HOMOMORPHISMS OF PRINCIPAL FIBRATIONS: APPLICATIONS TO CLASSIFICATION, INDUCED FIBRATIONS, AND THE EXTENSION PROBLEM

Dedicated to the memory of Tudor Ganea

BY

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A fibration

$$F \xrightarrow{i} E \xrightarrow{p} B$$

is called principal if there exists an associative multiplication

$$\mu : F \times F \to F$$

and an associative action

$$\varphi : F \times E \to E$$

such that the following diagram commutes.

$$\text{(I)} \qquad \begin{array}{ccccc} F \times F & \xrightarrow{1 \times i} & F \times E & \xrightarrow{* \times p} & B \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \varphi} & & \| \\ F & \xrightarrow{i} & E & \xrightarrow{p} & B \end{array}$$

A fibre preserving map

$$g : (\mu, \varphi, F \to E \to B) \to (\mu', \varphi', F' \to E' \to B')$$

between principal fibrations is called a homomorphism if

$$\text{(II)} \qquad \begin{array}{ccc} F \times E & \xrightarrow{(g|F) \times g} & F' \times E' \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi'} \\ E & \xrightarrow{g} & E' \end{array}$$

commutes.

Principal fibrations and their homomorphisms are easily seen to form a category. The Dold-Lashof construction [2] is a functor from this category to the category of universal principal quasifibrations and their homomorphisms.

Homotopy commutativity of diagram (II) is not sufficient to ensure the existence of a map between the associated universal quasifibrations. However one is able to give higher homotopy conditions which, if satisfied, permit

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the construction of such a map. Maps satisfying these higher homotopy conditions are called strong homotopy homomorphisms.

Two principal fibrations are called multiplicatively equivalent if there is a strong homotopy homomorphism from one to the other which is also a fibre homotopy equivalence. We prove that every principal fibration is multiplicatively equivalent to a fibration induced from a path fibration. Furthermore principal fibrations over X with fibre multiplicatively equivalent to F are classified by $[[X, B_F]]$ where $[[X, B_F]]$ is a quotient of the set of homotopy classes of maps, $X \to B_F$, and B_F is a classifying space for F.

Using the concept of strong homotopy homomorphism we give solutions to the following two classical problems of algebraic topology.

I. Given fibrations

$$F \xrightarrow{i} E \xrightarrow{p} B \quad \text{and} \quad F' \xrightarrow{i'} E' \xrightarrow{p'} B'$$

when does there exist $f : B \to B'$ such that

$$E \xrightarrow{p} B$$

is fibre homotopy equivalent to the fibration induced by f from

$$E' \xrightarrow{p'} B'?$$

In particular is $E \to B$ induced from a path fibration over B'? This is equivalent to Problem 10 of [10] which asks when an inclusion $A \subset X$ is (up to homotopy equivalence) the inclusion of a fibre into the total space of a fibration.

II. Given $f : A \to Y$ and $X \supset A$ when can f be extended to X?

It seems possible that the solutions given here are related to the work of Husseini [9]; however we do not see a direct translation.

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1. Preliminaries

We work in the intersection of the CW-category and the compactly generated category. In particular we assume the following.

(i) All spaces have the compactly generated topology.
(ii) Products are taken in the sense of this topology.
(iii) All spaces have the homotopy type of CW-complexes.
(iv) All spaces have a base point, $*$, which is an NDR in X.

For details of the compactly generated topology see [14].

Fibration will mean Serre fibration with *connected* base space.

A *principle fibration* is a triple

$$(\mu, \phi, F \xrightarrow{i} E \xrightarrow{p} B)$$

such that:

(i) $F \xrightarrow{i} E \xrightarrow{p} B$ is a fibration.
(ii) Diagram (I) commutes.
(iii) $\mu : F \times F \to F$ satisfies
 (a) μ is associative,
 (b) $\mu(*, x) = \mu(x, *) = x$ for all $x \in X$,
 (c) $\mu(x, \ ) : F \to F$ is a homotopy equivalence for all $x \in F$. (This follows from (b) if F is connected.)
(iv) $\phi : F \times E \to F$ satisfies
 (a) $\phi(\mu \times 1) = \phi(1 \times \phi)$,
 (b) $\phi(*, y) = y$ for all $y \in Y$.

We say that (F, μ) is an associative H-space if it satisfies the conditions of (iii).

2. Strong homotopy homomorphisms

Quasifibrations were defined in [3], principal quasifibrations and the Dold-Lashof construction were defined in [2]. We use a variant of the original construction We sketch the basic construction.

Let

$$(\mu, \varphi, F \xrightarrow{i} E \xrightarrow{p} B)$$

be a principal fibration. Set $E_0 = E$, $B_0 = B$, $i_0 = i$, $p_0 = p$, and $\varphi_0 = \varphi$. Assume inductively that E_n, B_n, i_n, p_n and φ_n have been constructed such that

$$(\mu, \varphi_n, F \xrightarrow{i_n} E_n \xrightarrow{p_n} B_n)$$

is a principal quasifibration. Let CX be the cone on X. Define

$$E_{n+1} = E_n \cup_{\varphi_n} F \times CE_n, \quad B_{n+1} = B_n \cup_{p_n} CE_n,$$

$p_{n+1} \mid E_n = p_n$ and $p_{n+1} \mid (F \times CE_n)$ is projection onto the second factor, i_{n+1} is the obvious inclusion, and

$$\varphi_{n+1} \mid (F \times E_n) = \varphi_n, \quad \varphi_{n+1} \mid (F \times F \times CE_n) = \mu \times 1.$$

Dold and Lashof prove that

$$(\mu, \varphi_{n+1}, F \xrightarrow{i_{n+1}} E_{n+1} \xrightarrow{p_{n+1}} B_{n+1})$$

is a principal quasifibration and that a homomorphism of principal fibrations

$$g : (\mu, \varphi, F \to E \to B) \to (\mu', \varphi', F' \to E' \to B')$$

induces a homomorphism of principal quasifibrations

$$g_{n+1} : (\mu, \varphi_{n+1}, F \to E_{n+1} \to B_{n+1}) \to (\mu', \varphi'_{n+1}, F' \to E'_{n+1} \to B'_{n+1}).$$

Let $D\bar{L}(E) = \bigcup_{n \geq 0} E_n$ and $DL(E) = \bigcup_{n \geq 0} B_n$ be given the weak topology.

