CERTAIN SEMIGROUPS OF LINEAR FRACTIONAL TRANSFORMATIONS CONTAIN ELEMENTS OF ARBITRARILY LARGE TRACE

BY

KENNETH B. STOLARSKY[1]

1. Introduction

The main purpose of this paper is to prove the

THEOREM. *Let Γ be a semigroup of linear fractional transformations acting on the Riemannsphere $\mathcal{S}$. Then if every punctured neighborhood of some $p \in \mathcal{S}$ contains both fixed points of infinitely many elliptic transformations of Γ, Γ contains elements whose trace is arbitrarily large in absolute value.*

In §3 this is used to give a new proof of the well-known fact that a discrete group of 2×2 elliptic matrices is finite [Lehner, pp. 91–92].

By definition, a semigroup Γ of linear fractional transformations consists of elements V such that for $z \in \mathcal{S}$

$$V(z) = (az + b)/(cz + d); \qquad a, b, c, d \ \text{complex}, \qquad ad - bc = 1.$$

With V it is convenient to associate the two matrices $\pm V'$,

$$V' = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

since whenever $V_1, V_2 \in \Gamma$, $V_1 V_2$ is then associated with $\pm V_1' V_2'$. The prime marks will be dropped for notational convenience; this causes no confusion.

For $V \in \Gamma$, let $\chi(V)$ denote the trace of V. If $\chi(V)$ is real, V is said to be elliptic, parabolic, or hyperbolic depending upon whether $|\chi(V)| < 2$, $=2$, or >2 respectively. It is well known that if V is elliptic, and has finite fixed points α_1, α_2, then $V(z) = z'$ where

$$(1) \quad (z' - \alpha_1)/(z' - \alpha_2) = \kappa(z - \alpha_1)/(z - \alpha_2); \qquad \kappa = e^{i\theta}, \quad 0 < \theta < 2\pi.$$

Thus

$$V = \begin{bmatrix} \dfrac{\kappa^{-1/2}\alpha_1 - \kappa^{1/2}\alpha_2}{\alpha_1 - \alpha_2} & -\left(\kappa^{-1/2} - \kappa^{1/2}\right)\dfrac{\alpha_1 \alpha_2}{\alpha_1 - \alpha_2} \\ \dfrac{\kappa^{-1/2} - \kappa^{1/2}}{\alpha_1 - \alpha_2} & \dfrac{\kappa^{1/2}\alpha_1 - \kappa^{-1/2}\alpha_2}{\alpha_1 - \alpha_2} \end{bmatrix}.$$

Here κ is called the multiplier of V.

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2. Proof of the theorem

LEMMA. *Let V_1, V_2 be elliptic transformations with finite fixed points and multipliers $\{\alpha_1, \alpha_2, e^{i\theta_1}\}$, $\{\beta_1, \beta_2, e^{i\theta_2}\}$ respectively. Then*

$$\chi(V_1 V_2) = 2\left[\frac{\mu_1}{\lambda}\cos\left(\frac{\theta_1 - \theta_2}{2}\right) - \frac{\mu_2}{\lambda}\cos\left(\frac{\theta_1 + \theta_2}{2}\right)\right] \tag{3}$$

where

$\mu = (\alpha_1 + \alpha_2)(\beta_1 + \beta_2) - 2\alpha_1\alpha_2 - 2\beta_1\beta_2, \qquad \lambda = (\alpha_1 - \alpha_2)(\beta_1 - \beta_2)$

$\mu_1 = (\alpha_1 - \beta_1)(\alpha_2 - \beta_2), \qquad$ *and* $\quad \mu_2 = (\alpha_1 - \beta_2)(\alpha_2 - \beta_1)$.

Proof. Let $\kappa = e^{i\theta_1}$, $\kappa' = e^{i\theta_2}$. A direct calculation from (2) yields

$$\begin{aligned}\lambda\chi(V_1 V_2) = {} & [(\kappa\kappa')^{1/2} + (\kappa\kappa')^{-1/2}](\alpha_1\beta_1 + \alpha_2\beta_2 - \alpha_1\alpha_2 - \beta_1\beta_2) \\ & + [\kappa^{1/2}\kappa'^{-1/2} + \kappa^{-1/2}\kappa'^{1/2}](\alpha_1\alpha_2 + \beta_1\beta_2 - \alpha_1\beta_2 - \alpha_2\beta_1).\end{aligned}$$

Formula (3) follows.

