ON THE RATIONAL COHOMOLOGY OF CLASSIFYING SPACES OF RING-VALUED FUNCTORS

BY

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1. Introduction

Consider a contravariant functor $t : \mathcal{W} \to \mathcal{R}$ from the category $\mathcal{W}$ of finite connected pointed CW complexes to the category $\mathcal{R}$ of (not necessarily associative) rings. Suppose that t has a CW classifying space B. Our objective is to study the effect of the ring structure on t upon the rational cohomology Hopf algebra $A = H^*(B; \mathbf{Q})$. As a consequence we find from (7.2) that if $t(X)$ is ring isomorphic to the integers $\mathbf{Z}$ for some X then B must be infinite dimensional; in fact, $H^r(B; \mathbf{Q})$ is non-zero for arbitrarily large r.

Now $t : \mathcal{W} \to \mathcal{R}$ induces on B the structure of a topological ring up to weak homotopy, the multiplication being a map $m : B \wedge B \to B$ from the smashed product; see (5.1). Sections 2–5 are devoted to definitions, elementary properties and examples of such *H-ring structures*. In turn m induces a cohomology algebra homomorphism $\theta : A \to A \otimes A$ satisfying certain properties which we axiomatize in §6 to arrive at the notion of a *secondary coproduct* θ on a Hopf algebra A. In particular, we show in (6.6) that θ is trivial if A is an exterior algebra. Next, we state our main result, Theorem (7.1), which under mild conditions on t guarantees the non-triviality of the associated θ. The proof of (7.1) given in §8 makes use of several general algebraic topology results which we establish in an Appendix, §9.

2. H-rings

Let B be an H-commutative group with addition map $a : B \times B \to B$, i.e. a satisfies the axioms of a commutative group up to homotopy. An *H-ring structure* on B is a map $m : B \wedge B \to B$ such that diagram

$$(2.1)\qquad \begin{array}{ccccc} (B \times B) \wedge B & \xrightarrow{\gamma} & (B \wedge B) \times (B \wedge B) & \xrightarrow{m \times m} & B \times B \\ \big\downarrow a \wedge 1 & & & & \big\downarrow a \\ B \wedge B & & \xrightarrow{\qquad m \qquad} & & B \end{array}$$

and its $1 \wedge a$ counterpart are homotopy commutative, where

$$\gamma((a, b) \wedge c) = (a \wedge c, b \wedge c).$$

Thus distributivity holds up to homotopy. Since any null homotopic m is an

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H-ring structure we shall be interested only in those m which are *nontrivial*, i.e. not null homotopic.

An H-ring B can be *homotopy associative* or *homotopy commutative*, the definitions of which should be evident. A point $\varepsilon \in B$ is a *multiplicative homotopy identity* if the compositions

$$B \xrightarrow{\kappa_j} B \wedge B \xrightarrow{m} B$$

for $j = 1, 2$ are homotopic to the identity map $1 : B \to B$, where $\kappa_1(b) = b \wedge \varepsilon$, $\kappa_2(b) = \varepsilon \wedge b$. In the present paper, which is restricted mainly to connected spaces, this notion will play no role because of the following result.

(2.2) Proposition. *A pathwise connected H-ring with multiplicative homotopy identity is contractible.*

Proof. Suppose that ε and m existed for the pathwise connected space B. Letting $\lambda : I \to B$ be path from ε to the base point $*$ of B, observe that $h_t(b) = b \wedge \lambda(t)$ defines a homotopy between κ_1 and the constant map 0. Hence $1 \simeq m\kappa_1 \simeq 0$ and B is contractible.

One may also consider *weak H-ring structures* on B by requiring diagram (2.1) to be weakly homotopy commutative instead of homotopy commutative.

3. Examples

(3.1) Let $B = \Omega S^1$. Regarding a loop in S^1 as a based map $\alpha : S^1 \to S^1$ define

$$\bar{m} : B \times B \to B \quad \text{by} \quad \bar{m}(\alpha, \beta) = \alpha \circ \beta : S^1 \to S^1.$$

Since $\bar{m}$ is continuous and $\bar{m}(B \vee B) = *$ there is induced a map $m : B \wedge B \to B$ with $m(\alpha \wedge \beta) = \alpha \circ \beta$ which is easily seen to be an H-ring structure.

