FOURIER-STIELTJES TRANSFORMS ON THE GENERALIZED LORENTZ GROUP

BY

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1. Introduction

The purpose of this note is to define the Fourier-Stieltjes transform and prove a uniqueness theorem for certain subalgebras of the measure algebra $M(G)$ of the generalized Lorentz group G. For an arbitrary semi-simple Lie group G with finite center such a definition was given in [1] for the algebra of measures stable for the action of K, K the compact constituent of the Iwasawa decomposition of G. In the formulation given below this algebra corresponds to $M^0(\chi)$, χ the trivial character of K. On the other hand, we have limited ourselves to the generalized Lorentz group G since various aspects of the harmonic analysis on this group needed for the definitions are well known [5]. The main result in this paper is the fact that the Fourier-Stieltjes transform $\hat{\mu}$ of a measure μ determines μ, that is, $\hat{\mu} = 0$ implies $\mu = 0$. This result was obtained in [1, P. 218] for the algebra $M^0(\chi)$, χ the trivial character of K. Our proof is similar to the one in [1] (cf. also [3, P. 680] where the same technique is employed in a different setting). For the convenience of the reader, we have gathered the necessary prerequisite material from [5] in a preliminary section.

2. Preliminaries

(A) Definition of the group G. Let G be the identity component of the orthogonal group associated with the indefinite quadratic form

$$-X_0^2 + X_1^2 + \cdots + X_n^2 \qquad (n \text{ an integer} \geq 2).$$

G is a real simple Lie group called the generalized Lorentz group. Hence G consists of all matrices $g \in GL\ (n+1, R)$ such that $tg \cdot J \cdot g = J$ ("t" = transpose) where

$$J = \begin{bmatrix} -1 & & & & 0 \\ & 1 & & & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & & & & 1 \end{bmatrix}$$

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and

$$g = \begin{bmatrix} g_{00} & g_{01} & \cdots & g_{0n} \\ g_{10} & g_{11} & \cdots & g_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ g_{n0} & g_{n1} & \cdots & g_{nn} \end{bmatrix}$$

with $g_{00} \geq 1$, $\det(g) = 1$. G admits an Iwasawa decomposition, $G = KA_+N$, where K is the maximal compact subgroup of rotations around the x_0 — axis, A_+ is a one-parameter subgroup of matrices of the form

$$a_t = \begin{bmatrix} \cosh t & \sinh t & 0 \\ \sinh t & \cosh t & \\ & 0 & I_{n-1} \end{bmatrix} \qquad (t \in R)$$

(I_{n-1} denoting the unit matrix of order $n - 1$), and N is a nilpotent group homeomorphic to R^{n-1}. Let M denote the centralizer of A_+ in K; then M may be identified with the rotations in the space $(X_2, X_3, \cdots, X_n)$ leaving fixed X_0 and X_1[5, P. 300]. If the Haar measure dg on G is suitably normalized, one has

$$\int_G f(g)\, dg = \int_K \int_R \int_N f(ka_t\, x)e^{(n-1)t}\, dk\, dt\, dx$$

where $g = k\, a_t x (k \in K, a_t \in A_+, x \in N)$, dk is the Haar measure on K of mass 1 and dt, dx are the Euclidean measures in R, resp. R^{n-1} ([5, P. 299]).

(B) The algebras $L^0(\chi)$. Denote by $\hat{K}$ the set of irreducible characters χ of K normalized in such a way that one has

$$\chi(k) = \chi * \chi(k) = \int_K \chi(kl^{-1})\chi(l)\, dl$$

("$*$" is convolution product). Hence for each $\chi \in \hat{K}$ there exists an irreducible unitary representation π in a unitary space E of finite dimension $d(\chi)$ such that

$$\chi(k) = d(\chi)\cdot \mathrm{Tr}\,(\pi(k)) \qquad (k \in K).$$

Similar normalizations and notations will be used for the set $\hat{M}$ of irreducible characters η of M.