Remark. If V_1 and V_2 have a fixed point in common, say $\alpha_1 = \beta_1$, it follows from (3) that $V_1 V_2$ is elliptic or parabolic, since $\chi(V_1 V_2) = 2\cos[(\theta_1 + \theta_2)/2]$.

To prove the theorem we clearly may assume p is finite. Let $S = \{V_n\}_{n=1}^{\infty}$ be a sequence of elliptic elements of Γ such that V_n has fixed points $\beta_1(n)$, $\beta_2(n)$, neither of which is p, multiplier $e^{i\theta(n)}$, $0 < \theta(n) < 2\pi$, and

$$\lim \beta_1(n) = \lim \beta_2(n) = p.$$

We may also assume $\lim \theta(n) = \theta_2$, $0 \leq \theta_2 \leq 2\pi$. For each $W \in S$ with fixed points α_1, α_2 and multiplier $e^{i\theta_1}$ there is a $\delta > 0$ such that $|\alpha_1 - p| > \delta$, $i = 1, 2$. Set $t(n) = (\beta_1(n) - \beta_2(n))\chi(WV_n)$ and $f(x) = |\cos x|$. Then by (3),

$$t = \limsup |t(n)| \geq \frac{2\delta^2}{|\alpha_1 - \alpha_2|}\left|f\left(\frac{\theta_1 - \theta_2}{2}\right) - f\left(\frac{\theta_1 + \theta_2}{2}\right)\right|.$$

Thus it suffices to show S can be chosen so that $t > 0$.

$f(x) = f(y)$ yields $x - y = \pi k$ or $x + y = \pi + \pi k$, where k is an integer, so the only cases which cause any difficulty are $\theta_2 = 0, \pi, 2\pi$ and $\theta_1 = \pi$. If $\theta_1 = \pi$, and this cannot be avoided by a new choice of W, then clearly $\theta_2 = \pi$. If $\theta_2 = 0$ or 2π, V_n is a rotation through an angle which tends to zero as $n \to \infty$, so there is an integer $m = m(n)$ such that V_n^m tends to a rotation through an angle of π as $n \to \infty$. Replace S by $S' = \{V_n^m\}$. Hence only the case $\theta_2 = \pi$ need be considered. Choose W so that $\theta_1 = \pi + \varepsilon$, $|\varepsilon| < \pi/2$. By (3),

$$\lim t(n) = 2\frac{(\alpha_1 - p)(\alpha_2 - p)}{(\alpha_1 - \alpha_2)}\left[\cos\left(\frac{\theta_1 - \pi}{2}\right) - \cos\left(\frac{\theta_1 + \pi}{2}\right)\right]$$

$$= 4\,\frac{(\alpha_1 - p)(\alpha_2 - p)}{(\alpha_1 - \alpha_2)} \cos \varepsilon/2 > 0,$$

and this proves the theorem.

3. An application

Topologize any set Γ of 2×2 matrices by embedding it in 4-space in the obvious manner. Let $S = \{V_n\} \subseteq \Gamma$ denote a sequence of *distinct* elliptic elements V_n having fixed points $\beta_1(n)$, $\beta_2(n)$, with $\lim \beta_1(n) = \beta_1$, $\lim \beta_2(n) = \beta_2$, β_1, β_2 possibly infinite. From (2), if $\beta_1 \neq \beta_2$ then some subsequence of S converges to an elliptic or parabolic transformation.

THEOREM. *A discrete group Γ of 2×2 elliptic matrices is finite.*

Proof. If Γ is infinite, choose S as above. Then $\beta_1 = \beta_2$, and by applying the theorem of §1 there is a subsequence S' of S such that $\beta_1(n) = \beta_1$ for all $V_n \in S'$. Only now is Γ required to be a group rather than a semigroup: the commutator $V_n V_m V_n^{-1} V_m^{-1}$ is parabolic and not the identity for $n \neq m$ [Lehner, p. 73], a contradiction.

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REFERENCE

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THE INSTITUTE FOR ADVANCED STUDY
PRINCETON, NEW JERSEY