Let $DL(p)$, $DL(i)$, $DL(\varphi)$, and $DG(g)$ be the obvious induced maps. By a proof similar to the one given in [13] one shows the following.

LEMMA 1. *Let $(\mu, \varphi, F \to E \to B)$ be a principal fibration.*
(a) *$(\mu, DL(\varphi), F \to D\bar{L}(E) \to DL(E))$ is a principal quasifibration.*
(b) *$D\bar{L}(E)$ is contractible.*

LEMMA 2. *Let $g : (\mu, \varphi, F \to E \to B) \to (\mu', \varphi', F' \to E' \to B')$ be a homomorphism of principal fibrations. Then*

$$DL(g) : (\mu, DL(\varphi), F \to D\bar{L}(E) \to DL(E)) \to (\mu', DL(\varphi'), F' \to D\bar{L}(E') \to DL(E'))$$

is a homomorphism of principal quasifibrations which extends g.

In his work on homotopy commutativity, Sugawara [15] introduced the idea of a strongly homotopy multiplicative map. We generalize this to the case of a principal fibration (or principal quasifibration).

DEFINITION 3.

$$g : (\mu, \varphi, F \to E \xrightarrow{p} B) \to (\mu', \varphi', F' \to E' \xrightarrow{p'} B')$$

is said to be a strong homotopy homomorphism if:
(a) g is a map of fibrations;
(b) there exist maps

$$M_n : F^n \times E \times I^n \to E', \qquad n = 0, 1, 2, \cdots,$$

such that

$$\begin{array}{ccc} F^n \times E \times I^n & \xrightarrow{M_n} & E' \\ \Big\downarrow {\scriptstyle * \times p \times *} & & \Big\downarrow {\scriptstyle p'} \\ * \times B \times * & \xrightarrow{g} & B' \end{array}$$

commutes and
(0) $M_0(y) = g_E(y)$
(1) $M_1(x, y, 0) = g_E\,\varphi(x, y)$, $M_1(x, y, 1) = \varphi'(g_F(x), g_E(y))$
$\vdots$
(n) $M_n(x_1, \cdots, x_n, y, t_1, \cdots, t_n)$

$$= M_{n-1}(x_1, \cdots, \mu(x_i, x_{i+1}), \cdots, x_n, y, t_1, \cdots, \hat{t}_i, \cdots, t_n), \qquad t_i = 0, i < n,$$

$$= M_{n-1}(x_1, \cdots, x_{n-1}, \varphi(x_n, y), t_1, \cdots, t_{n-1}), \qquad t_n = 0,$$

$$= \varphi'(M_{j-1}(x_1, \cdots, ix_j, t_1, \cdots, t_{j-1}), M_{n-j}(x_{j+1}, \cdots, x_n, y, t_{j+1}, \cdots, t_n)), \qquad t_j = 1.$$

LEMMA 4. *If* $g : (\mu, \varphi, F \to E \to B) \to (\mu', \varphi', F' \to E' \to B')$ *is a homomorphism of principal quasifibrations, then g is a strong homotopy homomorphism.*

Proof. Set $M_n(x_1, \cdots, x_n, t, t_1, \cdots, t_n) = g\varphi(x_1 \cdots x_n, y)$ where juxtaposition indicates multiplication in F.

LEMMA 5. *Let* $g : (\mu, \varphi, F \to E \to B) \to (\mu', \varphi', F' \to E' \to B')$ *be a strong homotopy homomorphism of principal quasifibrations; then g extends to a strong homotopy homomorphism*

$$g_1 : (\mu, \varphi_1, F \to E_1 \to B_1) \to (\mu', \varphi_1', F' \to E_1' \to B_1').$$

Proof. Set $(g_1)_{E_1} \mid E = g_E$ and for $(x, t, y) \in F \times CE$ set

$$\begin{aligned}(g_1)_{E_1}(x, t, y) &= g_E(ix), 2t, g_E(y), \quad 0 \le t \le \tfrac{1}{2},\\ &= M_1(x, y, t), \qquad \tfrac{1}{2} \le t \le 1.\end{aligned}$$

Set $(g_1)_{B_1} \mid B = g_B$ and for $(t, y) \in CE$ set

$$\begin{aligned}(g_1)_{B_1}(t, y) &= 2t, g_E(y), \quad 0 \le t \le \tfrac{1}{2},\\ &= g_B(p(y)), \quad \tfrac{1}{2} \le t \le 1.\end{aligned}$$

$M_n^1 \mid F^n \times E \times I = M_n$ and $M_n^1 \mid (F^n \times (F \times CE) \times I^n)$ is defined by

$$\begin{aligned}M_n^1(x_1, &\cdots, x_n, (x, t, y), t_1, \cdots, t_n)\\ &= 2t, \varphi'(M_n(x_1, \cdots, x_n, x, t_1, \cdots, t_n), g_E(y)), \quad 0 \le t \le \tfrac{1}{2}\\ &= M_{n+1}(x_1, \cdots, x_n, x, y, t_1, \cdots, t_n, 2 - 2t), \quad \tfrac{1}{2} \leqq t \leqq 1\end{aligned}$$

A straightforward computation shows that g_1 and M_n^1 have the desired properties.

By iterating the result of Lemma 5, one proves:

COROLLARY 6. *Let* $g : (\mu, \phi, F \to E \to B) \to (\mu', \phi', F' \to E' \to B')$ *be a strong homotopy homomorphism. Then g can be extended to a strong homotopy homomorphism*

$$\begin{aligned}DL(g) : (\mu, DL(\phi), F &\to D\bar{L}(E) \to DL(E)\\ &\to (\mu', DL(\phi'), F' \to D\bar{L}(E') \to DL(E')).\end{aligned}$$

Strictly speaking $DL(g)$ is incorrect notation since $DL(g)$ is dependent upon the choice of the maps $M_1, \cdots, M_n, \cdots$ which give g the structure of a strong homotopy homomorphism. Thus $DL(g)$ is not uniquely defined. For most of our applications the only property required of $DL(g)$ is that it extends g. Thus we shall continue to use the symbol $DL(g)$ to denote some extension of g.