Let $L = L(G)$ denote the algebra of continuous complex-valued functions with compact support on G, with convolution product being the multiplication. The subset $L^0(\chi)$ of L consisting of all functions $f \in L$ such that

(i) $\chi * f * \chi = f$

(ii) $f^0 = f$, where $f^0(g) = \int_K f(kgk^{-1})\, dk$

is a subalgebra of L and the map $f \to f^0 * \chi = \chi * f^0$ is a projection of L onto $L^0(\chi)$. Given $f \in L^0(\chi)$, define the Abel transform F_f of f by

$$F_f(t) = e^{(n-1)t/2} \int_K \int_N f(ka_t\, x)\pi(k^{-1})\, dk\, dx.$$

Hence F_f is a map from the real line R to the algebra of linear operators in the representation space E of π. Among other things, it is proved in [5, P. 309] that

(i) $L^0(\chi)$ is a commutative algebra for every $\chi \in \hat{K}$;
(ii) if $F_f(t) \equiv 0$, then $f(g) = 0$.

In addition, as a consequence of the fact that the restriction to M of every irreducible unitary representation of K decomposes into a direct sum of pairwise inequivalent irreducible representations of M, one is able to choose an orthonormal basis $(e_p : 1 \le p \le d(\chi))$ in E such that if

$$\pi(k)e_p = \sum_{q=1}^{d(\chi)} e_{qp}(k)e_q \qquad (k \in K),$$

then one has for $m \in M$,

$$(e_{pq}(m)) = \begin{bmatrix} f^1(m) & & & & & & \\ & \cdot & & & & & \\ & & \cdot & & & 0 & \\ & & & \cdot\ f^i(m) & & & \\ & 0 & & & \cdot & & \\ & & & & & \cdot & \\ & & & & & & \cdot\ f^\mu(m) \end{bmatrix},$$

where, for $1 \le j \le \mu$, $m \to f^j(m)$ is an irreducible representation of M, of dimension r_j, with $d(\chi) = r_1 + \cdots + r_\mu$. Let I_j be the set of integers p such that

$$r_1 + \cdots + r_{j-1} < p \le r_1 + \cdots + r_j$$

and let $\eta^j(m) = r_j \cdot \mathrm{Tr}(f^j(m))$, $m \in M$. In [5, P. 312] it is proved that with respect to the basis (e_p) the matrix of $F_f(t)$ assumes diagonal form with

$$F_f(t)_{pp} = e^{(n-1)t/2} \int_K \int_N f(ka_t\, x)\, \frac{\overline{\chi * \eta^j(k)}}{\chi * \eta^j(e)}\, dk\, dx$$

for $p \in I_j$(e is the identity element in G).

(C) Spherical functions of type χ. Fix a character χ and choose $\eta \in \hat{M}$ such that $\chi * \eta \ne 0$ (see (**B**)). Let s be a complex number and put

$$\alpha_{\chi,\eta,s}(g) = \frac{\overline{\chi * \eta(k)}}{\chi * \eta(e)}\, e^{-st}$$

if $g = ka_t x$, $k \in K$, $t \in R$, $x \in N$. Let

$$\zeta_{\chi,\eta,s}(g) = (\alpha_{\chi,\eta,s})^0(g) = \int_K \alpha_{\chi,\eta,s}(kgk^{-1})\, dk.$$

Then the functions $\zeta_{\chi,\eta,s}$ are spherical functions in the sense of Godement [2] and Takahashi [5, P. 315] proved

(i) if $\mathrm{Re}(s) = (n-1)/2$, $\zeta_{\chi,\eta,s}$ is positive definite;

(ii) for all $g_1, g_2 \in G$, one has

$$\int_K \zeta_{\chi,\eta,s}(kg_1k^{-1}g_2)\,dk = \zeta_{\chi,\eta,s}(g_1)\zeta_{\chi,\eta,s}(g_2);$$

(iii) the map

$$f \to \zeta_{\chi,\eta,s}(f) = \int_G f(g)\zeta_{\chi,\eta,s}(g)\,dg$$

is a homomorphism of $L^0(\chi)$ into C; C = complex numbers.

