SHRINKABILITY OF CERTAIN DECOMPOSITIONS OF E^3 THAT YIELD E^3

BY

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1. Introduction

In this paper, we shall study shrinkability conditions satisfied by certain types of pointlike decompositions of E^3. We shall show that if G is a point like decomposition of E^3 having a 0-dimensional set of nondegenerate elements and such that the associated decomposition space is homeomorphic to E^3, then G satisfies a well-known shrinkability condition. The results of this paper carry over, with essentially no changes, to cellular decompositions of arbitrary 3-manifolds with boundary.

In order to state our results precisely, we introduce some notation. If G is an upper semicontinuous decomposition of E^3, then E^3/G denotes the associated decomposition space, P denotes the projection map from E^3 onto E^3/G, and H_G denotes the union of all the nondegenerate elements of G.

Suppose that G is an upper semicontinuous decomposition of E^3 such that $P[H_G]$ is 0-dimensional. Then we shall say that G is *shrinkable* if and only if for each open set U containing H_G and each positive number ε, there is a homeomorphism h from E^3 onto E^3 such that (1) if $x \in E^3 - U$, $h(x) = x$, and (2) if $g \in G$, $(\operatorname{diam} h[g]) < \varepsilon$.

The importance of shrinkable decompositions is easily seen from the following theorem, due to Bing [7], [8]: If G is a monotone decomposition of E^3 such that $P[H_G]$ is 0-dimensional and G is shrinkable, then E^3/G is homeomorphic to E^3.

The main result of this paper is the following theorem which provides a converse, in the case of pointlike decompositions of E^3, to the theorem of Bing's stated above: *If G is a pointlike decomposition of E^3 such that $P[H_G]$ is 0-dimensional and E^3/G is homeomorphic to E^3, then G is shrinkable.* An analogous result holds for cellular decompositions of arbitrary 3-manifolds with boundary.

The significance of the two theorems stated above concerning shrinkabilty of decompositions of E^3 becomes clearer when it is pointed out that shrinkability provides one of the most commonly used criteria for deciding whether the space of some particular decomposition of E^3 is homeomorphic to E^3. Although the study of local properties of decomposition spaces is beginning to provide some different ways of showing that spaces of various decompositions of E^3 are topologically distinct from E^3, such methods seem as yet more difficult to apply than those involving shrinkability.

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Various special cases of the main result of this paper have been established previously. In [1], it was shown to hold in case $P[H_G]$ is countable. In [2], it was established in case $P[H_G]$ is a compact 0-dimensional set.

In Section 5, we show that if G is a shrinkable monotone decomposition of E^3 such that $P[H_G]$ is 0-dimensional and E^3/G is homeomorphic to E^3, then each element of G is cellular. A number of questions related to the problems studied in this paper are considered in Section 6.

2. Notation and terminology

The statement that M is a *3-manifold with boundary* means that M is a separable metric space such that each point of M has a neighborhood which is a 3-cell. A point x of a 3-manifold with boundary M is an *interior* point of M if and only if x has an open neighborhood in M which is an open 3-cell. The *interior* of M, Int M, is the set of all interior points of M. The *boundary* of M, *Bd* M, is M − Int M.

A subset X of a 3-manifold with boundary M is *cellular* in M if and only if there is a sequence $C_1, C_2, C_3, \cdots$ of 3-cells in M such that (1) for each i, $C_{i+1} \subset \text{Int } C_i$, and (2) $X = \bigcap_{i=1}^{\infty} C_i$. A cellular set in a 3-manifold with boundary M lies in Int M. The statement that G is a *cellular decomposition* of a 3-manifold with boundary M means that G is an upper semicontinuous decomposition of M into cellular sets.

A subset X of E^3 is *pointlike* if and only if X is a compact continuum such that $E^3 - X$ is homeomorphic to $E^3 - \{0\}$. G is a *pointlike decomposition* of E^3 if and only if G is an upper semicontinuous decomposition of E^3 into pointlike sets. It is well known that in E^3, "pointlike" and "cellular" are equivalent; see [13]. By a *monotone decomposition* of a 3-manifold with boundary M is meant an upper semicontinuous decomposition of M into compact continua.

If A is a set in a topological space, then Cl A denotes the closure of A and βA denotes the (topological) boundary of A. If X is a metric space, then a sequence $A_1, A_2, A_3, \cdots$ of sets in X is a *null sequence* if and only if for each positive number ε, there exists a positive integer n such that if $i > n$, then $(\text{diam } A_i) < \varepsilon$. If ε is a positive number and A is a subset of a metric space, then $V(\varepsilon, A)$ denotes the open ε-neighborhood of A.