3. Principal fibrations

In [11], J. P. Meyer used "principal fibration" to mean a fibration induced from the path fibration. This definition is equivalent to the one given in this paper under a suitable equivalence relation.

We first define what we mean by "multiplicatively equivalent" and prove that within the category in which we work every principal fibration is multiplicatively equivalent to a principal fibration in the sense of Meyer. We then give two classification theorems for principal fibrations over a fixed space X with fibre multiplicatively equivalent to a given H-space F.

The most important example of a principal fibration is that of the path fibration. Let

$$PY = \{f : [0, r] \to Y \mid r \geq 0, f(0) = *\}.$$

$\pi : PY \to Y$ is given by $\pi(\eta) = \eta(r)$ $(\eta : [0, r] \to Y)$. $\Omega Y \subset PY$ is

$$\{\eta \in PY \mid \pi(\eta) = *\}.$$

Define $\varphi : \Omega Y \times PY \to PY$ by $\varphi(\xi, \eta) = \xi * \eta$ where

$$\begin{aligned}(\xi * \eta)(t) &= \xi(t), & 0 \leq t \leq r,\\ &= \eta(t - r), & r \leq t \leq r + s,\end{aligned}$$

for $\xi : [0, r] \to Y$, $\eta : [0, s] \to Y$. Let $\mu : \Omega Y \times \Omega Y \to \Omega Y$ be the restriction of φ. Thus ΩY is an associative monoid and

$$(\mu, \varphi, \Omega Y \to PY \xrightarrow{\pi} Y)$$

is a principal fibration.

Let $(\mu, \varphi, F \to E \to B)$ be a principal fibration and let $f : B' \to B$. Set

$$E_f = \{(b, e) \in B' \times E \mid f(b') = p(e)\}.$$

Define $\varphi' : F \times E_f \to E_f$ by $\varphi'(x, (b, e)) = (b, \varphi(x, e))$. One easily sees that $(\mu, \varphi', F \to E_f \to B')$ is a principal fibration and

$$\begin{tikzcd} F \arrow[r] \arrow[d, equal] & E_f \arrow[r] \arrow[d] & B' \arrow[d, "f"] \\ F \arrow[r] & E \arrow[r] & B \end{tikzcd}$$

is a homomorphism of principal fibrations.

Definition 7. Two principal fibrations

$$(\mu, \phi, F \to E \to B) \quad \text{and} \quad (\mu', \phi', F' \to E' \to B)$$

are *multiplicatively related* if there is a strong homotopy homomorphism

$$\varepsilon : (\mu, \phi, F \to E \to B) \to (\mu', \phi', F' \to E' \to B)$$

such that $\varepsilon_B = 1_B$ and ε_E is a homotopy equivalence. Two principal fibrations over B are called *multiplicatively equivalent* if they lie in the same equivalence class under the equivalence relation generated by the above relation.

Two associative H-spaces (F, μ) and (F', μ') are *multiplicatively related* if the principal fibrations

$$(\mu, \mu, F \to F \to *) \quad \text{and} \quad (\mu', \mu', F' \to F' \to *)$$

are multiplicatively related.

Drachman [4] and Fuchs [5] have shown that if F and F' have the homotopy type of CW-complexes then multiplicatively related is an equivalence relation among H-spaces. In this case we call (F, μ) and (F', μ') multiplicatively equivalent.

It follows from the results of [1] that within the category in which we work multiplicatively equivalent fibrations are fibre homotopy equivalent. We do not know under what hypothesis multiplicatively related is an equivalence relation.

PROPOSITION 8.
(a) *If*

$$\begin{array}{ccccc} F & \longrightarrow & E & \xrightarrow{p} & B \\ \downarrow{\scriptstyle f_F} & & \downarrow{\scriptstyle f_E} & & \downarrow{\scriptstyle f_B} \\ F' & \longrightarrow & E' & \longrightarrow & B' \end{array}$$

commutes and f_F is a homotopy equivalence then $F \to E \to B$ is fibre homotopy equivalent to the fibration induced from $F' \to E' \to B'$ by f_B.

(b) *If in addition*

$$(\mu, \phi, F \to E \to B) \quad \textit{and} \quad (\mu', \phi', F' \to E' \to B')$$

are principal fibrations and f is a strong homotopy homomorphism then the equivalence of (a) *is a multiplicative equivalence.*

Proof. $E_f = \{(b, e) \in B \times E' \mid f_B(b) = p'(e)\}$. For $x \in E$ set

$$g(x) = (p(x), f_E(x)).$$

Since $f_B\, p(x) = p' f_E(x)$, $g : E \to E_f$ and

$$\begin{array}{ccccc} F & \to & E & \to & B \\ \downarrow{\scriptstyle f_F} & & \downarrow{\scriptstyle g} & & \| \\ F' & \to & E_f & \to & B \end{array}$$

commutes. The results of [1] then imply that g is a fibre homotopy equivalence.

Part (b) follows by observing that g is a strong homotopy homomorphism since f is.

THEOREM 9. *Let $(\mu, \phi, F \to E \to B)$ be a principal fibration.*

(a) *Let $i_B : B \to DL(E)$ be inclusion. $(\mu, \phi, F \to E \to B)$ is multiplicatively equivalent to the principal fibration induced by i_B from the path fibration over $DL(E)$.*

(b) *Let* $F = \Omega Y$ *and let* μ *be the standard multiplication on* ΩY *(see above). There exists a homotopy equivalence* $\varepsilon : DL(E) \to Y$ *such that* $\Omega Y \to E \to B$ *is multiplicatively equivalent to the fibration induced by* εi_B *from the path fibration over* Y.

