ENDOMORPHISM RINGS OF INDUCED LINEAR REPRESENTATIONS[1]

BY

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Introduction

Let ψ be a complex linear character of a subgroup H of a finite group G. In [2], C. W. Curtis and the author exhibited a basis and corresponding structure constants for the endomorphism ring E of a module affording the induced character ψ^G. In this paper we attack the same problem at characteristic p.

Section one establishes a relationship between the endomorphism ring E with an endomorphism ring at characteristic p related to ψ, while section two examines the decomposition theory of E relative to that of the group-algebra of G.

The following notations will be used throughout this paper:

G	a finite group of order $\mid G \mid$
H	a subgroup of G of order $\mid H \mid$
K	a p-adic number field containing the $\mid G \mid^{\text{th}}$ roots of 1
R	the ring of integers in K
P	the maximal ideal of R
F	the residue class field R/P
ψ	a linear representation of H in K
e	the idempotent $\mid H \mid^{-1} \sum_{h \in H} \psi(h^{-1})h$ in KH
M	the right KH-module eKH
N	the right KG-module eKG
E	the endomorphism ring $eKGe$.

Observe that the KH-module M affords the representation ψ, and $N = eKG \simeq eKH \otimes_{KH} KG = M^G$. Finally, $E = eKGe \simeq \mathrm{End}_{KG}(N)$, where we view E as operating on the left of N. For additional notation and terminology the reader may consult [2] and [3].

The following is a routine result about orders, modules and endomorphism rings which sets the stage for our discussion.

(0.1) PROPOSITION. *Let R be a noetherian domain with quotient field K and let A be a finite-dimensional K-algebra with R-order A'. Suppose L is a right A-module and L' a finitely generated right A'-submodule of L such that $L'K = L$. Then every A'-endomorphism of L' can be extended uniquely to an A-endomorphism of L, and under this embedding* $\mathrm{End}_{A'}(L')$ *is an R-order in* $\mathrm{End}_A(L)$.

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We include the following result [2, Theorem 2.2] giving a basis and structure constants for $E = eKGe$.

(0.2) THEOREM. *Assume G, H, ψ are as above. If $\{g_i\}$ is a set of representatives of the distinct (H,H)-double cosets HgH for which $\psi^g = \psi$ on $H^{(g)}$, then the set of $a_i = (\text{ind } g_i)eg_i e$ is a basis for E. Moreover if $a_i a_j = \sum_k \alpha_{ijk} a_k$, then the constants of structure α_{ijk} are all algebraic integers in K.*

Recall that $H^{(g)} = g^{-1}Hg \cap H$, ind $g = [H : H^{(g)}]$, and $\psi^g(h^g) = \psi(h)$ for $h \in H$, $g \in G$.

1. Modular endomorphism ring

Clearly N is an (E, KG)-bimodule. By Theorem 0.2, the set E' of all R-linear combinations of the elements $\{a_i\}$ is an R-order in E. Our aim is to reverse the idea of Proposition 0.1 and identify in N a right RG-module N' whose endomorphism ring is E'.

(1.1) LEMMA. *Let $G = \bigcup Hx_i$ (disjoint), $x_1 = 1$. Then $N = eKG$ has a K-basis $\mathcal{B} = \{ex_i\}$. Let $N' = \sum Rex_i$, i.e., N' is all R-linear combinations of elements of $\mathcal{B}$. Then restricting the domains of operators on the left and right to E' and RG respectively, N' is an (E', RG)-bimodule.*

Proof. Since $G = \bigcup Hx_i$ and $N = eKG$, $M^G \simeq N = \sum Mx_i$ (direct sum). But $M = eKH = K \cdot e$ because M is one dimensional. Hence $N = \sum K \cdot ex_i$ (direct sum) and $\mathcal{B}$ is a K-basis. Now suppose $g \in G$ and $ex_i \in \mathcal{B}$. Write $x_i g = hx_j$ for $h \in H$; then

$$(ex_i)g = e(x_i g) = e(hx_j) = (eh)x_j = \psi(h) \cdot ex_j \in R \cdot ex_j \subset N'.$$