It should be noted that ΩS^1 has the homotopy type of $\mathbf{Z}$ as H-rings. Indeed, one readily verifies that $\pi_0(m) : \pi_0(B) \times \pi_0(B) \to \pi_0(B)$ is multiplication of integers upon identifying $\pi_0(B)$ with $\mathbf{Z}$.

(3.2) Next, let G be an abelian group and let B be an Eilenberg-MacLane space of type (G, n), $n \geqq 1$, with the usual H-commutative group structure. *There is no non-trivial H-ring structure on B.* Indeed,

$$[m] \in [B \wedge B, B] \cong H^n(B \wedge B; G)$$

which is a zero group since B being $(n - 1)$-connected implies that $B \wedge B$ is $(2n - 1)$-connected.

In particular, this result applies to infinite real, complex and quaternionic projective spaces. Moreover, (3.1) shows $n \geqq 1$ is necessary.

Our final example will be used in conjunction with our main result, Theorem (7.1). First, consider the natural transformation

$$\alpha : H^n(X) \otimes H^n(Y) \to H^{2n}(X \wedge Y)$$

occurring in the reduced Künneth Theorem (9.2), $n \geqq 1$. Since $H^n(\ ) \cong [\ , K_n]$ where K_n denotes an Eilenberg-MacLane space of type $(\mathbf{Z}, n)$, we may apply Theorems (2.1) and (2.2) of [7] to obtain a representing homotopy-bilinear map $\varphi : K_n \wedge K_n \to K_{2n}$. Precisely, α corresponds to

$$\text{(3.3)} \qquad \begin{gathered} \alpha' : [X, K_n] \otimes [Y, K_n] \to [X \wedge Y, K_{2n}], \\ \alpha'([f] \otimes [g]) = [\varphi \circ (f \wedge g)], \end{gathered}$$

and the diagram

$$\text{(3.4)} \qquad \begin{tikzcd} (K_n \times K_n) \wedge K_n \arrow[r, "\gamma"] \arrow[d, "\mu \wedge 1"] & (K_n \wedge K_n) \times (K_n \wedge K_n) \arrow[r, "\varphi \times \varphi"] & K_{2n} \times K_{2n} \arrow[d, "\nu"] \\ K_n \wedge K_n \arrow[rr, "\varphi"] & & K_{2n} \end{tikzcd}$$

and its $1 \wedge \mu$ analogue are homotopy commutative, where μ and ν denote the H-structures.

Let $B = K_n \times K_{2n}$ have the product H-structure $a : B \times B \to B$, and define $m : B \wedge B \to B$ as the composition

$$(K_n \times K_{2n}) \wedge (K_n \times K_{2n}) \xrightarrow{p_1 \wedge p_1} K_n \wedge K_n \xrightarrow{\varphi} K_{2n} \xrightarrow{i_2} K_n \times K_{2n}.$$

(3.5) *The map m is an H-ring structure on B. Moreover, the group $[B \wedge B, B]$ is infinite cyclic on generator $[m]$.*

Proof.

$$\begin{aligned} a(m \times m)\gamma &= a(i_2 \times i_2)(\varphi \times \varphi)\{(p_1 \wedge p_1) \times (p_1 \wedge p_1)\}\gamma \\ &= i_2\, \nu(\varphi \times \varphi)\gamma\{(p_1 \times p_1) \wedge p_1\} \\ &\simeq i_2\, \varphi(\mu \wedge 1)\{(p_1 \times p_1) \wedge p_1\} \\ &= i_2\, \varphi(p_1 \wedge p_1)(a \wedge 1) \\ &= m(a \wedge 1) \end{aligned}$$

using diagram (3.4).

Next, consider the following commutative diagram:

$$\begin{tikzcd} H^n(B) \otimes H^n(B) \arrow[r, "\alpha"] & H^{2n}(B \wedge B) \cong [B \wedge B, K_{2n}] \arrow[r, "i_{2*}"] & {[B \wedge B, B]}. \\ H^n(K_n) \otimes H^n(K_n) \arrow[r, "\alpha"] \arrow[u, "p_1^* \otimes p_1^*"] & H^{2n}(K_n \wedge K_n) \cong [K_n \wedge K_n, K_{2n}] \arrow[u, "(p_1 \wedge p_1)^*"] \arrow[ur, "(p_1 \wedge p_1)^*"'] & \end{tikzcd}$$