Proof. Define $m_0 : E \to PB_1$ by setting

$$m_0(x)(s) = (s, x) \in B_1 = B \cup_p CE, \qquad x \in E,\ s \in I.$$

If $x \in F$, $(1, x) \sim p(x) = *$ in B_1. Thus $m_0 \mid F : F \to \Omega B_1$. Let

$$M_0 : E \to PDL(E)$$

be the composite of m_0 with the map induced by the inclusion $B_1 \subset DL(\mathrm{E})$. To prove that M_0 is a strong homotopy homomorphism we must define $\{M_n : F^n \times E \times I^n \to PDL(E)\}$ satisfying Definition 3. To do this we regard

$$B_n = B_{n-1} \cup_{\delta_n} F^n \times E \times I^n$$

where $\delta_n : F^n \times E \times \dot{I}^n \to B_{n-1}$ is given by

$$\begin{aligned}\delta_n(x_1, \cdots, x_n, y, t_1, \cdots, t_n) &= [x_1, \cdots, x_{j-1} * x_j, \cdots, y, t_1, \cdots, \hat{t}_j, \cdots, t_n], & t_j = 1,\\ &= [x_1, \cdots, ix_j, t_1, \cdots, t_{j-1}], & t_j = 0\end{aligned}$$

with the convention that $x_0 * x_1$ means omit x_1 and square brackets indicate equivalence class in B_{n-1}.

Define $m_n : F^n \times E \times I^n \to PB_{n+1}$ by

$$\begin{aligned}&m_n(x_1, \cdots, x_n, y, t_1, \cdots, t_n)(s)\\ &= [x_1, \cdots, x_n, y, s, 1 - t_1, \cdots, 1 - t_n], \qquad 0 \le s \le 1\\ &\ \vdots\\ &= [x_j, \cdots, x_n, y, s - \textstyle\sum_{k=1}^{j} t_k, 1 - t_{j+1}, \cdots, 1 - t_n],\\ &\qquad\qquad 1 + \textstyle\sum_{k=1}^{j-1} t_k \le s \le 1 + \sum_{k=1}^{j} t_k\\ &\ \vdots\\ &= [y, s - \textstyle\sum_{k=1}^{n} t_k], \qquad 1 + \sum_{k=1}^{n-1} t_k \le s \le 1 + \sum_{k=1}^{n} t_k.\end{aligned}$$

Set $M_n : F^n \times E \times I^n \to PDL(E)$ equal to the composite

$$F^n \times E \times I^n \xrightarrow{m_n} PB_{n+1} \subset PB_\infty = PDL(E).$$

For the fibration, $F \to F \to *$, set $DL(F) = B_E$. The maps defined above are denoted by $M_0^F, \cdots, M_n^F, \cdots$ in this case and appear in [4] where it is shown that $M_0^F : F \to \Omega B_E$ is a homotopy equivalence. (See also Fuchs [5] and Sugawara [15].) Let

$$i : (\mu, \mu, F \to F \to *) \to (\mu, \varphi, F \to E \to B)$$

be inclusion. Since i is a homomorphism it induces $DL(i) : B_F \to DL(E)$

which is a homotopy equivalence. $M_0 | F : F \to \Omega DL(E)$ is equal to the composite

$$F \xrightarrow{M_0^F} \Omega B_F \xrightarrow{\Omega DL(i)} \Omega DL(E)$$

and hence is a homotopy equivalence. Thus we have a commutative diagram

$$\begin{array}{ccc} F & \longrightarrow & \Omega DL(E) \\ \downarrow & & \downarrow \\ E & \xrightarrow{M_0} & PDL(E) \\ \downarrow & & \downarrow \\ B & \xrightarrow{i_B} & DL(E) \end{array}$$

with M_0 a strong homotopy homomorphism and $M_0 | F$ a homotopy equivalence. Part (a) then follows from Proposition 8.

To prove (b) consider the inclusion,

$$\begin{array}{ccccc} \Omega Y & \to & \Omega Y & \to & * \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle i} \\ \Omega Y & \to & PY & \to & Y \end{array}$$

As above this induces a homotopy equivalence $DL(i) : B_{\Omega Y} \to DL(PY)$. We prove in the appendix that $DL(PY)$ has Y as a strong deformation retract. Thus there is a homotopy equivalence $B_{\Omega Y} \to Y$. Also the inclusion

$$\begin{array}{ccccc} \Omega Y & \to & \Omega Y & \to & * \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle i'} \\ \Omega Y & \to & E & \to & B \end{array}$$

induces a homotopy equivalence $DL(i') : B_{\Omega Y} \to DL(E)$. Combining these results, there exists a homotopy equivalence $\varepsilon : DL(E) \to Y$. The map induced by ε on the path fibrations is a homomorphism and part (b) follows from (a) and Proposition 8.

Thus we have shown that in the category in which we work every principal fibration is multiplicatively equivalent to a fibration induced from a path fibration.

A stronger result is proven by Fuchs [6] who shows that every principal fibration with fibre ΩY is fibre homotopy equivalent to a fibration induced from Y and the equivalence is given by a (strict) homomorphism.

Let $\mathcal{P}F(X)$ be the set of equivalence classes of principal fibrations over X with fibre multiplicatively equivalent to (F, μ). Let $f, g : X \to Y$. We say f is homotopy equivalent to g ($f \sim\sim g$) if there exists a homotopy equivalence, $\varepsilon : Y \to Y$ such that f is homotopic to εg. In particular $f \sim g$ implies $f \sim\sim g$. Let $[[X, Y]]$ denote the set of these equivalence classes.

THEOREM 10. *The functors* $\mathcal{P}F(\ \ )$ *and* $[[\ \ , B_F]]$ *are naturally equivalent. In particular principal fibrations over* X *with fibre multiplicatively equivalent to* F *are classified up to multiplicative equivalence by* $[[X, B_F]]$.

Proof. Let $T : \mathcal{P}F(\ \ ) \to [[\ \ , B_F]]$ be defined as follows. Given a principal fibration

$$(\mu', \varphi', F' \to E' \to X)$$

with fibre multiplicatively equivalent to (F, μ) let $i_x : X \to DL(E')$ be inclusion and $h' : (F, \mu) \to (F', \mu')$ a strong homotopy homomorphism which is a homotopy equivalence (i.e. a multiplicative equivalence). As usual $DL(h') : B_F \to DL(E')$ is a homotopy equivalence. Let $\varepsilon(h')$ be a homotopy inverse to $DL(h')$. Set

$$T(X)(\mu', \varphi', F' \to E' \to X) = [[\varepsilon(h')i_x]]$$

where $[[\ \ ]]$ indicates equivalence class. We must show that if

$$(\mu'', \varphi'', F'' \to E'' \to X)$$

is multiplicatively equivalent to

$$(\mu', \varphi', F' \to E' \to X)$$

and $h'' : (F, \mu) \to (F'', \mu'')$ then

$$T(X)(\mu', \varphi', F' \to E' \to X) = T(X)(\mu'', \varphi'', F'' \to E'' \to X)$$

and moreover neither side depends upon the choice of a multiplicative equivalence (h' or h'').