3. Preliminary Results

The following two lemmas are corollaries of Lemmas 3 and 4, respectively, of [3].

LEMMA 1. *Suppose that G is a monotone decomposition of E^3 such that $P[H_G]$ is 0-dimensional, and $\mathfrak{U}$ is an open covering (in E^3) of H_G such that*

(1) *each set of $\mathfrak{U}$ is a union of elements of G and*

(2) *if B is any bounded subset of E^3, $\bigcup\{U : U \in \mathfrak{U}$ and U intersects $B\}$ is bounded.*

Then there exists an open (in E^3) covering $\mathcal{V}$ of H_G by mutually disjoint bounded sets such that

(1) *each set of $\mathcal{V}$ lies in some set of $\mathcal{U}$ and*

(2) *if B is any bounded set in E^3, then $\{V : V \in \mathcal{V}$ and V intersects $B\}$ is a null sequence.*

LEMMA 2. *Suppose that $\{V_1, V_2, V_3, \cdots\}$ is a sequence of mutually disjoint bounded open sets in E^3 such that if B is any bounded set in E^3, $\{V_i : V_i$ intersects $B\}$ is a null sequence. Suppose that for each i, h_i is a homeomorphism from Cl V_i onto Cl V_i such that $h_i \mid \beta V_i$ is the identity on βV_i. Let h be the function from E^3 into E^3 such that*

(1) *if $x \in E^3 - \bigcup_{i=1}^{\infty} V_i$, $h(x) = x$, and*

(2) *if i is a positive integer and $x \in V_i$, $h(x) = h_i(x)$.*

Then h is a homeomorphism from E^3 onto E^3.

The following result is established in [4].

THEOREM 1 OF [4]. *Suppose that M is a 3-manifold with boundary and G is a cellular decomposition of M such that M/G is a 3-manifold with boundary N. Suppose that U is an open set in* Int N *such that $P[H_G] \subset U$. Then there is a homeomorphism h from* Cl $P^{-1}[U]$ *onto* Cl U *such that $h \mid \beta P^{-1}[U] = P \mid \beta P^{-1}[U]$.*

4. The main result

THEOREM 1. *If G is a pointlike decomposition of E^3 such that $P[H_G]$ is 0-dimensional and E^3/G is homeomorphic to E^3, then G is shrinkable.*

Proof. Suppose U is an open set in E^3 containing H_G and ε is a positive number. With the aid of Lemma 1, it follows that there exists a covering $\{V_1, V_2, V_3, \cdots\}$ of H_G by mutually disjoint open sets in E^3 such that (1) for each i, $V_i \subset U$ and (2) if B is any bounded set in E^3, then $\{V_i : V_i$ *intersects* $B\}$ is a null sequence. Notice that for each i, βV_i and H_G are disjoint, V_i is a union of elements of G, and Cl V_i is compact.

Our first step is to construct, for each i, a homeomorphism h_i from Cl V_i onto Cl V_i such that $h_i \mid \beta V_i$ is the identity and h_i shrinks nondegenerate elements of G in V_i. Hence suppose i is some positive integer. Since by hypothesis, E^3/G is homeomorphic to E^3, then by Theorem 1 of [4], there is a homeomorphism f_i from Cl V_i onto Cl $P[V_i]$ such that $f_i \mid \beta V_i = P \mid \beta V_i$. Since Cl $P[V_i]$ is compact and f_i^{-1} is continuous, there is a positive number δ_i such that if A is any subset of $P[V_i]$ and (diam A) $< \delta_i$, then (diam $f^{-1}[A]$) $< \varepsilon$.

Since V_i is a union of elements of G, $P[V_i]$ is open. By arguments similar to those used to establish Lemmas 1, 2, and 3 of [3], it may be shown, since $P[V_i] \cap P[H_G]$ is 0-dimensional, that there exists an open covering of $P[V_i] \cap P[H_G]$ by mutually disjoint open sets $V_{i1}, V_{i2}, V_{i3}, \cdots$ such that (1) for each positive integer j, Cl $V_{ij} \subset P[V_i]$ and (diam V_{ij}) $< \delta_i$, and (2) V_{i1}, V_{i2}, V_{i3}, $\cdots$ is a null sequence.

If j is a positive integer, then by Theorem 1 of [4], there is a homeomorphism k_{ij} from Cl V_{ij} onto Cl $P^{-1}[V_{ij}]$ such that $k_{ij}| \beta V_{ij} = P^{-1} | \beta V_{ij}$. Observe that $k_{ij}^{-1} P^{-1}[V_{ij}] = V_{ij}$.