Thus N' is a right RG-module. Finally we compute the action of E' on elements of $\mathcal{B}$. For $f \in E'$, if $f(e) \in N'$ then $f(ex_i) = f(e)x_i \in N'$ since N' is a right RG-module. Hence it suffices to check what elements of E' do to e. Observe that $E = eKGe$ acts on N by left multiplication. Suppose $a_j \in E'$ is an R-basis element. Then $a_j = (\text{ind } g)ege$ where we write $g_j = g$ to simplify notation. Consider $a_j e = (\text{ind } g)ege^2 = a_j$. Then $a_j \in eKG = N$. Let $H = \bigcup H^{(g)}h_k$ (disjoint). Now $e = |H|^{-1} \sum_{h \in H} \psi(h^{-1})h$ so that

$$\begin{aligned}
a_j &= (\text{ind } g)eg\{|H|^{-1} \textstyle\sum_{h \in H} \psi(h^{-1})h\} \\
&= |H^{(g)}|^{-1} eg\{\textstyle\sum_{h \in H} \psi(h^{-1})h\} \\
&= |H^{(g)}|^{-1} eg\{\textstyle\sum_{h \in H^{(g)}, k} \psi(h_k^{-1} h^{-1}) hh_k\} \\
&= |H^{(g)}| \{\textstyle\sum_{h \in H^{(g)}, k} \psi(h_k^{-1})\psi(h^{-1}) eghh_k\} \\
&= |H^{(g)}|^{-1} \{\textstyle\sum_{h \in H^{(g)}, k} \psi(h_k^{-1})\psi(h^{-1}) eh^{g^{-1}} gh_k\} \\
&= |H^{(g)}|^{-1} \{\textstyle\sum_{h \in H^{(g)}, k} \psi(h_k^{-1})\psi(h^{-1})\psi(h^{g^{-1}}) egh_k\} \\
&\qquad \qquad \qquad \qquad \qquad \qquad (\text{since } h^{g^{-1}} \in H) \\
&= \textstyle\sum_k \psi(h_k^{-1}) egh_k \{|H^{(g)}|^{-1} \sum_{h \in H^{(g)}} \psi(h^{-1})\psi^g(h)\}.
\end{aligned}$$

But $\psi^g = \psi$ on $H^{(g)}$ so by the usual orthogonality relations,

$$|H^{(g)}|^{-1} \sum_{h \in H^{(g)}} \psi(h^{-1})\psi^g(h) = 1.$$

Thus

$$a_j e = a_j = \sum_k \psi(h_k^{-1}) egh_k \in N' \tag{1.2}$$

since each $\psi(h_k^{-1}) \in R$ and each $egh_k \in N'$. Now E' is generated over R by the set $\{a_j\}$, so (1.2) shows that N' is a left E'-module. Clearly then N' is an (E', RG)-bimodule, as desired.

Observe that $N' = eRG$ is a subset of eKG, and N' is independent of the choice of coset representatives. We will assume that N' and E' (see the beginning of this section) are fixed in what follows.

(1.3) LEMMA. *N' is a faithful left E'-module.*

Proof. Suppose $f \in E' \subset E = \mathrm{End}_{KG}(N)$. If $fN' = 0$ then $0 = K \cdot fN' = f \cdot KN' = fN$ so $f = 0$ since N is clearly a faithful left E-module. This proves the lemma.

Let θ be any RG-endomorphism of N'. Then by (0.1) there exists a unique KG-endomorphism θ^N of N which extends θ such that $\theta^N(kn) = k\theta(n)$ for $k \in K$, $n \in N'$.

(1.4) LEMMA. *Let $\theta \in \mathrm{End}_{RG}(N')$ and write $\theta^N = \sum_{j \in J} \beta_j a_j$, $\beta_j \in K$, where θ^N is the extension of θ to a KG-endomorphism of N. Then each $\beta_j \in R$ so that $\theta^N \in E'$.*

Proof. Since θ^N extends θ and $e \in N'$,

$$\sum_{j \in J} \beta_j a_j = \sum_{j \in J} \beta_j a_j(e) = \theta^N(e) \in N'.$$

By (0.2) the support of a_j lies in the double coset Hg_jH (viewing elements in KG as functions from G to K). For $i \neq j$ the support of a_i is disjoint from the support of a_j. Thus in examining $\sum_{j \in J} \beta_j a_j$ we need only consider one (H, H)-double coset at a time. Let j be fixed and write $g = g_j$. By (1.2), we have $a_j = \sum_k \psi(h_k^{-1}) egh_k$ where $H = \bigcup H^{(g)} h_k$ (disjoint). For each k write $gh_k = d_k x_{i(k)}$, $d_k \in H$, where $G = \bigcup Hx_i$ (disjoint) as in Lemma 1.1. (Then we also know that $\mathcal{B} = \{ex_i\}$ is an R-basis for N'.) We then obtain

$$a_j = \sum_k \psi(h_k^{-1}) egh_k = \sum_k \psi(h_k^{-1})\psi(d_k) ex_{i(k)}.$$

