ENVELOPES AND p-SIGNALIZERS OF FINITE GROUPS

BY

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The object of this paper is to obtain information about p-signalizers of finite groups, more particularly, to construct a characteristic subgroup which contains all the p-signalizers and is "small". As an application of the result obtained, and of independent interest, we obtain some information about groups whose c.f. are Suzuki groups.

If X is a group and p is a prime, $\mathbf{S}_p(X)$ is the join of all the normal p-solvable subgroups of X. Thus, $\mathbf{S}_2(X) = \mathbf{S}(X)$ is the largest solvable normal subgroup of X. The *socle* of X is $\mathbf{Soc}(X)$, the join of all the minimal normal subgroups of X; and $\mathbf{Soc}_p(X) = \mathbf{Soc}(X \bmod \mathbf{S}_p(X))$. Finally, $\mathbf{ESoc}(X)$ is the *extended socle* of X and is defined by $\mathbf{ESoc}(X) = \bigcap \mathbf{N}_X(M)$, where M ranges over all subgroups of $\mathbf{Soc}(X)$ which are subnormal in X;[1] and $\mathbf{ESoc}_p(X) = \mathbf{ESoc}(X \bmod \mathbf{S}_p(X))$.

If $1 = \mathbf{S}(X)$, then $\mathbf{Soc}(X) = S_1 \times \cdots \times S_n$, where each S_i is simple, $1 = \mathbf{C}_X(\mathbf{Soc}(X))$ and $\mathbf{ESoc}(X) = \bigcap_{i=1}^{n} \mathbf{N}_X(S_i)$.

Lemma 1. (a) *Suppose* $\mathbf{Soc}_p(G) \subseteq H \subseteq G$. *Then*

(i) $\mathbf{S}_p(G) = \mathbf{S}_p(H)$.
(ii) $\mathbf{Soc}_p(G) = \mathbf{Soc}_p(H)$.
(iii) $\mathbf{ESoc}_p(H) \subseteq \mathbf{ESoc}_p(G)$.

(b) *If* $H \lhd G$, *then*

(i) $\mathbf{S}_p(H) = H \cap \mathbf{S}_p(G)$.
(ii) $\mathbf{Soc}_p(H) \subseteq \mathbf{Soc}_p(G)$.
(iii) $\mathbf{ESoc}(H) \subseteq \mathbf{ESoc}_p(G)$.

Proof. (a) Since $\mathbf{S}_p(G)$ is a p-solvable normal subgroup of H, we have $\mathbf{S}_p(G) \subseteq \mathbf{S}_p(H)$. Since $\mathbf{Soc}_p(G) \cap \mathbf{S}_p(H)$ is a p-solvable normal subgroup of $\mathbf{Soc}_p(G)$, we have $\mathbf{Soc}_p(G) \cap \mathbf{S}_p(H) \subseteq \mathbf{S}_p(G)$.
Thus

$$[\mathbf{Soc}_p(G),\ \mathbf{S}_p(H)] \subseteq \mathbf{Soc}_p(G) \cap \mathbf{S}_p(H) \subseteq \mathbf{S}_p(G),$$

so $\mathbf{S}_p(H)$ centralizes $\mathbf{Soc}_p(G)/\mathbf{S}_p(G)$, whence $\mathbf{S}_p(H) \subseteq \mathbf{S}_p(G)$. This gives (i). In proving (ii), we may assume that $\mathbf{S}_p(G) = \mathbf{S}_p(H) = 1$. Thus, $\mathbf{Soc}_p(G) \lhd \mathbf{Soc}_p(H)$, and so $\mathbf{Soc}_p(G) = \mathbf{Soc}_p(H)$, since $\mathbf{C}_G(\mathbf{Soc}_p(G)) = 1$. This yields (ii). Clearly, $\mathbf{ESoc}_p(H) = \mathbf{ESoc}_p(G) \cap H$, so (iii) holds.

As for (b), since every characteristic subgroup of H is normal in G, (i), (ii), (iii) follow.

For each group G, let $\mathfrak{A}(G) = \{A \mid A \lhd G,\ \mathbf{C}_G(A) = \mathbf{Z}(A)\}$.

THEOREM 1. *Suppose p is a prime, P is a S_p-subgroup of G, $A \in \mathfrak{A}(P)$ and $Q \in \mathsf{M}_G(A; p')$. Then $Q \subseteq \mathbf{ESoc}_p(G_0)$, where $G_0 = \mathbf{C}_G(\mathbf{O}_{p',p}(G) \bmod \mathbf{O}_{p'}(G))$.*

[1] Observe that if $Y \subseteq \mathbf{Soc}(X)$ then $Y \lhd\lhd X$ if and only if $Y \lhd \mathbf{Soc}(X)$.

