INVARIANTS FOR COMMUTATIVE GROUP ALGEBRAS

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Let K be a commutative ring with identity and G an abelian group. Then the structure of KG as a K-algebra depends to some extent upon the primes p for which the torsion subgroup of G has non-trivial p-components and the relationship of these primes to the arithmetic of K. The case in which these primes are not invertible in K has been investigated in [2] and it was seen that the algebraic structure of these p-components is intimately connected with that of the algebra. If the ring K is especially nice, namely an integral extension ring of the integers, then it is shown in [3] that the isomorphism class of KG determines the isomorphism class of G, hence this latter class is a complete set of invariants for commutative group algebras over K.

In this paper we consider a case at the opposite extreme. Take K to be an algebraically closed field and G an abelian group having no element whose order is equal to the characteristic of K. Then all primes of the type mentioned above are invertible in this ring and so we should expect the structure of KG to be related only weakly to that of G. Of course when G is finite it is well known that KG is isomorphic to the direct product of n copies of K where n is the cardinality of G, hence in the finite case the cardinality of G (or the dimension of KG) constitutes a complete set of invariants. We shall show that in general, a complete set of invariants for the structure of KG consists of the cardinality of G_0 and the isomorphism class of G/G_0 (where G_0 is the torsion subgroup of G). Moreover we shall say something about how these invariants can be determined from the algebra.

For the rest of this paper, K will denote an algebraically closed field. In addition we shall tacitly assume that every group considered will have no element of order equal to the characteristic of K.

PROPOSITION. *Let G be an abelian group with torsion subgroup G_0. Then*

$$KG \cong KG_0 \otimes_K K(G/G_0).$$

Proof. Define $H = G_0 \times (G/G_0)$. Since $KH \cong KG_0 \otimes_K K(G/G_0)$, it will suffice to show that $KG \cong KH$. This will be accomplished by finding a group of units in KH which is isomorphic to G and is a K-basis for KH. First we must choose a certain generating set for G.

We wish to construct a family of subgroups of G, $\{G_\alpha\}$, indexed by some initial segment of ordinals. Start with G_0. Define $G_{\alpha+1}$ to be the subgroup generated by G_α and an element $g_\alpha \notin G_\alpha$ in case $G_\alpha \neq G$. If α is a limit ordinal, define $G_\alpha = \bigcup_{\beta<\alpha} G_\beta$. Then $G = \bigcup_\alpha G_\alpha$. Now for each α, let n_α be 0 in case $\langle g_\alpha \rangle \cap G_\alpha = \{1\}$, otherwise let n_α be the least positive integer such that

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$g_\alpha^{n_\alpha} \in G_\alpha$. Finally, for each α, we may choose v_α and t_α, where v_α is a word in $\{g_\beta \mid \beta < \alpha\}$ and $t_\alpha \in G_0$, such that $g_\alpha^{n_\alpha} = v_\alpha t_\alpha$. This is so since $G_0 \cup \{g_\beta \mid \beta < \alpha\}$ generates G_α. It now follows from the nature of the choices made that G is isomorphic to the group generated by $G_0 \cup \{g_\alpha\}$ subject to the relations $\{g_\alpha^{n_\alpha} = v_\alpha t_\alpha\}$.

We claim that G/G_0 can be given by generators $\{h_\alpha\}$ subject to relations $\{h_\alpha^{n_\alpha} = w_\alpha\}$ where w_α is the same word as v_α, but with the g's replaced by corresponding h's. Define h_α by $h_\alpha = g_\alpha G_0$. Then clearly $\{h_\alpha\}$ generates G/G_0 and $h_\alpha^{n_\alpha} = w_\alpha$ is satisfied for every α. To show that these are actually defining relations for G/G_0, it is sufficient to verify that if $0 < m < n_\alpha$ (or just $0 < m$ in case $n_\alpha = 0$), then $h_\alpha^m \notin G_\alpha/G_0$. But this follows immediately from the definition of n_α. Hence H is generated by $G_0 \cup \{h_\alpha\}$ subject to the relations $\{h_\alpha^{n_\alpha} = w_\alpha\}$.