The Künneth Theorem implies that $\tilde{H}^r(B) = 0$ for $r < n$ while p_1^* is an isomorphism in dimension n. Applying the reduced Künneth Theorem (9.2) we find that both functions α are isomorphisms. And so is i_{2*} because

$$[B \wedge B, B] \cong [B \wedge B, K_n] \oplus [B \wedge B, K_{2n}]$$

and

$$[B \wedge B, K_n] \cong H^n(B \wedge B) = 0.$$

Since the lower left hand corner is isomorphic to $\mathbf{Z}$ we conclude that

$$[B \wedge B, B] \cong \mathbf{Z}.$$

Moreover, the lower route takes generator $1 \otimes 1$ to $[m] = i_{2*}(p_1 \wedge p_1)^*[\varphi]$ because (3.3) implies that $\alpha'([1] \otimes [1]) = [\varphi]$, so the proof of (3.5) is complete.

We shall return to this example in Sections 5, 6 and 7.

4. Quasi H-rings

We have chosen to define an H-ring structure on B as a map on the smashed product. If instead we choose cartesian product we have the notion of a *quasi H-ring structure* $\bar{m} : B \times B \to B$ for which we require the homotopy commutativity of the following diagram and its $1 \times a$ analogue:

$$\text{(4.1)} \quad \begin{array}{ccccc} B \times B \times B \times B & \xrightarrow{1 \times T \times 1} & B \times B \times B \times B & \xrightarrow{\bar{m} \times \bar{m}} & B \times B \\ \uparrow{\scriptstyle 1 \times 1 \times \Delta} & & & & \downarrow{\scriptstyle a} \\ B \times B \times B & \xrightarrow{a \times 1} & B \times B & \xrightarrow{\bar{m}} & B \end{array}$$

The two notions of H-rings are nearly identical in view of the next result. First, recall that a space B is *well pointed* if the pair $(B, *)$ has the homotopy extension property.

(4.2) THEOREM. (1) *If $m : B \wedge B \to B$ is an H-ring structure on B, then*

$$\bar{m} = m \circ q : B \times B \to B \wedge B \to B$$

is a quasi H-ring structure.

(2) *If $\bar{m} : B \times B \to B$ is a quasi H-ring structure on a well pointed space B, then there exists an H-ring structure $m : B \wedge B \to B$ such that $m \circ q \simeq \bar{m}$.*

Proof of (1). Let $q' : (B \times B) \times B \to (B \times B) \wedge B$. Then (4.1) implies that

$$\begin{aligned} a(\bar{m} \times \bar{m})(1 \times T \times 1)(1 \times 1 \times \Delta) &= a(m \times m)\gamma q' \\ &\simeq m(a \wedge 1)q' = \bar{m}(a \times 1). \end{aligned}$$

In order to establish (2) we need a preliminary result.

(4.3) LEMMA. *If B is a quasi H-ring, then $\bar{m} \circ i_j$ is null homotopic, $j = 1, 2$.*

Proof. Let $\iota : B \to B$ be the homotopy inversion map and let $j_3 : B \to B \times B \times B$ be injection onto the third factor. Then

$$
\begin{aligned}
\bar{m}i_2 &\simeq ai_1\,\bar{m}i_2 \quad (a \text{ is an } H\text{-structure})\\
&= a(1 \times 0)\Delta\bar{m}i_2\\
&\simeq a(1 \times a)(1 \times 1 \times \iota)(1 \times \Delta)\Delta\bar{m}i_2 \quad (\iota \text{ is homotopy inversion})\\
&\simeq a(a \times 1)(1 \times 1 \times \iota)(1 \times \Delta)\Delta\bar{m}i_2 \quad (a \text{ is h.a.})\\
&= a(a \times 1)(\bar{m} \times \bar{m} \times 1)(1 \times T \times 1 \times 1)(1 \times 1 \times \Delta \times 1)\\
&\qquad\qquad \cdot (j_3 \times \iota)(1 \times \bar{m}i_2)\Delta\\
&\simeq a(\bar{m} \times 1)(a \times 1 \times 1)(i_3 \times \iota)(1 \times \bar{m}i_2)\Delta \quad (\text{by } 4.1)\\
&= a(1 \times \iota)\Delta\bar{m}i_2\\
&\simeq 0 \quad (\iota \text{ is homotopy inversion}).
\end{aligned}
$$

Proof of (2). Since we can assume that the homotopies in (4.3) are based (cf. Satz 4.11 in [1]), we have that

$$B \vee B \overset{j}{\subset} B \times B \xrightarrow{\bar{m}} B \text{ is}$$

null homotopic. But B is well pointed, so $(B \times B, B \vee B)$ has the homotopy extension property by [1, Satz 3.14′]. Hence there exists a map $\tilde{m} \simeq \bar{m}$ with $\tilde{m}(B \vee B) = *$. Factoring $\tilde{m}$ through $B \wedge B$ we get a map $m : B \wedge B \to B$ with $mq \simeq \bar{m}$.