Proof. We proceed by lexicographic induction on the ordered triple $(|G|, |P:A|, |Q|)$, and by way of contradiction. The induction hypothesis gives

$$\mathbf{O}_{p'}(G) = 1. \tag{1}$$

Let $H = \mathbf{O}_p(G) \subseteq P$ and let $H_1 = [H, Q]$. Suppose $H_1 \neq 1$. Let $Q_0 = \mathbf{C}_Q(A)$. Since $AH \subseteq P$, it follows that $\mathbf{C}_H(A) \subseteq \mathbf{Z}(A)$. Hence, Q_0 centralizes $\mathbf{C}_{H_1}(A)$, and so Q_0 centralizes H_1. Let $Q_1 = [Q, A]$. We will show that Q_1 centralizes H. This follows from the induction hypothesis if $Q_1 \subset Q$, so suppose $Q_1 = Q$. In any case, $[H, A] \subseteq A$, and so $[H, A, Q] \subseteq H \cap Q = 1$. This violates Lemma 5.16 of [3]. We conclude that

$$Q \subseteq \mathbf{C}_G(H) = G_0. \tag{2}$$

Our induction hypothesis, together with Lemma 1, gives

$$G = G_0 A. \tag{3}$$

Let $\tilde{A} = A \cdot \mathbf{O}_p(G)$, so that $\tilde{A} \in \mathfrak{A}(P)$. Since $Q \in \mathsf{N}(\tilde{A}; p')$, our induction hypothesis gives $A = \tilde{A}$.

Let $G_1 = \mathbf{Soc}_p(G_0)$. Now $\mathbf{S}_p(G_0) = \mathbf{Z}(G_0)$ is a p-group and $G_1/\mathbf{Z}(G_0) = S_1 \times \cdots \times S_m$, where each S_i is a simple non abelian group of order divisible by p. Since $Q \not\subseteq \mathbf{ESoc}_p(G_0)$, we may assume that Q does not normalize S_1.

By our induction hypothesis, every proper subgroup of Q which admits A normalizes S_1. Hence, Q is a q-group for some prime $q \neq p$, and $Q \cap \mathbf{N}_G(S_1) = Q_1 \supseteq \mathbf{D}(Q)$, while $Q/\mathbf{D}(Q)$ is an irreducible A-group.

Let $S_i = L_i/\mathbf{Z}(G_0)$, and let G_2 be the normal closure of L_1 in G. Thus, $Q \not\subseteq \mathbf{ESoc}_p(G_2 QA)$, and so

$$G = G_2 QA, \quad G_1 = G_2, \quad L_1 \simeq L_i, \qquad i = 1, 2, \cdots, m. \tag{4}$$

Let $P_i = P \cap L_i$. Since $L_i \lhd \lhd G$, we see that

$$P_i \text{ is an } S_p\text{-subgroup of } L_i, \qquad i = 1, \cdots, m. \tag{5}$$

We proceed to show that

$$A \text{ normalizes } L_i, \qquad i = 1, \cdots, m. \tag{6}$$

Suppose false. Choose $a \in A$ such that $L_i^a = L_j$ with $j \neq i$. Thus, for each $x \in P_i$, $x^a \in L_j$, so

$$y = x^{-1} \cdot x^a = [x, a] \in L_i L_j \cap A.$$

For each $z \in Q$, we get $[y, z] \in Q \cap G_1 = \mathbf{D}(Q)$, so y centralizes $Q/\mathbf{D}(Q)$. Hence $[y, z] = 1$. Hence Q normalizes $L_i L_j$, and so $Q \cap \mathbf{N}(L_i)$ is of index at most 2 in Q. First, suppose q is odd. Then Q normalizes L_i, so Q normalizes L_1, as QA permutes $\{L_1, \cdots, L_m\}$ transitively. This is not the case, so $|Q:Q \cap \mathbf{N}(L_i)| = 2$. Thus, $\{L_i, L_j\}$ is permuted transitively by Q, so we may assume that

$$\{L_1, L_2\}, \{L_3, L_4\}, \cdots, \{L_{2n-1}, L_{2n}\}$$

are the orbits of $\{L_1, \cdots, L_m\}$ under Q, $2n = m$, and that $i = 1, j = 2$. On the other hand, a is an arbitrary element of A which does not normalize L_1, and so we conclude that A normalizes $L_1 L_2$. Hence $m = 2, n = 1$, and so A normalizes L_1 and L_2, p being odd. We conclude that (6) holds.

Since $A \subseteq \bigcap_{i=1}^m \mathbf{N}(L_i)$, so also $[A, Q] \subseteq \bigcap_{i=1}^m \mathbf{N}(L_i)$. This implies that $[A, Q] \subseteq \mathbf{D}(Q)$, and so

$$AQ = A \times Q. \tag{7}$$

Since P is a S_p-subgroup of $\mathbf{N}_G(A)$, $P \cap \mathbf{C}_G(A) = \mathbf{Z}(A)$ is a S_p-subgroup of $\mathbf{C}_G(A)$. Hence $\mathbf{C}_G(A) = \mathbf{Z}(A) \times D$, where $D = \mathbf{O}_{p'}(\mathbf{C}_G(A)) \supseteq Q$. By induction, we get $A = P$.