Next construct units c_α in KG_0 of augmentation 1 (augmentation means sum of coefficients as an element of KH). Start with $c_0 = 1$. Now suppose c_β has been defined for all $\beta < \alpha$. Let w_α' be the same word as w_α, but with the h's replaced by corresponding c's. We shall show that $c_\alpha \in KG_0$ can be chosen such that $c_\alpha^{n_\alpha} = w_\alpha' t_\alpha$ and c_α is a unit of augmentation 1. Now there is a finite subgroup G_0' of G_0 such that $w_\alpha' t_\alpha \in KG_0'$. Because of our hypotheses on K and G, we know that $KG_0' \cong K^q$ where $q = |G_0'|$. But we may take arbitrary roots in K^q, hence there exists $c \in KG_0$ such that $c^{n_\alpha} = w_\alpha' t_\alpha$. If the augmentation of c is $a \in K$, then the augmentation of $w_\alpha' t_\alpha$ being 1 implies $a^{n_\alpha} = 1$. Therefore $c_\alpha = a^{-1}c$ satisfies the desired conditions. (Note it is a unit since a product of units.)

Now define elements $f_\alpha \in KH$ by $f_\alpha = h_\alpha c_\alpha$. Let u_α be the same word as v_α, but with the g's replaced by corresponding f's. Then $w_\alpha w_\alpha' = u_\alpha$ and so $f_\alpha^{n_\alpha} = u_\alpha t_\alpha$. Since each f_α is a unit in KH, we may consider the group of units in KH generated by G_0 and $\{f_\alpha\}$, call it U. For each α, let U_α be the subgroup generated by G_0 and $\{f_\beta \mid \beta < \alpha\}$. In order to show that U is isomorphic to G (where f_α corresponds to g_α and u_α to v_α), we must show that $f_\alpha^m \notin U_\alpha$ for $0 < m < n_\alpha$ (or just $0 < m$ in case $n_\alpha = 0$). So consider the map $\varphi : KH \to K(G/G_0)$ induced by the projection of H onto G/G_0. Then since each c_α has augmentation 1, we have $\varphi(c_\alpha) = 1$ so $\varphi(f_\alpha) = h_\alpha$. Therefore $f_\alpha^m \in U_\alpha$ would imply $h_\alpha^m \in G_\alpha/G_0$ contrary to fact. Hence $U \cong G$.

All that remains is to show that U is a basis for KH. The linear subspace generated by U is the same as the subalgebra generated, hence c_α and f_α in this subalgebra imply h_α is in it and so the subspace is all of KH. Now let $\alpha_1, \cdots, \alpha_n$ be finitely many indices and define V to be the group of units generated by G_0 and $f_{\alpha_1}, \cdots, f_{\alpha_n}$. Then to show U is a K-independent set, it suffices to show such a V is, since any finite subset of U is contained in such a V. In addition, define W to be the group of units generated by G_0 and $h_{\alpha_1}, \cdots, h_{\alpha_n}$. Because G_0 is the torsion subgroup of W, we may choose words $y_1, \cdots, y_k$ in $h_{\alpha_1}, \cdots, h_{\alpha_n}$ such that

$$W = G_0 \times \langle y_1 \rangle \times \cdots \times \langle y_k \rangle$$

as an inner direct product where each $\langle y_i \rangle$ is infinite cyclic. Now let x_i be the same word as y_i, but with the h's replaced by corresponding f's. Then if φ is as previously, we have

$$\varphi(V) = \langle y_1 \rangle \times \cdots \times \langle y_k \rangle$$

and G_0 is the kernel of φ. Moreover $\varphi(x_i) = y_i$ for each i allows us to conclude that

$$V = G_0 \times \langle x_1 \rangle \times \cdots \times \langle x_k \rangle$$

as an inner direct product where each $\langle x_i \rangle$ is infinite cyclic. Let γ be a K-linear combination of elements from V and suppose that $\gamma = 0$. We can write

$$\gamma = \textstyle\sum_{(i)} \beta_{(i)} x_1^{i_1} \cdots x_k^{i_k}$$

for certain $\beta_{(i)} \in KG_0$. It follows that

$$\gamma = \textstyle\sum_{(i)} \beta_{(i)} c_{(i)} y_1^{i_1} \cdots y_k^{i_k}$$

for certain units $c_{(i)} \in KG_0$. Since $y_1, \cdots, y_k$ are algebraically independent over KG_0 (because of the decomposition of W), we must have $\beta_{(i)} c_{(i)} = 0$ for all (i). But then $\beta_{(i)} = 0$ for all (i) and hence the coefficients of γ, which are the coefficients of the various β's, are all zero. Therefore V is a K-independent set. ∎

It will be convenient to express the following lemma in terms of K-algebras of a certain type. We require all idempotents to be non-zero.