To show that m is an H-ring structure observe that

$$
\begin{aligned}
a(m \times m)\gamma q' &= a(m \times m)(q \times q)(1 \times T \times 1)(1 \times 1 \times \Delta)\\
&\simeq a(\bar{m} \times \bar{m})(1 \times T \times 1)(1 \times 1 \times \Delta)\\
&\simeq \bar{m}(a \times 1) \quad \text{by } (4.1)\\
&\simeq mq(a \times 1)\\
&= m(a \wedge 1)q'.
\end{aligned}
$$

Hence

$$q'^*[a(m \times m)\gamma] = q'^*[m(a \wedge 1)]$$

under

$$q'^* : [(B \times B) \wedge B, B] \to [(B \times B) \times B, B].$$

Since B is well pointed, so is $B \times B$ and hence $((B \times B) \times B, (B \times B) \vee B)$ has the homotopy extension property. It follows [3, Satz 16] that the mapping cone C_j of $j : (B \times B) \vee B \subset (B \times B) \times B$ has the homotopy type of $(B \times B) \wedge B$. Applying $[\ \ , B]$ to the Puppe sequence of j and using the fact that Qj is null homotopic [3, p. 329] we conclude that q'^* is a monomorphism. Therefore $[a(m \times m)\gamma] = [m(a \wedge 1)]$, i.e. diagram (2.1) is homotopy commutative, so the proof of (4.2) is complete.

Quasi H-rings are of some interest because of

(4.4) PROPOSITION. *The loop space of a quasi H-ring is a quasi H-ring.*

To see this merely observe that $\Omega\bar{m} : \Omega B \times \Omega B \to \Omega B$ us a quasi H-ring structure since the loop space functor preserves homotopies.

The weak analogue of (4.4) is also true.

5. Ring-valued functors

Consider the set-valued functor $t(\quad) = [\quad, B] : \mathcal{W} \to \mathcal{S}$ where B is pathwise connected.

(5.1) THEOREM. *A weak H-ring structure on B induces a ring-valued functor $t : \mathcal{W} \to \mathcal{R}$. Conversely, if B is a countable CW complex, then any functor $t : \mathcal{W} \to \mathcal{R}$ arises in this way. Moreover, in this case we have the following:*

(1) *t does not map to the category of rings-with-identity if t is non-zero;*

(2) *the ring $t(S^r)$ has trivial multiplication for each $r > 0$.*

A similar result holds if the domain of t is the category of all CW complexes in which case the word "weak" can be deleted.

The first part of (5.1) is essentially (3.1) of [7] while statement (1) may be derived from either (2.2) or the following argument: The constant map $0 : X \to Y$ induces a morphism of rings with identity. Hence $[0] \in [X, B]$ must be the identity element and therefore $[f] = [f][0] = [0]$.

To prove (2) recall that the product $[f][g] \in [X, B]$ is the homotopy class of the composition

$$X \xrightarrow{\Delta'} X \wedge X \xrightarrow{f \wedge g} B \wedge B \xrightarrow{m} B$$

where $\Delta'(x) = x \wedge x$. Since $S^r \wedge S^r = S^{2r}$ we see that $\Delta' \simeq 0$ for $r > 0$ and the proof is complete.

Let's specialize (5.1) to example (3.5) of an H-ring structure on $B = K_n \times K_{2n}$.

(5.2) $[X, B] \cong H^n(X) \oplus H^{2n}(X)$ *as groups with multiplication m_* given by*

$$m_*((\alpha \oplus \beta) \otimes (\alpha' \oplus \beta')) = 0 \oplus (\alpha \cup \alpha').$$