Now $Hx_{i(k)} = Hx_{i(m)}$ implies $Hgh_k = Hgh_m$ which implies h_k and h_m are in the same right coset of $H^{(g)}$, so $k = m$; $k \to i(k)$ is therefore one-to-one. Now since $\mathcal{B}$ is an R-basis for N' and $\sum_j \beta_j a_j \in N'$ the above formula implies that $\beta_j a_j \in N'$ for each j. Clearly

$$\beta_j a_j = \sum_k \beta_j \psi(h_k^{-1})\psi(d_k) ex_{i(k)},$$

so since $\mathcal{B} = \{ex_i\}$ is an R-basis for N' and $k \to i(k)$ is one-to-one, each $\beta_j \psi(h_k^{-1})\psi(d_k) \in R$. But $\psi(h_k^{-1})\psi(d_k)$ is a unit in R for each k, and so each $\beta_j \in R$. This shows that $\theta^N \in E'$, and proves the lemma.

The preceding lemmas combine to prove the following:

(1.5) THEOREM. $E' \cong \mathrm{End}_{RG}(N')$.

Proof. For $f \in E' \subset E$ define the restriction $f_{N'}$ of f to N'. By (1.1), $f_{N'} \in \mathrm{End}_{RG}(N')$. Lemma 1.3 implies that $f \rightarrow f_{N'}$ is a monomorphism. Finally (1.4) shows that the mapping is onto $\mathrm{End}_{A'}(N')$. This proves the theorem.

For the remainder of the paper we set $E'' = E'/PE'$, the P-residue class algebra of E', and $N'' = N'/PN'$.

It is obvious that N'' is an (E'', FG)-bimodule since $FG \simeq RG/PG$. The following allows us to identify E'' as a subalgebra of $\mathrm{End}_{FG}(N'')$.

(1.6) LEMMA. *N'' is a faithful left E''-module.*

Proof. Since R is a principal ideal domain, $P = \pi R$ for some $0 \neq \pi \in R$. Thus $E'' = E'/\pi E'$, $N'' = N'/\pi N'$, etc. Suppose $\theta + \pi E' \in E''$ with $\theta \in E' \subset E$ and assume $(\theta + \pi E')N'' = 0$, i.e., $\theta N' \subset \pi N'$. Consider $\pi^{-1}\theta \in E$. Then $(\pi^{-1}\theta)N' \subset \pi^{-1}(\pi N') = N'$ so by the proof of (1.4), $\pi^{-1}\theta \in E'$. But then $\theta = \pi(\pi^{-1}\theta) \in \pi E'$ so $\theta + \pi E' = 0$ in E''. We conclude that N'' is faithful.

(1.7) COROLLARY. *There is an algebra monomorphism of E'' into $\mathrm{End}_{FG}(N'')$.*

We wish to know the structure of $\mathrm{End}_{FG}(N'')$ in order to examine the structure of N''. In particular we would like to know when the monomorphism of (1.7) is actually an isomorphism. This is just a dimensionality problem which we proceed to settle.

Since ψ defined on H has values in R we can consider the residue class function $\varphi : H \rightarrow F^*$ defined by $\varphi(h) = \psi(h) + P$. (Each $\psi(h)$ is a unit in R so $\psi(h) \notin P$ for all $h \in H$.) Clearly φ is a linear representation of H in $F = R/P$. Moreover M'' is a right FH-module which affords the representation φ defined above, where $M' = eRH$ and $M'' = M'/PM'$.

(1.8) LEMMA. *As right FG-modules, $(M'')^G \cong N''$.*

Proof. By definition, $(M'')^G = M'' \otimes_{FH} FG$. Define

$$f : M'' \times FG \rightarrow N''$$

via $f(re + PM', a) = (re + PN')a$, $r \in R$, $a \in FG$. (Recall that $M' = R \cdot e$.) This is well defined since $PM' \subset PN'$. Clearly f is FH-balanced. Thus there is an FG-homomorphism

$$\hat{f} : M'' \otimes_{FH} FG \rightarrow N''.$$

But N'' is generated over FG by $e + PN' = \hat{f}(e + PM' \otimes 1)$ so $\hat{f}$ is an epimorphism. Finally since M'' is one dimensional over F, the dimension $((M'')^G:F)$ is just $[G:H]$ which in turn is the dimension $(N'':F)$. Thus $\hat{f}$ is an isomorphism.

(1.9) COROLLARY. *The F-dimension of* $\operatorname{End}_{FG}(N'')$ *is the number of* (H,H)-*double cosets* HgH *in* G *such that* $\varphi^g = \varphi$ *on* $H^{(g)}$.

Proof. Since $N'' \cong (M''^G)$ by (1.8) and M'' has character φ we may apply the Intertwining Number Theorem [3, (44.5)] to obtain the desired result.