LEMMA 1. *Let A and B be two commutative K-algebras which are algebraic, have trival nilradicals, and are such that every idempotent decomposes into a sum of two orthogonal idempotents. Then if A and B are both of countable dimension, they are isomorphic.*

Proof. We shall show that A is a certain direct limit of subalgebras. But this direct system will be seen to be independent of A up to isomorphism. Hence we will conclude $A \cong B$.

A has infinitely many idempotents from the hypotheses. To see that there are countably many, select a countable K-basis for A, say $a_1, a_2, \cdots$. Let A_i be the subalgebra generated by $a_1, \cdots, a_i$. Then $A_1 \subseteq A_2 \subseteq \cdots$ and $A = \bigcup_i A_i$. Moreover, A algebraic implies that each A_i is finite dimensional, hence we may conclude that $A_i \cong K^{n_i}$ since K is algebraically closed and A_i has trivial nilradical. In particular, A_i has only finitely many idempotents and so A has countably many. Let $f_1, f_2, \cdots$ be the distinct idempotents of A which are different from 1.

We now want to choose certain idempotents in A. These idempotents will be written e_α where the index α is a finite sequence of 1's and 2's. We put $|\alpha|$ equal to the number of terms in the sequence. To begin with, let $1 = e_1 + e_2$ be a decomposition of 1 into orthogonal idempotents. This defines e_1 and e_2. Now suppose that e_α is defined for all α with $|\alpha| = n$ and that, moreover, $1 = \sum_{|\alpha|=n} e_\alpha$ is a decomposition of 1 into orthogonal idempotents. For each α with $|\alpha| = n$, consider

$$e_\alpha = e_\alpha f_n + e_\alpha(1 - f_n).$$

Suppose first that both right hand summands are non-zero. Then define $e_{\alpha 1} = e_\alpha f_n$ and $e_{\alpha 2} = e_\alpha(1 - f_n)$. In this case, let us say $\alpha \in I_1$. If either summand above is zero, then take $e_\alpha = e_{\alpha 1} + e_{\alpha 2}$ to be any decomposition of e_α. In case $e_\alpha f_n = 0$, let us say $\alpha \in I_2$, and in case $e_\alpha(1 - f_n) = 0$, let us say $\alpha \in I_3$. Hence we have chosen e_α for $|\alpha| = n + 1$ and moreover it is clear that $1 = \sum_{|\alpha|=n+1} e_\alpha$ is a decomposition of 1 into orthogonal idempotents.

Now let $S_n = \oplus_{|\alpha|=n} Ke_\alpha$. Then $S_1 \subseteq S_2 \subseteq \cdots$ are subalgebras of A and the inclusion map $S_n \rightarrow S_{n+1}$ is determined by the decompositions $e_\alpha = e_{\alpha 1} + e_{\alpha 2}$ for all α with $|\alpha| = n$. The direct system of $\{S_n\}$ under the inclusion maps is therefore independent of A up to isomorphism. We will be finished if we can show that the limit, $S = \bigcup_n S_n$, is A. By the local structure of A which we have previously examined, it is sufficient to show that $f_n \in S$ for all n. But we have

$$\begin{aligned} f_n = f_n\left(\textstyle\sum_{|\alpha|=n} e_\alpha\right) &= \textstyle\sum_{|\alpha|=n} e_\alpha f_n = \sum_{|\alpha|=n,\alpha\in I_1} e_\alpha f_n + \sum_{|\alpha|=n,\alpha\in I_3} e_\alpha \\ &= \textstyle\sum_{|\alpha|=n,\alpha\in I_1} e_{\alpha 1} + \sum_{|\alpha|=n,\alpha\in I_3} e_\alpha . \end{aligned}$$

This is an element of S_{n+1}. ∎

COROLLARY. *Let G and H be countably infinite torsion abelian groups. Then $KG \cong KH$.*

Proof. Given $\alpha \in KG$, then $\alpha \in KG_1$ for some finite subgroup $G_1 \subseteq G$. Hence α is contained in a finite-dimensional subalgebra, and moreover it cannot be nilpotent unless zero. Suppose α is idempotent. We may select a finite subgroup $G_2 \supseteq G_1$, but $G_2 \neq G_1$. Every minimal idempotent in KG_1 decomposes in KG_2, hence so does α. The hypotheses of the lemma are therefore satisfied by KG (and KH). ∎

This corollary has been proved by S. D. Berman (see [1, Theorem 5]). The author has not seen Berman's proof, but it seems reasonable to include the lemma for completeness and since the approach to the problem may differ. Of course there is a bonus result implicit in our considerations, namely that any algebra of countable dimension which satisfies the "local" conditions of the lemma is seen to satisfy the "global" conclusion that it is a group algebra. It would be interesting to know whether the restriction on the dimension can be dropped.

