6.2) such that

$$f_r(\theta) = \sum_{n=0}^{\infty} (1/2\pi) \int_0^{2\pi} P_r^{(n)}(\theta - t) g_n(t)\, dt.$$

Thus

$$P_r * \tilde{f} = f_r = \sum_{n=0}^{\infty} P_r^{(n)} * g_n = \sum_{n=0}^{\infty} P_r * g_n^{(n)} = P_r * \sum_{n=0}^{\infty} g_n^{(n)}$$

for all $r < 1$. The last equality makes use of the first part of this proof. Applying the uniqueness part of Theorem 4 we find $\tilde{f} = \sum_{n=0}^{\infty} g_n^{(n)}$.

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