(1.10) THEOREM. *The following statements are equivalent*:

(a) *The F-algebras* E'' *and* $\operatorname{End}_{FG}(N'')$ *are isomorphic.*

(b) *For each* $g \in G$, *if* $\varphi^g = \varphi$ *on* $H^{(g)}$ *then* $\psi^g = \psi$ *on* $H^{(g)}$.

(c) *For each* $g \in G$, *if* $\psi^g = \psi$ *on the p-regular elements of* $H^{(g)}$, *then* $\psi^g = \psi$ *on* $H^{(g)}$.

Proof. By (1.7), $E'' \simeq \operatorname{End}_{FG}(N'')$ if and only if the dimensions $(E'':F)$ and $(\operatorname{End}_{FG}(N''):F)$ are equal. But $(E'':F) = (E:K)$ and by (0.2) this is the number of double cosets HgH for which $\psi^g = \psi$ on $H^{(g)}$. Clearly $\psi^g = \psi$ on $H^{(g)}$ implies $\varphi^g = \varphi$ on $H^{(g)}$, so that by Corollary 1.9, $E'' \simeq \operatorname{End}_{FG}(N'')$ if and only if (b) holds.

Let m be the p'-part of $|H|$. By assumption, K contains a primitive m^{th} root of unity (contained also in R) which reduces modulo P to a primitive m^{th} root of unity in F. Moreover $w \leftrightarrow w + P$ is a group isomorphism of m^{th} roots of unity between K and F.

Assume first that $\varphi^g = \varphi$ on $H^{(g)}$. If $h \in H^{(g)}$ is p-regular then $\psi^g(h)$ and $\psi(h)$ are m^{th} roots of units in K such that

$$\psi^g(h) + P = \varphi^g(h) = \varphi(h) = \psi(h) + P.$$

By the isomorphism $w \leftrightarrow w + P$ we conclude that $\psi^g(h) = \psi(h)$. Therefore $\psi^g = \psi$ on the p-regular elements of $H^{(g)}$. On the other hand assume $\psi^g = \psi$ on the p-regular elements of $H^{(g)}$. Choose any $h \in H^{(g)}$ and write $h = h_1 h_2$ where h_1 is p-regular and h_2 is p-singular. Since both φ^g and φ are homomorphisms of $H^{(g)}$ into F and F has characteristic p, both contain the p-singular elements in their kernels. Therefore

$$\varphi^g(h) = \varphi^g(h_1) = \psi^g(h_1) + P = \psi(h_1) + P = \varphi(h_1) = \varphi(h),$$

so

$$\varphi^g = \varphi \quad \text{on} \quad H^{(g)}.$$

This proves the equivalence of (b) and (c)

(1.11) COROLLARY. *If p is relatively prime to* $|H|$ *then* $E'' \simeq \operatorname{End}_{FG}(N'')$.

(1.12) COROLLARY. *If H is a p-group then* $E'' \simeq \operatorname{End}_{FG}(N'')$ *if and only if* $\psi^g = \psi$ *on* $H^{(g)}$ *for all* $g \in G$.

Examples. Let G be a group and suppose h is an element of G of order p. Let H be the subgroup of G generated by h and assume $C_G(H) = N_G(H)$, where $C_G(H)$ and $N_G(H)$ are the centralizer and normalizer of H, respectively. Then for each $g \in G$ either $h^g = h$ or $H^{(g)} = \{1\}$. (Note that $C_G(H) = N_G(H)$ if G is a p-group.) Thus for ψ *any* linear KH-character, $\psi^g = \psi$ on $H^{(g)}$ for all $g \in G$. Note that the corresponding FH-character φ is the 1-character since H is a p-group. Thus there may be many KH-characters ψ which reduce to the same FH-character φ.

Now let G be the dihedral group of order 8, H the cyclic normal subgroup of order 4. Let χ be the irreducible KG-character of degree 2. Then $\chi_H = \psi + \psi^g$ for ψ some linear character of H and $g \in G$, $g \notin H$. Moreover $\psi^g \neq \psi$. By Corollary 1.12, $E'' \ncong \mathrm{End}_{FG}(N'')$ for p equal to 2, since in this case H is a 2-group.

We show, as a sort of converse to the preceding development, that if we start with a linear representation φ of H in F there is a representation ψ of H in K such that ψ reduces modulo P to φ and which satisfies the compatibility condition (c) of Theorem 1.10.

(1.13) PROPOSITION. *Let φ be a linear FH-character. Then there exists a linear KH-character ψ such that $\psi(h) + P = \varphi(h)$ for all $h \in H$ and which satisfies condition* (c) *in* (1.10).