LEMMA 2. *Let G be a torsion abelian group with subgroup H of index n. Then $KG \cong (KH)^n$ as KH-algebras.*

Proof. Choose a finite subgroup G_1 such that $G_1 H = G$ and put $H_1 = G_1 \cap H$. Then $(G_1{:}H_1) = n$ and it is known that $KG_1 \cong (KH_1)^n$ as KH_1-algebras. Let $\alpha_1, \cdots, \alpha_n$ be orthogonal idempotents in KG_1 giving such a decomposition as a KH_1-algebra. Note that if $\beta \in KH_1$ and $\beta\alpha_i = 0$ for some i, then $\beta = 0$. We claim that $KG = \oplus_1^n KH \cdot \alpha_i$. This follows since

$$\textstyle\sum_1^n KH \cdot \alpha_i \supseteq KH \cdot \sum_1^n KH_1 \cdot \alpha_i = KH \cdot KG_1 = KG$$

and because the α's are orthogonal. Let $\beta \in KH$ be such that $\beta\alpha_i = 0$. If we can show this implies $\beta = 0$, then $KH \cdot \alpha_i \cong KH$ and we will be finished. Choose $\{h_j\}$ to be a complete family of representatives of cosets of H_1 in H. Then $\beta = \sum_j h_j \beta_j$ for certain $\beta_j \in KH_1$. Now $0 = \beta\alpha_i = \sum_j h_j \beta_j \alpha_i$ implies $\beta_j \alpha_i = 0$ for all j since $\beta_j \alpha_i \in KG_1$ and $\{h_j\}$ are coset representatives of G_1 in G. Hence $\beta_j = 0$ for all j as remarked earlier and so $\beta = 0$. ∎

LEMMA 3. *Let G and H be p-primary abelian groups such that $|G| = |H|$. Then $KG \cong KH$.*

Proof. If G and H are finite, then the result is true, hence we may assume both are infinite. It suffices to consider $G = \oplus_I Z_p$ where I is an index set such that $|I| = |H|$ (for then $|G| = |H|$). Let J be an index set with $|J| > |I|$ and put $M = \oplus_J Z_p$. Consider triples $(M_\alpha, \varphi_\alpha, H_\alpha)$ where M_α is a subgroup of M, H_α is a subgroup of H, and $\varphi_\alpha : KM_\alpha \to KH_\alpha$ is a K-isomorphism. Order in the obvious fashion and select a maximal such triple by Zorn's lemma, call it (M', φ', H'). We claim $H' = H$. If not, let H'' be generated by H' and an element of H outside H'. Then $(H'':H') = p^r$ for some $r > 0$ and $KH'' \cong (KH')^{p^r}$ as KH'-algebras. But $KM' \cong KH'$ implies $|M'| = |H'| < |M|$ and hence there is a subgroup M'' of M such that $M'' \supseteq M'$ and $(M'':M') = p^r$. We have $KM'' \cong (KM')^{p^r}$ as KM'-algebras and therefore there exists an isomorphism $\varphi'' : KM'' \to KH''$ extending φ'. This contradicts the maximality of (M', φ', H') and so $H' = H$. But now $|M'| = |H| = |I|$ implies $M' \cong \oplus_I Z_p$ and so $M' \cong G$. Therefore $KG \cong KH$. ∎

LEMMA 4. *Let G be a p-primary and H a q-primary abelian group. Then* $|G| = |H|$ implies $KG \cong KH$.