3. Fourier-Stieltjes transforms

(A) The algebra $M^\circ(\chi)$. The symbol $M(G)$ will stand for the algebra (under convolution $*$) of all complex regular Borel measures on G with compact support. $M(G)$ is a normed algebra under the norm

$$\| \mu \| = \int_G d\,|\mu|\,(g)$$

($|\mu|$ being the total variation of μ). Measures ν on K (i.e. elements of $M(K)$) will be identified with elements in $M(G)$ by

$$f \to \int_K f(k)\,d\nu(k) \qquad (f \in L(G)).$$

Similarly elements in $L(G)$ will sometimes be identified with $f\,dg$ in $M(G)$. If $\mu \in M(G)$, then μ^0 can be defined by the "weak" integral:

$$\mu^0 = \int_K \varepsilon_k * \mu * (\varepsilon_{k^{-1}})\,dk$$

(ε_k being the unit mass at k). One has

$$(\mu_1^0 * \mu_2)^0 = (\mu_1 * \mu_2^0)^0 = \mu_1^0 * \mu_2^0 .$$

Definition. $M^0(\chi)$ will consist of all measures $\mu \in M(G)$ such that (i) $\mu = \mu^0$; (ii) $\mu = \chi * \mu * \chi$.

Evidently $M^0(\chi)$ is a subalgebra of $M(G)$ and in order to determine $\mu(f)$ ($f \in L(G)$) it is enough to know $\mu(f)$ for $f \in L^0(\chi)$.

Lemma 1. *$M^0(\chi)$ is a commutative algebra.*

Proof. This follows at once from the fact that $L^0(\chi)$ is commutative and weakly dense in $M^0(\chi)$.

Definition. Let μ be a measure in $M^0(\chi)$. Let $r \in R$, $\eta \in \hat{M}$ such that $\chi * \eta \neq 0$, and put $s = (n-1)/2 + \sqrt{-1}\,r$. Then the Fourier-Stieltjes transform $\hat{\mu}$ is defined by

$$\hat{\mu}(r, \eta) = \int_G \zeta_{\chi,\eta,s}(g)\, d\mu(g).$$

Thus $\hat{\mu}$ is a map from the Cartesian product of the line R with the finite set of characters η such that $\chi * \eta \neq 0$. Since $\mathrm{Re}(s) = (n - 1)/2$, $\zeta_{\chi,\eta,s}$ is positive definite and so $|\hat{\mu}(r, \eta)| \leq \|\mu\|$. In addition, the usual argument employing the regularity of μ shows that if η is fixed and $r_j \to r_0$, then

$$\hat{\mu}(r_j, \eta) \to \hat{\mu}(r_0, \eta).$$

LEMMA 2. *If $\sigma = \mu * \nu$, then $\hat{\sigma} = \hat{\mu}\cdot\hat{\nu}$. Hence the map $\mu \to \hat{\mu}(r, \eta)$ is, for each (r, η), a complex homomorphism of $M^0(\chi)$.*

Proof. The proof depends on the functional equation satisfied by the $\zeta_{\chi,\eta,s}$ (see 2, part (C)). We have

$$\begin{aligned}\hat{\sigma}(r, \eta) &= (\mu * \nu)^\wedge\,(r, \eta)\\ &= \int_G \int_G \zeta_{\chi,\eta,s}\,(g_1 g_2)\, d\mu\,(g_1)\, d\nu\,(g_2).\end{aligned}$$

And, since $\mu = \mu^0$,

$$\begin{aligned}\int_G \zeta_{\chi,\eta,s}\,(g_1 g_2)\, d\mu\,(g_1) &= \int_G \zeta^0_{\chi,\eta,s}\,(g_1 g_2)\, d\mu\,(g_1)\\ &= \int_G \int_K \zeta_{\chi,\eta,s}\,(kg_1 k^{-1} g_2)\, d\mu\,(g_1)\\ &= \int_G \zeta_{\chi,\eta,s}\,(g_1)\zeta_{\chi,\eta,s}\,(g_2)\, d\mu\,(g_1).\end{aligned}$$

The assertion is now clear.