Proof. Let H' be the derived group of H and write $H/H' = H_1 \oplus H_2$ where $|H_1|$ is prime to p and $|H_2|$ is a power of p. Since φ is a linear character of H, φ factors through H/H'. Also $\varphi(h_2) = 1$ for all $h_2 \in H_2$ since H_2 is a p-group and F has characteristic p. The elements of H_1 are all p-regular, so to each $h_1 \in H_1$ we correspond $\psi(h_1) \in K$ uniquely defined by $\psi(h_1) + P = \varphi(h_1)$. (See the proof of Theorem 1.10.) Since $w \leftrightarrow w + P$ is a group-isomorphism between the $|H_1|^{\text{th}}$ roots of unity in K and F, $\psi : H_1 \to K$ is a homomorphism. This pulls back to a homomorphism $\psi : H \to K$ in the natural way. Clearly ψ is determined by what it does to the p'-elements of H, so if $\psi^g = \psi$ on the p-regular elements of $H^{(g)}$, then $\psi^g = \psi$ on $H^{(g)}$. Notice that $\psi(h) + P = \varphi(h)$ for all $h \in H$ by construction, concluding the proof.

The reduction to the residue class algebras given above enable us to examine the representation induced from a linear representation of H at characteristic p by looking at the corresponding situation at characteristic zero: Proposition 1.13 shows how to construct a suitable representation at characteristic zero, and Theorem 0.2 gives the structure of the endomorphism ring.

2. Modular decomposition theory

Throughout this section we assume the hypotheses and notation of Section 1.

(2.1) LEMMA. *$M''(N'')$ is isomorphic to a right ideal in FH* (*respectively* FG).

Proof. Let $x = \sum_{h \in H} \varphi(h^{-1})h \in FH$. Since φ is linear, $xFH = xF$ and is isomorphic to M''. Similar to the proof of (1.8) we have that $xFG \cong (xFH)^G \cong (M'')^G \cong N''$.

(2.2) THEOREM. *The right FG-module N'' is FG-projective if and only if p is relatively prime to $|H|$.*

Proof. If p is relatively prime to $|H|$ then $\varepsilon = |H|^{-1}x$ is an idempotent in FH—where x is defined in the proof of (2.1)—and $\varepsilon FH = xFH \cong M''$. But then $N'' \cong (M'')^G \cong \varepsilon FH$ which is FG-projective. Conversely suppose N'' is FG-projective. Then N'' is an FG-direct summand of a finitely generated free FG-module, say $N'' \oplus N_1 \cong \sum_\alpha FG$ (direct sum). But $(FG)_H$ is a free KH-module so that $(N'')_H \oplus (N_1)_H \cong \sum_\beta FH$ (direct sum). Thus $(N'')_H$ is FH-projective. By [3, (63.6)], M'' is an FH-direct summand of $(N'')_H \cong ((M'')^G)_H$. Hence M'' is FH-projective. Now FH is quasi-Frobenius so M'' is FH-injective [3, (58.14)], and since M'' is isomorphic to the right ideal xFH of FH as in the proof of (2.1) we may conclude that $xFH = xF$ is a direct summand of FH. Thus there is a non-zero idempotent ε in xF such that $\varepsilon FH = xF$, say $\varepsilon = x\alpha$ for some $\alpha \in F$. But then

$$x\alpha = \varepsilon = \varepsilon^2 = x^2\alpha^2 = |H|\cdot x\alpha^2$$

(since $x^2 = |H|\cdot x$) and $\varepsilon \neq 0$ so $|H| \neq 0$ in F. Thus p is relatively prime to $|H|$. This proves the lemma.

We next consider the relationships between the various E-, E'-, and E''-modules. For convenience, define $A = KG$, $A' = RG$, $A'' = A'/PA' \cong FG$. Recall that, by hypothesis, R is a complete local ring. The point of view here is influenced by Swan [4], and roughly parallels the theory for group-algebras.

Let ${}_{E''}\mathcal{P}$ and ${}_{E'}\mathcal{P}$ denote the categories of finitely generated projective left E''- and E'-modules respectively, ${}_E\mathfrak{M}$ and ${}_{E''}\mathfrak{M}$ the categories of all finitely generated left E- and E''-modules respectively. Similarly define ${}_{A''}\mathcal{P}$, etc. For S any ring and $\mathfrak{N}$ a category of S-modules let $\mathcal{G}(\mathfrak{N})$ denote the Grothendieck group of $\mathfrak{N}$, i.e., the abelian group generated by all $[T]$ with $T \in \mathfrak{N}$ and with relations $[T] = [U] + [V]$ whenever there is an exact sequence $0 \to U \to T \to V \to 0$ of S-modules in $\mathfrak{N}$.