It clearly suffices to assume that there is a multiplicative equivalence

$$g : (\mu', \varphi', F' \to E' \to X) \to (\mu'', \varphi'', F'' \to E'' \to X)$$

Since g is a multiplicative equivalence it induces a homotopy equivalence

$$DL(g) : DL(E') \to DL(E'').$$

Let $i_x' : X \to DL(E'')$. Since $DL(h)\varepsilon(h') \sim 1_{DL(E')}$ and $DL(g)i_x = i_x'$, one has

$$\varepsilon(h'')DL(g)DL(h')(\varepsilon(h')i_x) \sim \varepsilon(h'')i_x'$$

with $\varepsilon(h'')DL(g)DL(h)$ a homotopy equivalence. Thus

$$[[\varepsilon(h')i_x]] = [[\varepsilon(h'')i_x']].$$

The above argument applied to the identity homomorphism shows $T(X)$ is independent of the choice of h'.

Let $S : [[\ \ , B_F]] \to \mathcal{P}F(X)$ be given by $S(X)(f) = [\Omega B_F \to E_f \to X]$. ($\Omega B_F$ is multiplicatively equivalent to F via $\{M_j^F\}$ constructed in the proof of Theorem 9.)

To show that S is independent of the choice of representative of $[[f]]$ let $\varepsilon : B_F \to B_F$ be a homotopy equivalence and let $\varepsilon g \sim f$. Let $h : X \to B_F^I$ be such that $h(x)(0) = \varepsilon g(x)$ and $h(x)(1) = f(x)$.

$$E_f = \{(x, \eta) \in X \times PB_F \mid f(x) = \pi\eta\}$$

and

$$E_g = \{(x, \eta) \in X \times PB_F \mid g(x) = \pi\eta\}.$$

Define $\bar{\varepsilon} : E_g \to E_f$ by $\bar{\varepsilon}(x, \eta) = (x, P(\varepsilon)\eta * h(x))$. Since $\pi(P(\varepsilon)\eta) = \varepsilon\pi(\eta) = \varepsilon g(x) = h(x)(0)$ and $\pi P(\varepsilon)\eta * h(x) = h(x)(1) = f(x)$ this is well defined. $\bar{\varepsilon}$ is clearly a homomorphism and $\bar{\varepsilon} \mid \Omega B_F = \Omega\varepsilon$ which is a homotopy equivalence. Thus $S : [[\ \ , B_F]] \to PF(\ \ )$.

It remains to show that ST and TS are the respective identity functors. We first consider ST.

$$ST[\mu', \varphi', F' \to E' \to X] = [\Omega B_F \to E_{\varepsilon(h)i_x} \to X]$$

where h is a multiplicative equivalence $(F, \mu) \to (F', \mu')$. By Theorem 9(a) $(\mu', \varphi', F' \to E' \to X)$ is multiplicatively equivalent to $(\Omega B_F \to E_{i_x} \to X)$. Since $\varepsilon(h)$ is a homotopy equivalence $F' \to E' \to X$ is multiplicatively equivalent to $\Omega B_F \to E_{\varepsilon(h)i_x} \to X$ and $ST = 1$.

To show $TS = 1$ consider

$$\begin{array}{ccccc}
F & \xrightarrow{M_0^F} & \Omega B_F & = & \Omega B_F \\
\downarrow & & \downarrow & & \downarrow \\
F & \longrightarrow & E_f & \longrightarrow & PB_F \\
\downarrow & & \downarrow & & \downarrow \\
* & \longrightarrow & X & \xrightarrow{f} & B_F .
\end{array}$$

Both maps are strong homotopy homomorphisms and induce a commutative diagram:

$$\begin{array}{ccccc}
* & \longrightarrow & X & \xrightarrow{f} & B_F \\
\downarrow & & \downarrow{\scriptstyle i_x} & & \downarrow{\scriptstyle i_{B_F}} \\
B_F & \xrightarrow{DL(M_0^F)} & DL(E_f) & \xrightarrow{DL(f)} & DL(PB_F) .
\end{array}$$

It is proven in the appendix that i_{B_F} is a homotopy equivalence. Let r be a homotopy inverse. As usual $DL(f)$ is a homotopy equivalence

$$TS(f) = [[\varepsilon(DL(M_0^F))i_x]].$$

Since $rDL(f)DL(M_0^F)\varepsilon(DL(M_0^F))i_x \sim rDL(f)i_x \sim f$ and $rDL(f)DL(M_0^F)$ is a homotopy equivalence it follows that

$$[[\varepsilon(DL(M_0^F))i_x]] = [[f]] \quad \text{and} \quad TS = 1.$$

This completes the proof of Theorem 10.

In general $[[\ \ , Y]]$ is not representable. This is seen by considering $[[S^1, S^1]]$ = non-negative integers. If $[[\ \ , S^1]]$ were representable, $[[S^1, S^1]]$ would be a group. Thus in some sense $[\ \ , B_F]$ ($[X, Y]$ = the set of homotopy classes of maps $X \to Y$) is a 'nicer' functor than $[[\ \ , B_F]]$. To see what $[\ \ , B_F]$ classifies we must introduce a stronger equivalence relation on principal fibrations.

Until this point we have defined strong homotopy homomorphism to mean a map $M_0 : E \to E'$ such that there exist $M_1, \cdots, M_n, \cdots$ satisfying Definition 3. For the remainder of this section a strong homotopy homomorphism is a collection of maps $\{M_j\}$ satisfying Definition 3. Given $\{M_j\}$ the induced map $DL(\{M_j\}) : DL(E) \to DL(E')$ is functorial.

Composition and homotopies of strong homomorphisms may be defined as the obvious generalizations of the H-space situation ([4] and [5]). One then easily shows that if $\{M_j\} \sim \{M_j'\}$ then

$$DL(\{M_j\}) \sim DL(\{M_j'\}).$$

By a principal fibration over X with fibre multiplicatively equivalent to (F, μ) we mean a principal fibration $(\mu', \varphi', F' \to E' \to X)$ and a strong homotopy homomorphism $\{M_j'\} : (F, \mu) \to (F', \mu')$ with M_0 a homotopy equivalence.