Next we prove that $\hat{\mu}$ determines μ.

THEOREM 1. *Suppose $\mu_1, \mu_2 \in M^0(\chi)$ and $\hat{\mu}_1 = \hat{\mu}_2$. Then $\mu_1 = \mu_2$.*

Proof. It is plainly enough to prove that $\hat{\mu} = 0$ implies $\mu = 0$ Suppose first that μ is absolutely continuous with respect to dg, that is, $d\mu = f\, dg$ with $f \in L^0(\chi)$. We have

$$\begin{aligned}\hat{\mu}(r, \eta) &= \int_G \zeta_{\chi,\eta,s}\,(g)\, d\mu\,(g)\\ &= \int_G \zeta_{\chi,\eta,s}\,(g) f(g)\, dg\\ &= \int_K \int_R \int_N f(ka_t\, x)\, \frac{\overline{(\chi * \eta)(k)}}{\chi * \eta(e)}\, e^{-st} e^{(n-1)t}\, dk\, dt\, dx\\ &= \int_R \left\{ e^{\frac{(n - 1)t}{2}} \int_K \int_N f(ka_t\, x)\, \frac{\overline{(\chi * \eta)(k)}}{\chi * \eta(e)}\, dk\, dx \right\} e^{-\sqrt{-1}rt}\, dt\\ &= \int_R F_f(t)_{pp}\, e^{-\sqrt{-1}rt}\, dt.\end{aligned}$$

Here p is any element in the set I_j determined by η (see 2, part (**B**)). In [5, P. 309] it is shown that $F_f(t)$ is a continuous function of t with compact support and hence for each p, $F_f(t)_{pp} \in L^1(dt)$. Moreover the above calculation shows that for $p \in I_j$ $\hat{\mu}(r, \eta)$ is just the Fourier transform of $E_f(t)_{pp}$ and since $\hat{\mu}(r, \eta) = 0$ we must have $F_f(t)_{pp} = 0$. Letting η range over the set of characters such that $\chi * \eta \not\equiv 0$, we conclude $F_f(t) \equiv 0$ which in turn implies $f = 0$ (2, part (**B**)). Hence $\mu = 0$.

In order to complete the proof, let f_j be an approximate identity in $L(G)$, that is, f_j is a sequence of functions in $L(G)$ such that

(i) $f_j \geq 0, j = 1, 2, \cdots$;
(ii) $\int_G f_j(g)\, dg = 1, j = 1, = ,\cdots$;
(iii) if C is any compact subset of G containing e, then $\int_{G-C} f(g)\, dg \rightarrow 0$ as $j \rightarrow \infty$.

Let $\nu_j = \chi * f_j^0$. Then the arguments of the preceding paragraph imply $\mu * \nu_j = 0$ since

$$(\mu * \nu_j)^\wedge = \hat{\mu}\hat{\nu}_j = 0$$

and ν_j is absolutely continuous with respect to dg. On the other hand, given any $f \in L^0(\chi)$ we have $f_j * f \rightarrow f$ uniformly on compacta and so

$$\mu * \nu_j(f) = \mu * \chi * f_j^0(f) = \mu(\chi * f_j^0 * f) = \mu(\chi * (f_j * f)^0) \rightarrow \mu(\chi * f^0) = \mu(f).$$

But since $\mu * \nu_j = 0$ we must have $\mu = 0$ too. This completes the proof.

Remark. The algebra $M(G)$ admits a natural adjoint map $\mu \rightarrow \mu^*$ under which each algebra $M^0(\chi)$ is stable. One may view each algebra $M^0(\chi)$ as a set of measures possessing certain symmetry properties. It would be of interest to know whether the algebras $M^0(\chi)$ are symmetric in the technical sense (cf. [4, P. 104]).

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