(2.3) *Construction.* Consider the (E'', A'')-bimodule N''. Then $(N'')^* = \mathrm{Hom}_{A''}(N'', A'')$ is an (A'', E'')-bimodule. Moreover the functors

$$U'' = N'' \otimes_{A''} \_\_ \quad \text{and} \quad V'' = (N'')^* \otimes_{E''} \_\_$$

take A''-modules to E''-modules and E''-modulus to A''-modules respectively. Similarly define $U = N \otimes_A \_\_$ and $V = N^* \otimes_E \_\_$ where $N^* = \mathrm{Hom}_A(N, A)$.

Consider the rectangle

$$\mathbf{T}' \qquad \begin{array}{ccc} \mathcal{G}({}_{E''}\mathcal{P}) & \xrightarrow{c'} & \mathcal{G}({}_{E''}\mathfrak{M}) \\ {\scriptstyle e_1'}\big\downarrow & & \big\uparrow{\scriptstyle d'} \\ \mathcal{G}({}_{E'}\mathcal{P}) & \xrightarrow[e_2']{} & \mathcal{G}({}_E\mathfrak{M}) \end{array}$$

defined as follows:

(a) For $Q \in {}_{E''}\mathcal{P}$ define $c'([Q])$ to be $[Q]$, viewed as an element of $\mathcal{G}({}_{E''}\mathfrak{M})$.

(b) For $Q \in {}_{E''}\mathcal{P}$ let Q' be the projective E'-module such that $Q' + PQ' \cong Q$ (see [3, (77.11)]), and define $e_1'([Q]) = [Q']$.

(c) For $Q' \in {}_{E'}\mathcal{P}$ define $e_2'([Q']) = [K \otimes_R Q']$, an element of $\mathcal{G}({}_E\mathfrak{M})$.

(d) For $L \in {}_E\mathfrak{M}$ let L' be any R-free order E'-module contained in L, and define $d'([L]) = [L' + PL']$ in $\mathcal{G}({}_{E''}\mathfrak{M})$.

The above maps are all well defined and the rectangle commutes. (See [3, Chapter 12]).

Now consider the rectangle

T
$$\begin{array}{ccc} \mathcal{G}({}_{A''}\mathcal{P}) & \xrightarrow{c} & \mathcal{G}({}_{A''}\mathfrak{M}) \\ {\scriptstyle e_1}\downarrow & & \uparrow{\scriptstyle d} \\ \mathcal{G}({}_{A'}\mathcal{P}) & \xrightarrow[e_2]{} & \mathcal{G}({}_{A}\mathfrak{M}) \end{array}$$

defined analogously.

We attempt to relate the rectangles **T** and **T**$'$ using the functors U'', V'', U, and V. First note that $N = eA$ and so $N^* \cong Ae$. (Therefore N^* is left A-projective.) Moreover $U(L) = N \otimes {}_AL \cong eA \otimes_A L \cong eL$ as left E-modules, $E = eAe$. If we define $\hat{U}$ from $\mathcal{G}({}_A\mathfrak{M})$ to $\mathcal{G}({}_E\mathfrak{M})$ via $\hat{U}([L]) = [eL] = [U(L)]$ we have an epimorphism [2, Theorem 1.1] of abelian groups. Moreover $\hat{V}$ from $\mathcal{G}({}_E\mathfrak{M})$ to $\mathcal{G}({}_A\mathfrak{M})$ via $\hat{V}([L]) = [V(L)]$ is a splitting map for $\hat{U}$; i.e., the composition $\hat{U} \circ \hat{V}$ is the identity map of $\mathcal{G}({}_E\mathfrak{M})$.

To relate the rectangles **T** and **T**$'$ further we desire to factor the map c' in some way through c, using the functors U'' and V''.