$(\{M_j'\}, \mu', \varphi', F' \to E' \to X)$ is said to be multiplicatively related to $(\{M_j''\}, \mu'', \varphi'', F'' \to E'' \to X)$ if there is a strong homotopy equivalence

$$\{N_j\} : (\mu', \varphi', F' \to E' \to X) \to (\mu'', \varphi'', F'' \to E'' \to X)$$

such that

(i) N_0 is a fibre homotopy equivalence, and
(ii) $\{N_j\}\{M_j'\} \sim \{M_j''\}$

Let $\mathcal{PF}(X)$ be the set of equivalence classes under the equivalence relation generated by the above relation. The proof of Theorem 10 is then easily modified to show:

THEOREM 11. *There is a natural equivalence of functors between* $\mathcal{PF}(\ \ )$ *and* $[\ \ , B_F]$.

Remark. Under a different equivalence relation Fuchs [6] shows that principal fibrations over X with fibre ΩY are classified by $[X, Y]$.

4. Induced fibrations

Let

$$F \xrightarrow{i} E \xrightarrow{p} B \quad \text{and} \quad F' \xrightarrow{i'} E' \xrightarrow{p'} B'$$

be given fibrations. When does there exist $f : B \to B'$ such that $p : E \to B$ is equivalent to the fibration induced from $p' : E' \to B'$ by f? In this section we relate the obstructions to the existence of f to the existence of a strong homotopy homomorphism of principal fibrations.

Let $\Omega B \to E_p \to E$ and $\Omega B' \to E_{p'} \to E'$ be the principal fibrations induced by p and p' from the path fibrations over B and B' respectively.

THEOREM 12. *There exists $f : B \to B'$ such that $F \to E \to B$ is equivalent to the fibration induced by f from $F' \to E' \to B'$ if and only if there exists a strong homotopy homomorphism of principal fibrations*

$$\begin{tikzcd} \Omega B \arrow[r] \arrow[d] & E_p \arrow[r] \arrow[d, "g"] & E \arrow[d, "h"] \\ \Omega B' \arrow[r] & E_p' \arrow[r] & E' \end{tikzcd}$$

with g a homotopy equivalence.

Proof. If there exists such an f then we have that

$$\begin{tikzcd} E \arrow[r, "\alpha"] \arrow[d, "p"] & E_f \arrow[r, "\pi"] \arrow[d] & E' \arrow[d, "p'"] \\ B \arrow[r, equal] & B \arrow[r, "f"] & B' \end{tikzcd}$$

commutes, where α is a homotopy equivalence and $\pi : E_f \to E$ is the canonical projection. $\pi\alpha : E \to E'$ and $Pf : PB \to PB'$ define a map $E_p \to E_{p'}$ which is easily seen to be a homomorphism of principal fibrations. Since $\pi\alpha \mid F$ is a homotopy equivalence so is the map $E_p \to E_{p'}$.

Now assume the existence of a strong homotopy homomorphism as given above. Let $DL(PB)$ and $DL(PB')$ be the spaces associated with the (principal) path fibrations over B and B' respectively. Let

$$r : DL(PB) \to B \quad \text{and} \quad r' : DL(PB') \to B'$$

be the retractions given in the appendix. As usual

$$DL(p) : DL(E_p) \to DL(PB)$$

is a homotopy equivalence. Let $\alpha : DL(PB) \to DL(E_p)$ be a homotopy inverse. Since (g, h) is a strong homotopy homomorphism there exists

$$DL(h) : DL(E_p) \to DL(E_{p'}).$$

Define $f : B \to B'$ to be the composite

$$B \to DL(PB) \xrightarrow{\alpha} DL(E_p) \xrightarrow{DL(h)}$$

$$DL(E_{p'}) \xrightarrow{DL(p')} DL(PB') \xrightarrow{r'} B'.$$

It follows from the definition of f that the following diagram homotopy commutes.

$$\begin{array}{ccc} DL(E_p) & \xrightarrow{DL(h)} & DL(E_{p'}) \\ \Big\downarrow{\scriptstyle DL(p)} & & \Big\downarrow{\scriptstyle DL(p')} \\ DL(PB) & & DL(PB') \\ \Big\downarrow{\scriptstyle r} & & \Big\downarrow{\scriptstyle r'} \\ B & \xrightarrow{f} & B' \end{array}$$

Let $E \cup CF \to E \cup C(E_p) \to DL(E_p)$ be the obvious inclusions. Since

$$F \xrightarrow{j} E_p \longrightarrow PB$$

is constant, one sees that

$$E \cup CF \to E \cup C(E_p) \to DL(E_p) \xrightarrow{DL(p)} DL(PB) \xrightarrow{r} B$$

is p on E and constant on CF. Call this map $\bar{p}$. A similar statement holds with respect to $\bar{p}' : E' \cup CF' \to B'$.

We note that

$$F' \xrightarrow{j'} E_{p'} \xrightarrow{\pi'} PB'$$

is a fibration since

$$E' \xrightarrow{p'} B'$$

is. Since $\pi' gj : F \to PB'$ is constant, $(g \mid F) : F \to F'$. Call this map h^F.

The composite

$$F \xrightarrow{j} E_p \to E$$

is the inclusion map. Thus $h \mid F = h^F$. Since j, j', and g are homotopy equivalences so is h^F. Let

$$\bar{h} : E \cup CF \to E' \cup CF'$$

be the map induced by h. By the above remarks,

$$\begin{array}{ccc} E \cup CF & \xrightarrow{\bar{h}} & E' \cup CF' \\ \Big\downarrow{\scriptstyle \bar{p}} & & \Big\downarrow{\scriptstyle \bar{p}'} \\ B & \xrightarrow{f} & B' \end{array}$$

homotopy commutes

Let $H : (E \cup CF) \times I \to B'$ be a homotopy with

$$H \mid (E \cup CF) \times 0 = \bar{p}'\bar{h}, \qquad H \mid (E \cup CF) \times 1 = f\bar{p}.$$

Let $H_E = H \mid E \times I$. By the homotopy lifting property there exists

$$\bar{H} : E \times I \to E'$$

such that $p\bar{H} = H_E$ and $\bar{H} \mid E \times 0 = h$. Let $\beta = \bar{H} \mid E \times 1$. Thus $p'\beta = fp$ and $\beta \sim h$. To complete the proof it suffices by Proposition 8 to show that $(\beta \mid F) : F \to F'$ is a homotopy equivalence. To do this we show $(\beta \mid F) \sim h^F$.