Now define a function h_i as follows: (1) If $x \in \beta V_i$, $h_i(x) = x$. (2) If $x \in V_i - \bigcup_{i=1}^{\infty} P^{-1}[V_{ij}]$, then $h_i(x) = f_i^{-1}P(x)$. (3) If j is a positive integer and $x \in P^{-1}[V_{ij}]$, then $h_i(x) = f_i^{-1} k_{ij}^{-1}(x)$.

It is easily verified that h_i is well defined, from Cl V_i into Cl V_i, and is one-to-one. By an argument similar to that given for Lemma 4 of [4], it may be shown that both h_i and h_i^{-1} are continuous. The following argument shows that h_i has Cl V_i as its range. Let Q be a 3-cell containing Cl V_i. Define a function h_i^* from Q into Q as follows: (1) If $x \in V_i$, $h_i^*(x) = h_i(x)$. (2) If $x \in Q - V_i$, $h_i^*(x) = x$. It is easily seen that h_i^* is a continuous function from Q into Q and $h_i^* \mid$ Bd Q is the identity. If h_i does not have all of Cl V_i as its range, there would exist a retraction from Q onto Bd Q. Consequently, the range of h_i is Cl V_i.

Define a function h as follows: (1) If $x \in E^3 - \bigcup_{i=1}^{\infty} V_i$, $h(x) = x$. (2) If i is a positive integer and $x \in V_i$, $h(x) = h_i(x)$. By Lemma 2, h is a homeomorphism from E^3 onto E^3.

It is clear that if $x \in E^3 - U$, $h(x) = x$. In order to complete the proof of Theorem 1, we need only to show that if $g \in G$, then (diam $h[G]$) $< \varepsilon$. Suppose that g is a nondegenerate element of G. There is some positive integer i such that $g \subset V_i$. There is a positive integer j such that $P[g] \subset V_{ij}$. First we shall show that

$$h_i[g] \subset f_i^{-1}[V_{ij}].$$

Clearly $g \subset P^{-1}[V_{ij}]$. Now $h_i P^{-1}[V_{ij}] = f_i^{-1} k_{ij}^{-1} P^{-1}[V_{ij}]$, but $k_{ij}^{-1} P^{-1}[V_{ij}] = V_{ij}$. Hence $h_i P^{-1}[V_{ij}] = f_i^{-1}[V_{ij}]$, so $h_i[g] \subset f_i^{-1}[V_{ij}]$. Now by construction, (diam V_{ij}) $< \delta_i$ and hence (diam $f_i^{-1}[V_{ij}]$) $< \varepsilon$. Therefore

$$(\text{diam } h_i[g]) < \varepsilon$$

and since $h[g] = h_i[g]$, it follows that (diam $h[g]$) $< \varepsilon$. Hence if g is any element of G, (diam $h[g]$) $< \varepsilon$.

Consequently, G is shrinkable, and Theorem 1 is proved.

5. Cellularity of elements of G

Suppose G is a monotone decomposition of E^3 such that (1) E^3/G is homeomorphic to E^3 and (2) $P[H_G]$ is 0-dimensional. It is not known whether, under this hypothesis, each element of G is cellular. Indeed, if (2) above is replaced by "$P[H_G]$ is compact and 0-dimensional," it is not known whether each element of G is cellular.[2] Some information is available in cases where additional hypotheses are satisfied. For the case where $P[H_G]$ is countable,

[2] See Section 6.

see [10], and for the case where $P[H_G]$ lies in a compact 0-dimensional set, see [2] and [6]. If each element of G is a compact absolute retract or, indeed, satisfies certain weaker hypotheses, then each element of G is cellular; no hypothesis concerning the dimension of $P[H_G]$ is necessary. See [11] and [5] for these and related results.

There is an example due to Bing [9] of a monotone decomposition G of E^3 such that E^3/G is homeomorphic to E^3, $P[H_G]$ is an arc, but each nondegenerate element of G is non-cellular. This shows that, in the case of monotone decompositions of E^3, some condition on $P[H_G]$ is necessary.

It follows from results of [2] and [7] that if G is a monotone shrinkable decomposition of E^3 such that $P[H_G]$ is a compact 0-dimensional set, then each element of G is cellular. Our next result extends this to the case where $P[H_G]$ is 0-dimensional.