(2.4) LEMMA. *The following are equivalent:*

(a) *The functor V'' takes projective E''-modules to projective A''-modules.*

(b) *The functor U'' takes exact sequences of A''-modules to exact sequences of E''-modules.*

(c) *N'' is right A''-projective.*

(d) *N'' is right A''-flat.*

(e) *The prime p is relatively prime to $|H|$.*

Proof. The equivalence of (c) through (e) follows from Theorem 2.2 and the fact that in Artinian rings flat is equivalent to projective (see [1, Theorem 3.3 (c)]). The equivalence of (b) and (d) is by definition. Now suppose V'' takes projective E''-modules to projective A''-modules. Then in particular $V''(E'') = (N'')^* \otimes_{E''} E'' \cong (N'')^*$ is left A''-projective. But then $N'' \cong ((N'')^*)^*$ is right A''-projective. Thus (a) implies (c). Conversely suppose N'' is right A''-projective. Then by (2.2), p is relatively prime to $|H|$ and so $\varepsilon = e + PA'$ is an idempotent in A'', and $\varepsilon A'' \cong N''$. Thus $E'' \cong \varepsilon A''\varepsilon$ and $(N'')^* \cong A''\varepsilon$. Let f be an idempotent in $E'' = \varepsilon A''\varepsilon$ (by identification). Then

$$V''(E''f) = (N'')^* \otimes_{E''} E''f \cong A''\varepsilon \otimes_{\varepsilon A''\varepsilon} (\varepsilon A''\varepsilon)f \cong A''f,$$

which is clearly A''-projective. Thus (c) implies (a). This proves the theorem.

(2.5) THEOREM. *The maps*

$$\hat{U}'' : \mathcal{G}({}_{A''}\mathfrak{M}) \to \mathcal{G}({}_{E''}\mathfrak{M}) \quad \text{and} \quad \hat{V}'' : \mathcal{G}({}_{E''}\mathcal{P}) \to \mathcal{G}({}_{A''}\mathcal{P})$$

given by $\hat{U}''([L]) = [U''(L)]$ *and* $\hat{V}''([Q]) = [V''(Q)]$ *are well defined if and only if* p *is relatively prime to* $|H|$, *and in this case the following diagram commutes*:

$$\begin{tikzcd}
\mathcal{G}({}_{A''}\mathcal{P}) \arrow[rrr, "c"] \arrow[dr, "e_1"] & & & \mathcal{G}({}_{A''}\mathfrak{M}) \arrow[ddd, "\hat{U}''"] \\
& \mathcal{G}({}_{A''}\mathcal{P}) \arrow[r, "e_2"] & \mathcal{G}({}_{A}\mathfrak{M}) \arrow[ur, "d"] & \\
& \mathcal{G}({}_{E'}\mathcal{P}) \arrow[r, "e_2'"'] & \mathcal{G}({}_{E}\mathfrak{M}) \arrow[u, "\hat{V}"] \arrow[dr, "d'"'] & \\
\mathcal{G}({}_{E''}\mathcal{P}) \arrow[uuu, "\hat{V}''"] \arrow[ur, "e_1'"] \arrow[rrr, "c'"'] & & & \mathcal{G}({}_{E''}\mathfrak{M})
\end{tikzcd}$$

Proof. The first part of the theorem follows immediately from (2.4). Commutativity is easy to check.

One can interpret Theorem 2.5 as giving some information about the transformations c' and d' in terms of the corresponding transformations c and d. One can also use the above relationships to obtain information about the block decomposition of E'' in terms of the decomposition in A''.

(2.6) COROLLARY. *Let* $M_1, \cdots, M_r$ *and* $N_1, \cdots, N_s$ *be complete sets of (non-isomorphic) simple modules in* ${}_{A}\mathfrak{M}$ *and* ${}_{A''}\mathfrak{M}$, *respectively, arranged so that*

$$U(M_1), \cdots, U(M_{r'}) \quad \text{and} \quad U''(N_1), \cdots, U''(N_{s'})$$

are complete sets of simple modules in ${}_{E}\mathfrak{M}$ *and* ${}_{E''}\mathfrak{M}$, *respectively. Then*

$$[M_1], \cdots, [M_r] \quad \text{and} \quad [N_1], \cdots, [N_s]$$

are bases for $\mathcal{G}({}_{A}\mathfrak{M})$ *and* $G({}_{A''}\mathfrak{M})$, *respectively, and if*

$$d[M_i] = \textstyle\sum_{j=1}^{s} d_{ij}[N_j] \qquad (1 \leqq i \leqq r)$$

then

$$d'[U(M_i)] = \textstyle\sum_{j=1}^{s'} d_{ij}[U''(N_j)] \qquad (1 \leqq i \leqq r').$$

Proof. By (2.5), $d' = \hat{U}''d\hat{V}$, so that for $1 \leqq i \leqq r'$,

$$\begin{aligned} d'[U(M_i)] &= \hat{U}''d\hat{V}[U(M_i)] \\ &= \hat{U}''d[VU(M_i)] \\ &= \hat{U}''d[M_i] \end{aligned}$$

$$= \hat{U}'' \sum_{j=1}^{s} d_{ij}[N_j]$$
$$= \sum_{j=1}^{s'} d_{ij}[U''(N_j)].$$

Here one checks that $VU(M_i) \simeq M_i$ for $1 \leqq i \leqq r'$, and that $U''(N_j) = 0$ unless $1 \leqq j \leqq s'$.