Let $H \mid (CF \times I) : F \times I \times I \to B$ (where the last factor corresponds to the cone). By the homotopy lifting property there exists

$$\tilde{H} : F \times I \times I \to E$$

such that $p\tilde{H} = H \mid (CF \times I)$ and $\tilde{H} \mid F \times I \times 0 = \bar{H}$.

$$p\tilde{H} \mid (F \times 0 \times I) = H \mid (CF \times 0) = \bar{p}'\bar{h} \mid CF = *,$$

$$p\tilde{H} \mid (F \times 1 \times I) = H \mid (CF \times 1) = f\bar{p} \mid CF = *,$$

$$p\tilde{H} \mid F \times I \times 1 = H \mid * = *.$$

Thus $\tilde{H}$ restricted to these edges maps $F \to F'$ and yields the desired homotopy of $\beta \mid F$ and h^F. This completes the proof of the theorem.

We note that although we assumed the existence of a strong homotopy homomorphism it follows from the proof of the theorem that this hypothesis implies the existence of a (strict) homomorphism of principal fibrations.

Remark. The Dold-Lashof construction applied to the principal fibration $\Omega B \to E_p \to E$ yields the iterated fibre spaces, E_n, studied by Ganea in [7] and [8]. Theorem 12 extends the work of the first half of [8].

A particular case of interest is that in which $F' \to E' \to B'$ is the path fibration over B'. The inclusion map

$$\begin{array}{ccccc} \Omega B' & \to & \Omega B' & \to & * \\ \downarrow & & \downarrow & & \downarrow \\ \Omega B' & \to & E_\pi & \to & PB \end{array}$$

is a homomorphism of principal fibrations. Thus we have:

COROLLARY 13. *$F \to E \to B$ is equivalent to a fibration induced from $\pi : PB' \to B'$ if there exists a homotopy equivalence $\theta : E_p \to \Omega B'$ such that*

$$\begin{array}{ccccc} \Omega B & \xrightarrow{j} & E_p & \longrightarrow & E \\ \Big\downarrow{\scriptstyle \theta j} & & \Big\downarrow{\scriptstyle \theta} & & \Big\downarrow \\ \Omega B' & \longrightarrow & \Omega B' & \longrightarrow & * \end{array}$$

is a strong homotopy homomorphism.

Let $\lambda : E_p \to F$ be the composite of the "lift" map and evaluation at 0. Consider

$$\begin{array}{ccc} \Omega B \times E_p & \xrightarrow{\bar{\psi}} & E_p \\ \downarrow{\scriptstyle 1 \times \lambda} & & \downarrow{\scriptstyle \lambda} \\ \Omega B \times F & \xrightarrow{\psi} & F \end{array}$$

where $\bar{\psi}(\xi, (x, \eta)) = (x, \xi * \eta)$ and $\psi(\xi, x) = \lambda(x, \xi)$. We may assume this diagram commutes. (If not we can replace $F \to E \to B$ by an equivalent fibration with this property.)

Let $\theta : F \to \Omega B'$ be a homotopy equivalence and set

$$\bar{\theta} = \theta\lambda : E_p \to \Omega B'.$$

The commutativity of the above diagram implies that if

$$\begin{array}{ccc} \Omega B \times F & \xrightarrow{\psi} & F \\ \downarrow{\scriptstyle \theta\partial \times \theta} & & \downarrow{\scriptstyle \theta} \\ \Omega B' \times \Omega B' & \xrightarrow{\mu} & \Omega B' \end{array}$$

commutes, so does

$$\begin{array}{ccc} \Omega B \times E_p & \xrightarrow{\bar{\psi}} & E_p \\ \downarrow{\scriptstyle \bar{\theta} i \times \bar{\theta}} & & \downarrow{\scriptstyle \bar{\theta}} \\ \Omega B' \times \Omega B' & \xrightarrow{\mu} & \Omega B' \end{array}$$

where $\partial : \Omega B \to F$ is $\lambda \mid (* \times \Omega B)$. Combining this with the above, one has:

COROLLARY 14. (*a*) *Let $\theta : F \to \Omega B'$ be a homotopy equivalence. $p : E \to B$ is equivalent to a fibration induced from the path fibration over B' if*

$$\begin{array}{ccc} \Omega B \times F & \xrightarrow{\psi} & F \\ \downarrow{\scriptstyle \theta\partial \times \theta} & & \downarrow{\scriptstyle \theta} \\ \Omega B' \times \Omega B' & \xrightarrow{\mu} & \Omega B' \end{array}$$

commutes.

(b) *In particular if $F = \Omega B'$, then $p : E \to B$ is equivalent to a fibration induced from the path fibration if $\psi = \mu(\partial \times 1)$.*

5. The extension problem

The extension problem asks when a given map $f : A \to Y$ defined on a subcomplex A of X can be extended to X.

As an indication of the pervasiveness of the notion of strong homotopy homomorphism we give a solution to the extension problem in these terms.

Let $i : A \to X$ and let

$$\Omega X \to E_i \xrightarrow{p} A$$

be the fibration induces from the path fibration over X by i. The first obstruction to extending f is the homotopy class $[fp] \in [E_i, Y]$. If this is zero, there exists a fibre map

$$\begin{tikzcd} \Omega X \arrow[r] \arrow[d] & E_i \arrow[r, "p"] \arrow[d, "h"] & A \arrow[d, "f"] \\ \Omega Y \arrow[r] & PY \arrow[r, "\pi"] & Y. \end{tikzcd}$$

THEOREM 15. *$f : A \to Y$ can be extended to X if and only if there is a null homotopy, h, of fp which is a strong homotopy homorphism of principal fibrations.*

Proof. If f can be extended to $F : X \to Y$, we have

$$\begin{tikzcd} \Omega X \arrow[r, equal] \arrow[d] & \Omega X \arrow[r, "\Omega F"] \arrow[d] & \Omega Y \arrow[d] \\ E_i \arrow[r] \arrow[d] & PX \arrow[r, "PF"] \arrow[d] & PY \arrow[d] \\ A \arrow[r, "i"] & X \arrow[r, "F"] & Y \end{tikzcd}$$

is a homomorphism of principal fibrations.