Proof. The first part is clear. As for the multiplicative structure consider the following commutative diagram:

$$\begin{array}{ccccccc}
H^n(X) \otimes H^n(X) & & \xrightarrow{\quad\alpha\quad} & & H^{2n}(X \wedge X) & \xrightarrow{\Delta'^*} & H^{2n}(X) \\
\wr\Vert & & & & \wr\Vert & & \wr\Vert \\
[X, K_n] \otimes [X, K_n] & \xrightarrow{\wedge} & [X \wedge X, K_n \wedge K_n] & \xrightarrow{\varphi^*} & [X \wedge X, K_{2n}] & \xrightarrow{\Delta'^*} & [X, K_{2n}] \\
\uparrow{\scriptstyle p_{1*} \otimes p_{1*}} & & \uparrow{\scriptstyle (p_1 \wedge p_1)_*} & & \downarrow{\scriptstyle i_{2*}} & & \downarrow{\scriptstyle i_{2*}} \\
[X, B] \otimes [X, B] & \xrightarrow{\wedge} & [X \wedge X, B \wedge B] & \xrightarrow{m_*} & [X \wedge X, B] & \xrightarrow{\Delta'^*} & [X, B]
\end{array}$$

Since p_{1*} is projection of $H^n \oplus H^{2n}$ onto H^n and $\Delta'^*\alpha$ is cup product, we find that the restriction of m_* to $H^n \otimes H^n$ is cup product. On $H^n \otimes H^{2n}$, m_* is zero because the map

$$K_n \wedge K_{2n} \xrightarrow{i_1 \wedge i_2} B \wedge B \xrightarrow{m} B$$

is constant. Similarly for $H^{2n} \otimes H^n$ and $H^{2n} \otimes H^{2n}$.

6. Secondary coproducts

Let A be a Hopf algebra over the rational numbers $\mathbf{Q}$ (for convenience) with product $\varphi : A \otimes A \to A$ and coproduct $\psi : A \to A \otimes A$, tensor products being over $\mathbf{Q}$. A *secondary coproduct* on A is a morphism of augmented algebras with unit $\theta : A \to A \otimes A$ such that the diagram

$$(6.1)\qquad \begin{array}{ccccc} A & \xrightarrow{\theta} & A \otimes A & \xrightarrow{\psi \otimes 1} & A \otimes A \otimes A \\ \downarrow{\scriptstyle \psi} & & & & \uparrow{\scriptstyle 1 \otimes 1 \otimes \varphi} \\ A \otimes A & \xrightarrow{\theta \otimes \theta} & A \otimes A \otimes A \otimes A & \xrightarrow{1 \otimes T \otimes 1} & A \otimes A \otimes A \otimes A \end{array}$$

and its $1 \otimes \psi$ counterpart are commutative.

This definition is motivated by

(6.2) PROPOSITION. *If B is a weak quasi H-ring, then the Hopf algebra $A = H^*(B; \mathbf{Q})$ has $\bar{m}^*$ as secondary coproduct.*

Indeed, diagram (6.1) is a consequence of applying $H^*(\ ; \mathbf{Q})$ to diagram (4.1) and using the weak homotopy axiom [6, Theorem 11.1].

As usual let $\bar{A} = \ker \{\varepsilon : A \to \mathbf{Q}\}$ and decompose A into the direct sum $\bar{A} + \mathbf{Q}$.

(6.3) PROPOSITION. *If $\theta : A \to A \otimes A$ is a secondary coproduct and A is connected, then $\theta \mid \mathbf{Q} : \mathbf{Q} \cong \mathbf{Q} \otimes \mathbf{Q}$ while $\theta(\bar{A}) \subset \bar{A} \otimes \bar{A}$.*

Notice that in the case of (6.2) this result follows from (4.2) and the reduced Künneth Theorem (9.2).

Proof. Now $\theta(x) = a(1 \otimes 1) + b(1 \otimes u) + c(v \otimes 1) + dw$, an element of $\mathbf{Q} \otimes \mathbf{Q} + \mathbf{Q} \otimes \bar{A} + \bar{A} \otimes \mathbf{Q} + \bar{A} \otimes \bar{A}$. If $x \in \mathbf{Q}$ a chase of the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\theta} & A \otimes A \\ \downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle \epsilon \otimes 1} \\ \mathbf{Q} & \xrightarrow{\eta} & A \cong \mathbf{Q} \otimes A \end{array}$$

shows that $x(1 \otimes 1) = a(1 \otimes 1) + b(1 \otimes u)$. Hence $b = 0$. Similarly, $c = 0$ while $d = 0$ since A is connected. If $x \in \bar{A}$ then the diagram implies that $a = b = 0$, and $c = 0$ similarly.