(2.7) Corollary. *Let $N_1, \cdots, N_s$ be as in (2.6), and let $Q_1, \cdots, Q_s$ be a complete set of (non-isomorphic) indecomposable modules in ${}_{A''}\mathcal{P}$, arranged so that $Q_i/JQ_i \simeq N_i$ for $1 \leqq i \leqq s$ ($J = J(A'')$). Then $U''(Q_1), \cdots, U''(Q_{s'})$ is a complete set of indecomposable modules in ${}_{E''}\mathcal{P}$, and if*

$$c[Q_i] = \sum_{j=1}^{s} c_{ij}[N_j] \qquad (1 \leqq i \leqq s)$$

then

$$c'[U''(Q_i)] = \sum_{j=1}^{s'} c_{ij}[U''(N_j)] \qquad (1 \leqq i \leqq s').$$

Proof. Similar to the proof of (2.6).

Example. Let $G = D_6$, the dihedral group of order 12, generated by elements a and b with relations $a^6 = b^2 = baba = 1$. Let $H = \{1, b\}$, a subgroup of order 2. For K take $\mathbf{Q}(w)$, w a primitive 12^{th} root of 1, so that K is a splitting field for KG. Finally, let $p = 3$. Then $|H| = 2$, and 2 is prime to 3 so the theorems of this section apply.

Let ψ be the 1-character of H; then $e = 1/2(1 + b)$ is the idempotent in KH which corresponds to ψ, and $eKGe = E$ is the endomorphism ring of eKG.

$A = KG$ has 6 simple left modules, say $M_1, \cdots, M_6$, four of dimension one and two of dimension two. Of these, two one-dimensional modules, say M_1 and M_2, and both two-dimensional modules, say M_3 and M_4, map to simple E-modules under U. Now $A'' = FG$ has four simple modules, say $N_1, \cdots, N_4$, all of which are one dimensional, and all are "reduced" from the one-dimensional left A-modules M_1, M_2, M_5, M_6, at characteristic zero. Of these, two map to simple E''-modules under U'', namely N_1 and N_2 (those reduced from M_1 and M_2). Let $Q_1, \cdots, Q_4$ be the indecomposable projective left A''-modules, arranged so that $Q_i/JQ_i \simeq N_i$ for $1 \leqq i \leqq 4$, where $J = J(A'')$.

Matrices for the maps c and d may be given as follows:

$$\mathrm{d} = \begin{array}{c} \\ M_1 \\ M_2 \\ M_3 \\ M_4 \\ M_5 \\ M_6 \end{array} \begin{array}{c} \begin{array}{cccc} N_1 & N_2 & N_3 & N_4 \end{array} \\ \left[\begin{array}{cc|cc} 1 & 0 & & \\ 0 & 1 & & \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \hline & & 1 & 0 \\ & & 0 & 1 \end{array}\right] \end{array}$$

$$\mathbf{c} = \begin{array}{c} \\ Q_1 \\ Q_2 \\ \\ Q_3 \\ Q_4 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} N_1 & N_2 & N_3 & N_4 \end{array} \\ \left[\begin{array}{cc|cc} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ \hline 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{array}\right] \end{array}.$$

By (2.6) and (2.7), the corresponding matrices for $\mathbf{c}'$ and $\mathbf{d}'$ are merely the upper left-hand submatrices of the matrices for $\mathbf{c}$ and $\mathbf{d}$—we have been careful to arrange the modules so that they appear in the proper order required by (2.6) and (2.7).

$$\mathbf{d}' = \begin{array}{c} \\ eM_1 \\ eM_2 \\ eM_3 \\ eM_4 \end{array}\!\!\begin{array}{c} \begin{array}{cc} eN_1 & eN_2 \end{array} \\ \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{array}\right] \end{array}$$

$$\mathbf{c}' = \begin{array}{c} \\ eQ_1 \\ eQ_2 \end{array}\!\!\begin{array}{c} \begin{array}{cc} eN_1 & eN_2 \end{array} \\ \left[\begin{array}{cc} 2 & 0 \\ 0 & 2 \end{array}\right] \end{array}$$

Here eL denotes $U(L)$ or $U''(L)$, whichever is appropriate.

Observe from the matrix for $\mathbf{c}'$, that E'' has exactly two blocks. One checks that $(E:K) = (E'':F) = 4$, so each block of E'' consists of a single indecomposable projective, and that each such indecomposable projective has exactly two one-dimensional composition factors (which are isomorphic).

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