Proof. By the previous lemma it suffices to consider the case $G = \oplus_I Z_p$ and $H = \oplus_J Z_q$ where I and J are infinite index sets such that $|I| = |J|$. Consider triples $(G_\alpha, \varphi_\alpha, H_\alpha)$ where G_α is a subgroup of G, H_α a subgroup of H, and $\varphi_\alpha : KG_\alpha \to KH_\alpha$ a K-isomorphism. As before select a maximal triple (G', φ', H'). Suppose first that $(G:G') = \infty$ and $(H:H') = \infty$. Then there exist subgroups $G'' \supseteq G'$ and $H'' \supseteq H'$ such that G''/G' and H''/H' are countably infinite torsion groups. Moreover G' and H' are direct summands of G'' and H'' respectively so that we may write $G'' = G' \times L$ and $H'' = H' \times M$ as inner direct products for some subgroups $L \cong G''/G'$ and $M \cong H''/H'$. By the corollary to Lemma 1, there is a K-isomorphism $\psi : KL \to KM$. Hence φ' and ψ induce a natural isomorphism from $KG' \otimes_K KL$ to $KH' \otimes_K KM$. By combining this with the natural isomorphisms of KG'' and KH'' with the corresponding tensor products above, we get an isomorphism $\varphi'' : KG'' \to KH''$ which extends φ'. By contradiction, one of the indices, say $(G:G')$, must be finite. But then $|G| = |G'|$ and so $G \cong G'$ since the dimension of a vector space over a finite field is determined by the cardinality of the vector space. Further we have $|H'| = |G'| = |G| = |H|$, hence $H \cong H'$. Therefore $KG \cong KH$. ▮

Let G be an abelian group. Then the maximal algebraic subalgebra of KG is KG_0 where G_0 is the torsion subgroup of G (see [3, corollary to Lemma 2]). Hence the cardinal number $|G_0|$ is algebraically characterized as the dimension of this subalgebra. Now let $i:KG \to K$ be a "splitting" (i.e., a K-homomorphism) and let I be the ideal of KG generated by the intersection of the maximal algebraic subalgebra with the kernel of i. Then G/G_0 is isomorphic to the group of units in KG/I modulo the multiplicative group of K (see [3, corollary to Proposition 4]). Hence G/G_0 can be deduced algebraically from KG, although not canonically.

THEOREM. *Let K be an algebraically closed field and G an abelian group with torsion subgroup G_0 having no element of order equal to the characteristic of K. Then a complete set of invariants for KG as a K-algebra is $|G_0|$ and the isomorphism class of G/G_0.*

Proof. From the preceding discussion we see that if $KG \cong KH$ for another such group H, then $|G_0| = |H_0|$ and $G/G_0 \cong H/H_0$. Conversely assume that $|G_0| = |H_0|$ and $G/G_0 \cong H/H_0$. We must show that $KG \cong KH$. By the proposition, it suffices to show that $KG_0 \cong KH_0$. In case G_0 and H_0 are finite, it is trivial so we may assume they are infinite.

Let q denote a fixed prime (different from the characteristic of K) and let p denote arbitrary primes. It will suffice to show that $KG_0 \cong K(\oplus_I Z_q)$ where I is an index set such that $|I| = |G_0|$. Write $G_0 = \oplus_p G_p$ where G_p is the p-primary component of G_0. Let $P_1 = \{p \mid |G_p| < \infty\}$ and $P_2 = \{p \mid |G_p| = \infty\}$. For each $p \in P_2$ let J_p be an index set satisfying $|J_p| = |G_p|$. First suppose that $P_2 = \emptyset$. Then G_0 is countably infinite and so by the corollary to Lemma 1 we have $KG_0 \cong K(\oplus_I Z_q)$. So now we may suppose that $P_2 \neq \emptyset$. Then by Lemma 4,

$$K(\oplus_{P_2} G_p) \cong \otimes_{P_2} KG_p \cong \otimes_{P_2} K(\oplus_{J_p} Z_p) \cong K(\oplus_{P_2} \oplus_{J_p} Z_q) \cong K(\oplus_J Z_q)$$

where $|J| = \sup_{P_2} |J_p| = |G_0| = |I|$. Hence

$$K(\oplus_{P_2} G_p) \cong K(\oplus_I Z_q)$$

and we are finished if $P_1 = \emptyset$. So assume finally that $P_1 \neq \emptyset$. Partition I into a family of subsets $\{I_p \mid p \in P_1\}$ such that every I_p is infinite. Then we have

$$\begin{aligned} KG_0 &\cong K(\oplus_{P_1} G_p) \otimes K(\oplus_{P_2} G_p) \cong K(\oplus_{P_1} G_p) \otimes K(\oplus_I Z_q) \\ &\cong K(\oplus_{P_1}(G_p \oplus \oplus_{I_p} Z_q)) \cong K(\oplus_{P_1}(G_p \oplus \oplus_{I_p} Z_p)) \cong K(\oplus_{P_1} \oplus_{I_p} Z_p) \\ &\cong K(\oplus_{P_1} \oplus_{I_p} Z_q) \cong K(\oplus_I Z_q) \end{aligned}$$

since $p \in P_1$ implies $|G_p \oplus \oplus_{I_p} Z_p| = |\oplus_{I_p} Z_p|$. ▮

References

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