A finite set $\{x_i\}$ of elements of A is *independent in $A \otimes A$* if $\sum a_i \otimes x_i = 0 = \sum x_i \otimes b_i$ implies that all a_i and b_j are 0.

(6.4) PROPOSITION. *Suppose that* $\theta(x) = \sum x_i' \otimes x_i''$ *where the* $\{x_i'\}$ *and* $\{x_i''\}$ *are independent in* $A \otimes A$. *If* x *is primitive, then so is each* x_i', x_i''.

Proof. Since $\psi(x) = x \otimes 1 + 1 \otimes x$ and $\theta(1) = 1 \otimes 1$, a chase of diagram (6.1) yields

$$\sum \psi(x_i') \otimes x_i'' = \sum (x_i' \otimes 1 + 1 \otimes x_i') \otimes x_i''.$$

The primitiveness of x_i' then follows from independence.

Next, we investigate the possible secondary coproducts in specific Hopf algebras.

(6.5) PROPOSITION. *If* A *is a polynomial algebra on a single generator of positive degree, then* $\theta : \bar{A} \to \bar{A} \otimes \bar{A}$ *is zero.*

Indeed, $\bar{A} \otimes \bar{A}$ is zero in the dimension in which the generator appears and θ is an algebra morphism.

On the other hand, if A is a polynomial algebra on more than one generator θ need not be trivial [8, Prop. (4.6)].

(6.6) THEOREM. *If* A *is an exterior algebra on generators* $x_1, \cdots, x_n$ *of odd degree, there is no non-trivial secondary coproduct on* A.

Proof. Assuming that $\deg x_1 \leqq \deg x_2 \leqq \cdots \leqq \deg x_n$, we show by induction that $\theta(x_i) = 0, 1 \leqq i \leqq n$. Now $\theta(x_1) = 0$ because $\bar{A}_j = 0$ for $j < \deg x_1$. So assume that $\theta(x_1) = \cdots = \theta(x_{r-1}) = 0$ and suppose that

$$\theta(x_r) = \sum y_i' \otimes y_i'' \in (\bar{A} \otimes \bar{A})_q, \quad q = \deg x_r,$$

where the y_i' and y_i'' are nonzero.

Since the generators of A are of odd degree, at least one of y_i', y_i'', say y_i', is degenerate in the generators x_j; and no squares appear because A is an exterior algebra. Writing $\psi(w) = w \otimes 1 + 1 \otimes w + \bar{\psi}(w)$ one easily checks that $\bar{\psi}(y_i') \neq 0$ because of degeneracy. Hence we obtain the following.

(6.7) At least one of $\bar{\psi}(y_i')$, $\bar{\psi}(y_i'')$ is nonzero for each i.

Next, chase x_r around diagram (6.1) and its analogue. If

$$\bar{\psi}(x_r) = \sum z_j' \otimes z_j'',$$

the z_j', z_j'' are expressible solely in terms of generators $x_1, \cdots, x_{r-1}$. The induction hypothesis implies that $(\theta \otimes \theta)\bar{\psi}(x_r) = 0$ and we obtain formula

$$\sum y_i' \otimes \bar{\psi} y_i'' = 0 = \sum \bar{\psi} y_i' \otimes y_i''. \tag{6.8}$$

The above two facts will be used to derive a contradiction. Without loss of generality we can assume that all y_i' appearing in (6.7) and (6.8) have the same degree s. We then have two cases.

Case 1. Some $\bar{\psi} y_k'' \neq 0$. Without loss of generality we may assume that the y_i' are linearly independent in A_s. It follows from an elementary argument that the non-zero $y_i' \otimes \bar{\psi} y_i''$ are linearly independent too. But this contradicts (6.8).

Case 2. All $\bar{\psi}y_i'' = 0$. Then (6.7) implies that all $\bar{\psi}y_i' \neq 0$, and an argument parallel to the above yields another contradiction.

We therefore have $\theta(x_r) = 0$, so the proof of (6.6) is complete.

Although θ may be trivial on two algebras it need not be trivial on their tensor product. For example, it is easy to prove the following result (in which x need not be primitive).

(6.9) PROPOSITION. *Let* $A = \mathbf{Q}[x] \otimes \Lambda[y]$ *where* $\deg y = r$ *is odd,* $\deg x = 2r$. *Then any secondary coproduct* θ *on* A *is given by*

$$\theta(x) = a(y \otimes y), \quad a \in \mathbf{Q}$$

$$\theta(y) = \theta(x^k) = \theta(x^k \otimes y) = 0, \quad k \geqq 2.$$

If we replace the exterior algebra $\Lambda[y]$ by $\mathbf{Q}[y]$ and make r even, then $\theta(x)$ and $\theta(y)$ are unchanged.