THEOREM 2. *Suppose that G is a monotone shrinkable decomposition of E^3 such that $P[H_G]$ is 0-dimensional. Then each element of G is cellular.*

Proof. Suppose that g is an element of G. We shall first show that if U is any open set in E^3 containing g, then there is a 3-cell C such that $g \subset \operatorname{Int} C$ and $C \subset U$. Let U be an open set in E^3 containing g. Let V be an open set in E^3 containing H_G such that (1) each component of V is bounded and (2) if V_0 is the component of V containing g, then $\operatorname{Cl} V_0 \subset U$. Let W be an open set in E^3 containing H_G such that $W \subset V$ and if W_0 is the component of W containing g, then $\operatorname{Cl} W_0 \subset V_0$.

Let $\{C_1, C_2, \cdots, C_n\}$ be a finite set of 3-cells in E^3 such that $\{\operatorname{Int} C_1, \operatorname{Int} C_2, \cdots, \operatorname{Int} C_n\}$ covers $\operatorname{Cl} W_0$ and each of $C_1, C_2, \cdots$ and C_n lies in V_0. There exists a positive number ε such that any subset of $\operatorname{Cl} W_0$ of diameter less than ε lies in some one of $\operatorname{Int} C_1, \operatorname{Int} C_2, \cdots$, and $\operatorname{Int} C_n$.

Since G is shrinkable, there is a homeomorphism h from E^3 onto E^3 such that (1) if $x \in E^3 - W$, $h(x) = x$ and (2) if $g \varepsilon G$, $(\operatorname{diam} h[g]) < \varepsilon$. Since $h \mid E^3 - W$ is the identity and $V \subset W$, then $h \mid E^3 - V$ is the identity. Since both V_0 and W_0 are bounded, it follows by an argument similar to one used in the proof of Theorem 1, that $h[\operatorname{Cl} V_0] = \operatorname{Cl} V_0$ and $h[\operatorname{Cl} W_0] = \operatorname{Cl} W_0$. Since $g \subset V_0$, there is a positive integer i such that $i \leq n$ and $h[g] \subset \operatorname{Int} C_i$.

Let C denote $h^{-1}[C_i]$. Clearly C is a 3-cell and $g \subset \operatorname{Int} C$. Since $C_i \subset W_0$ and $h[\operatorname{Cl} W_0] = \operatorname{Cl} W_0$, it follows that $h^{-1}[C_i] \subset \operatorname{Cl} W_0$. Therefore

$$h^{-1}[C_i] \subset U,$$

and hence $C \subset U$.

We can now show that g is cellular. There is a 3-cell D_1 such that

$$g \subset \operatorname{Int} D_1$$

and $D_1 \subset V(1, g)$. There is a 3-cell D_2 such that

$$g \subset \operatorname{Int} D_2 \quad \text{and} \quad D_2 \subset (\operatorname{Int} D_1) \cap V(1/2, g).$$

Suppose that k is a positive integer and there is a 3-cell D_k such that $g \subset \operatorname{Int} D_k$ and $D_k \subset V(1/k, g)$. Then there is a 3-cell D_{k+1} such that

$$g \subset \operatorname{Int} D_{k+1} \quad \text{and} \quad D_{k+1} \subset (\operatorname{Int} D_k) \cap V(1/k+1, g).$$

It is easily seen that (1) for each positive integer m, $D_{m+1} \subset \operatorname{Int} D_m$ and (2) $g = \bigcap_{i=1}^{\infty} D_i$. Hence g is cellular. This establishes Theorem 2.

6. Questions

The following two questions of considerable interest are closely connected with the results of Section 5 and were mentioned there.

1. Suppose G is a monotone decomposition of E^3 such that (1) E^3/G is homeomorphic to E^3 and (2) $P[H_G]$ is 0-dimensional. Then is each element of G cellular?

2. Suppose G is a monotone decomposition of E^3 such that (1) E^3/G is homeomorphic to E^3 and (2) $P[H_G]$ is compact and 0-dimensional. Then is each element of G cellular? (Added in proof. Recently, D. R. McMillan, Jr. and, independently, H. W. Lambert have answered the question affirmatively.)

It should be posible to define a notion of "shrinkable" for arbitrary decompositions of E^3 (or 3-manifolds, or metric spaces). The definition used in this paper is not useful unless $P[H_G]$ is 0-dimensional.

3. Is there a definition of "shrinkable" for decompositions of E^3 such that the following are theorems? (a) If G is a monotone shrinkable decomposition of E^3, then E^3/G is homeomorphic to E^3. (b) If G is a pointlike decomposition of E^3 such that E^3/G is homeomorphic to E^3, then G is shrinkable.

McAuley has considered shrinkability conditions for arbitrary decompositions of E^3 (and other spaces); see [12] and [13]. In connection with part (b) of question 3 above, it has been shown in [4] that in the case of cellular decompositions of 3-manifolds into 3-manifolds, the projection map can be approximated arbitrarily closely by homeomorphisms.

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