On the other hand let h be a strong homotopy homomorphism. Thus there exists $DL(f) : DL(E_i) \to DL(PY)$ which extends f.

Consider

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"] & DL(E_i) \arrow[d, "DL(i)"] \\ X \arrow[r] & DL(PX). \end{tikzcd}$$

Since i is a cofibration and $DL(i)$ is a homotopy equivalence, by Theorem 7.6.22 of [12] there exists $j : X \to DL(E_i)$ such that $j \mid A$ is inclusion. Set

$F : X \to Y$ equal to the composite

$$X \xrightarrow{j} DL(E_i) \xrightarrow{DL(f)} DL(PY) \xrightarrow{r} Y$$

where r is the retraction given in the appendix. It is immediate that $F \mid A = f$.

Appendix

THEOREM. *$DL(PY)$ has Y as a strong deformation retract.*

Proof. For $\eta \in PY$ and $t \in I$ let $\eta_t \in PY$ be defined by $\eta_t(s) = \eta(ts)$. Define $\theta_n : (\Omega Y)^n \times I^{n+1} \times PY \to PY$ inductively by

$$\theta_0(t, \eta) = \eta_t,$$

$$\theta_n(\xi_1, \cdots, \xi_n, t_1, \cdots, t_{n+1}, \eta) = (\xi_n * \theta_{n-1}(\xi_1, \cdots, \xi_{n-1}, t_1, \cdots, t_n, \eta))_{t_{n+1}}.$$

Identify $(PY)_n$ as a quotient of $(\Omega Y)^n \times I^n \times PY$ by identifying

$$(\xi_1, \cdots, \xi_n, t_1, \cdots, t_n, \eta)$$

with

$$(t_n, y, \xi_n) \in C(PY)_{n-1} \times \Omega Y$$

where y is the point identified with

$$(\xi_1, \cdots, \xi_{n-1}, t_1, \cdots, t_{n-1}, \eta).$$

Define a retraction $r_n : Y_n \to Y$ inductively by

$$r_0 = 1, \quad r_n \mid Y_{n-1} = r_{n-1},$$

$$r_n(s, \xi_1, \cdots, \xi_{n-1}, t_1, \cdots, t_{n-1}, \eta) = \pi\theta_{n-1}(\xi_1, \cdots, \xi_{n-1}, t_1, \cdots, t_{n-1}, s, \eta).$$

Routine verification shows that r_n is well defined. Clearly

$$Y \xrightarrow{j} Y_n \xrightarrow{r_n} Y$$

is the identity. Thus it remains to show that

$$Y_n \xrightarrow{r_n} Y \xrightarrow{j} Y_n$$

is homotopic to the identity rel Y. Define a homotopy by

$$h_u(s, \xi_1, \cdots, \xi_{n-1}, t_1, \cdots, t_{n-1}, \eta) = (s_1, *, \cdots, *, 1, \cdots, 1, \theta_{n-1}(\xi_1, \cdots, \xi_{n-1}, t_1, \cdots, t_{n-1}, 2(s-1)u + 1, \eta)) \quad \text{if } 0 \leq u \leq \tfrac{1}{2},$$

$$= (s + (1 - s)(2u - 1), *, \cdots, *, 1, \cdots, 1,$$
$$\theta_{n-1}(\xi_1, \cdots, \xi_{n-1}, t_1, \cdots, t_{n-1}, s, \eta))$$
$$\text{if } \tfrac{1}{2} \leq u \leq 1;$$

$$h_1(s, \xi_1, \cdots, \xi_{n-1}, t_1, \cdots, t_{n-1}, \eta)$$
$$= (1, *, \cdots, *, *, 1, \cdots, 1, \theta_{n-1}(\xi_1, \cdots, \xi_{n-1}, t_1, \cdots, t_{n-1}, s, \eta))$$
$$= j(\pi\theta_{n-1}(\xi_1, \cdots, \xi_{n-1}, t_1, \cdots, t_{n-1}, s, \eta))$$
$$= jr_n(s, \xi_1, \cdots, \xi_{n-1}, t_1, \cdots, t_{n-1}, \eta);$$

$$h_0(s, \xi_1, \cdots, \xi_{n-1}, t_1, \cdots, t_{n-1}, \eta)$$
$$= (s, *, \cdots, *, 1, \cdots, 1, \theta_{n-1}(\xi_1, \cdots, \xi_{n-1}, t_1, \cdots, t_{n-1}, 1, \eta))$$
$$= (s, *, \cdots, *, \xi_{n-1}, 1, \cdots, 1, \theta_{n-2}(\xi_1, \cdots, \xi_{n-2}, t_1, \cdots, t_{n-1}, \eta))$$
$$= (s, *, \cdots, *, \xi_{n-2}, \xi_{n-1}, 1, \cdots, 1, \theta_{n-3}(\xi_1, \cdots, \xi_{n-3}, t_1, \cdots, t_{n-2}, \eta)_{t_{n-1}}).$$

By a homotopy similar to the one given above h_0 is homotopic to the function

$$h_0'(s, \xi_1, \cdots, \xi_{n-1}, t_1, \cdots, t_{n-1}, \eta)$$
$$= (s, *, \cdots, *, \xi_{n-2}, \xi_{n-1}, 1, \cdots, 1, t_{n-1}, \theta_{n-3}(\xi_1, \cdots, \xi_{n-3}, t_1, \cdots, t_{n-2}, \eta))$$
$$= (s, *, \cdot, *, \xi_{n-3}, \xi_{n-2}, \xi_{n-1}, 1, \cdot, 1, t_{n-1}, \theta_{n-4}(\xi_1, \cdot, \xi_{n-4}, t_1, \cdot, t_{n-3}, \eta)_{t_{n-2}}).$$

Iterating this procedure one eventually has $h_0 \sim 1$. Furthermore if $s = 1$, h_u does not depend on u. A similar statement holds with respect to the iterated homotopies if $t_1 = \cdots = t_{n-1} = 1$. Thus the combined homotopy is rel Y.

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