For reference in the next section we show how (6.9) arises geometrically from example (3.5) in which $B = K_n \times K_{2n}$. Using Prop. 4, p. 501, of [4] and the Künneth Theorem we have that

$$\begin{aligned} A = H^*(B; \mathbf{Q}) &\cong \mathbf{Q}[x] \otimes \Lambda[y], \quad n \text{ odd} \\ &\cong \mathbf{Q}[x] \otimes \mathbf{Q}[y], \quad n \text{ even} \end{aligned}$$

where $\deg y = n$, $\deg x = 2n$.

(6.10) $\theta = m^* : A_{2n} \to (\bar{A} \otimes \bar{A})_{2n}$ *takes* x *to* $\pm y \otimes y$.

Proof. The reduced Künneth Theorem implies that

$$\alpha : H^n(K_n) \otimes H^n(K_n) \to H^{2n}(K_n \wedge K_n)$$

is an isomorphism of $\mathbf{Z} \otimes \mathbf{Z}$ with $\mathbf{Z}$ (tensor products over $\mathbf{Z}$ here). Identifying $H^n(\quad)$ with $[\quad, K_n]$ and noting from (3.3) that $\varphi : K_n \wedge K_n \to K_{2n}$ is a representative of $\alpha([1] \otimes [1])$, we find that $[\varphi]$ generates $[K_n \wedge K_n, K_{2n}]$. Hence

$$\varphi^* : H^{2n}(K_{2n}) \cong H^{2n}(K_n \wedge K_n).$$

Passing to rational coefficients and using the definition of m, (6.10) follows.

7. The main result and applications

(7.1) THEOREM. *Let* $t : \mathcal{W} \to \mathcal{R}$ *be a representable contravariant functor whose classifying space* B *is a connected CW complex with* $H_*(B)$ *of finite type. Suppose that* B *has a representing weak H-ring structure, and let* $A = H^*(B; \mathbf{Q})$ *be the associated Hopf algebra with secondary coproduct* $\theta : \bar{A} \to \bar{A} \otimes \bar{A}$.

If there exists a finite complex X *such that* $t(X)$ *contains a multiplicatively decomposable element of infinite additive order, then* θ *is non-trivial. In fact,* $\theta : A_r \to (A \otimes A)_r$ *is non-zero for some* r *satisfying*

$$2(\text{connectivity of } X) + 2 \leqq r \leqq 2 \dim X.$$

The proof will occupy the next section.

It should be noted that *the lower bound for r is best possible.* Indeed, consider example (3.5) in which $B = K_n \times K_{2n}$. (6.10) shows that $\theta \neq 0$ in dimension $2n$ while (5.2) implies that $t(S^n \times S^n) \cong \mathbf{Z} + \mathbf{Z} + \mathbf{Z}$ as groups, the generator of the third copy of $\mathbf{Z}$ being the product of the generators of the first two copies.

As an application of (7.1) and (6.6) we derive the following result which roughly states that classifying spaces of many ring-valued functors must be infinite dimensional.

(7.2) THEOREM. *Let $t : \mathcal{W} \to \mathcal{R}$ be a contravariant functor with a connected countable CW complex B as classifying space where $H_*(B)$ is of finite type. Suppose that $\mathbf{Z}$ is a ring direct summand for some $t(X)$. Then B does not have the homotopy type of a finite dimensional complex; in fact, $H^r(B; \mathbf{Q})$ is non-zero for arbitrarily large r.*

Proof. By (5.1) B has an induced weak H-ring structure. Applying (7.1) we deduce that θ is nonzero. By the Leray Structure Theorem [5, p. 268] the Hopf algebra A is a tensor product of a polynomial algebra on even-dimensional generators and an exterior algebra on odd-dimensional generators. Our result then follows from the observation that if the polynomial algebra were not present we would have a contradiction to (6.6).

To illustrate the use of the bounds on r in (7.1) we prove

(7.3) PROPOSITION. *Let $m : BU \wedge BU \to BU$ represent the usual product on $\tilde{K}$, where BU denotes the classifying space of the infinite unitary group, and set $A = H^*(BU; \mathbf{Q})$. For every natural number n, there exists r, $4n \leqq r \leqq 8n$, such that $\theta = m^* : A_r \to (\bar{A} \otimes \bar{A})_r$ is nonzero. In particular, θ is nonzero in arbitrarily high dimensions.*

An explicit calculation of θ appears as (4.6) of [8].