Since $A = P$, it follows that Q centralizes P_i. Hence, for each $x \in Q$, $L_i \cap L_i^x \supseteq P_i$, and so $L_i \cap L_i^x \supset \mathbf{Z}(G_1)$. This implies that $L_i = L_i^x$, for all $x \in Q$, $i = 1, \cdots, m$, and completes the proof.

For each group G, let $\mathcal{E}(G)$, the *envelope* of G, be the class of all groups H which have a normal subgroup K such that

(1) $K/\mathbf{Z}(K) \simeq G$.

(2) $\mathbf{C}_H(K/\mathbf{Z}(K)) = \mathbf{Z}(K)$.

For each prime p, let $\mathcal{S}_p$ be the class of all groups G such that if $H \in \mathcal{E}(G)$, H_p is an S_p-subgroup of H and $A \in \mathfrak{A}(H_p)$, then $\mathbf{O}_{p'}(H)$ contains every element of $\text{И}_H(A; p')$. Let $\tilde{\mathcal{S}}_p$ be the class of all groups G such that if $H \in \mathcal{E}(G)$ and U is a p-subgroup of H, then $\mathbf{O}_{p'}(\mathbf{C}_H(U)) \subseteq \mathbf{O}_{p'}(H)$.

LEMMA 2. *If $q = 2^n > 2$ is an odd power of 2, then $Sz(q) \in \mathcal{S}_2 \cap \tilde{\mathcal{S}}_2$.*

Proof. Choose $H \in \mathcal{E}(G)$, where $G = Sz(q)$. In proving the lemma, we may assume that $\mathbf{O}_{2'}(H) = 1$. Let K be a normal subgroup of H such that $K/\mathbf{Z}(K) \simeq G$ and $\mathbf{C}_H(K/\mathbf{Z}(K)) = \mathbf{Z}(K)$. Let P be a S_2-subgroup of H. As is well known [2], $|H:K|$ is odd, so $P \subseteq K$.

Case 1. $q > 8$. By a result of Alperin and Gorenstein [1], $K = K' \times \mathbf{Z}(K)$. Thus, for each $A \in \mathfrak{A}(P)$, we have $A = A_1 \times \mathbf{Z}(K)$, where $A_1 = A \cap K' \in \mathfrak{A}(P \cap K')$. In particular, A contains $\mathbf{Z}(P \cap K')$. By a result of Suzuki [2], 1 is the only element of $\text{И}_H(A; 2')$, and so $G \in \mathcal{S}_2$.

If U is a 2-subgroup of H, we may assume $U \subseteq P$. Thus, $C_P(U)$ contains $Z(P \cap K')$, and by the result of Suzuki alluded to above, 1 is the only element of $\text{И}_H(C_P(U); 2')$, and so $G \in \tilde{\mathcal{S}}_2$.

Case 2. $q = 8$. Let $L = K'$. Thus, K is a central product of L and $\mathbf{Z}(K)$, and L is perfect. Let $P_1 = P \cap L$. If $A \in \mathfrak{A}(P)$, then A is a central product of A_1 and $\mathbf{Z}(K)$, where $A_1 = A \cap P_1 \in \mathfrak{A}(P_1)$. If $\mathbf{Z}(L) = 1$, the argument in Case 1 applies, so suppose $\mathbf{Z}(L) \neq 1$. Set $Z = \mathbf{Z}(L)$.

By the result of Alperin and Gorenstein, we get that Z is elementary of order

2 or 4, and in addition, $Z = \mathbf{Z}(P_1)$. Also, $P_1' = Z \times E$, where E is elementary of order 8. By a basic property of P_1/Z, each of its normal subgroups either contains P_1'/Z or is contained in P_1'/Z. Since $A_1 \lhd P_1$ and $\mathbf{C}_{P_1}(A_1) = \mathbf{Z}(A_1)$, we get that $A_1 \supseteq P_1'$. Hence, 1 is the only element of $\mathsf{M}_H(A; 2')$, so $G \in \mathcal{S}_2$.

Now suppose U is a 2-subgroup of H. We may assume that $U \subseteq P$.

First, suppose $\mathbf{C}_H(U) \subseteq K$. If $U \subseteq \mathbf{Z}(K)$, clearly $\mathbf{O}_{2'}(\mathbf{C}_H(U)) = 1$, so suppose $U \nsubseteq \mathbf{Z}(K)$. Since $K/\mathbf{Z}(K)$ is a CIT-group, it follows that $\mathbf{C}_H(U) = \mathbf{C}_K(U)$ is a 2-group, so $\mathbf{O}_{2'}(\mathbf{C}_H(U)) = 1$.

Finally, suppose $\mathbf{C}_H(U) \nsubseteq K$. In this case, $H/\mathbf{Z}(K) \simeq \mathrm{Aut}\ Sz(8)$. As observed by Alperin and Gorenstein, H/K acts non-trivially on $\mathbf{Z}(L) = Z$, and so $\mathbf{O}_{2'}(\mathbf{C}(U)) \subseteq K$, whence $\mathbf{O}_{2'}(\mathbf{C}(U)) = 1$.

Combining the various cases gives $G \in \tilde{\mathcal{S}}_2$ and completes the proof.

Bibliography

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