Proof. Let $f : S^{4n-1} \to S^{2n}$ be a map with Hopf invariant $h_f = 2$; see p. 200 of [2], for example. In the exact sequence [2, p. 196]

$$0 \leftarrow \tilde{K}(S^{2n}) \xleftarrow{\varphi} \tilde{K}(C_f) \xleftarrow{\psi} \tilde{K}(S^{4n}) \leftarrow 0.$$

the ends are isomorphic to $\mathbf{Z}$; let b_f be the ψ-image of a generator and let $\varphi(a_f)$ be a generator. Then $a_f^2 = h_f b_f$ implies that a_f^2 has infinite order. Therefore (7.3) follows from (7.1) applied to $X = C_f$.

Notice that the proof does not make use of the fact that A is the polynomial algebra in the universal Chern classes.

8. Proof of (7.1)

We adopt the notation of (7.1) in which $m : B \wedge B \to B$ is the weak H-ring structure on B. Without loss of generality we may assume that m is cellular. If $K \subset B$ is a subcomplex, write $m_K = m \mid K \wedge K$. We then have

(8.1) PROPOSITION. *There exists a space $X \in \mathcal{W}$ such that the ring $[X, B]$*

contains a multiplicatively decomposable element of infinite additive order if and only if there exists finite subcomplex $K \subset B$ such that $[m_K] \in [K \wedge K, B]$ is of infinite order. Moreover, we may assume that $\dim K \leqq \dim X$.

Proof. Observe that if

$$p_i' : K \times K \to K \subset B$$

denotes projection and

$$q : K \times K \to K \wedge K$$

the identification map, then $q^* : [K \wedge K, B] \to [K \times K, B]$ takes $[m_K]$ to $[p_1'][p_2']$ sincd $[m]$ represents the ring structure on $[\ \ , B]$. Moreover, q^* is injective as is seen by applying the functor $[\ \ , B]$ to the Puppe sequence of $i : K \vee K \subset K \times K$ and using the fact that $Qi \simeq 0$ [3, p. 329]. In particular, if $[m_K]$ has infinite order then so does $[p_1'][p_2'] \in [X, B]$ where $X = K \times K$.

Conversely, suppose that $[f][g] \in [X, B]$ has infinite order. Since we may assume that f and g are cellular, there is a finite connected subcomplex $K \subset B$ containing the compact set $f(X) \cup g(X)$ with $\dim K \leqq \dim X$. Writing $(f, g)x = (fx, gx) \in K \times K$, the ring homomorphism

$$[K \wedge K, B] \xrightarrow{q^*} [K \times K, B] \xrightarrow{(f, g)^*} [X, B]$$

takes $[m_K]$ to $[f][g]$ because $q^*[m_K] = [p_1'][p_2']$. Therefore a contradiction would arise if $[m_K]$ had finite order.

(8.2) PROPOSITION. *$[h] \in [Y, B]$ is of infinite order if and only if $[h] \otimes 1$ is nonzero in $[Y, B] \otimes \mathbf{Q}$.*

Proof. By (9.1), the abelian group $[Y, B]$ is finitely generated and is therefore of the form $F \oplus T$ where F is free abelian and T is a torsion group. Write $[h] = \alpha \oplus \beta \in F \oplus T$. If $[h] \otimes 1 = 0$ then $\alpha \otimes 1 = 0$, hence $[h] = \beta \in T$. On the other hand, if $n[h] = 0$ then $[h] \otimes 1 = n[h] \otimes (1/n) = 0$ and the proof is complete.

Recall that a contravariant functor $t : \mathcal{W} \to \mathcal{A}$ to the category of abelian groups is *half exact* if the sequence $A \subset X \to X/A$ in $\mathcal{W}$ induces an exact sequence $tA \leftarrow tX \leftarrow t(X/A)$ in $\mathcal{A}$.

(8.3) PROPOSITION (Dold). *If $t : \mathcal{W} \to \mathcal{A}$ is half exact, there exists a natural equivalence ("generalized Chern character")*

$$t(\ \ ) \otimes \mathbf{Q} \cong \textstyle\prod_{n=1}^{\infty} H^n(\ \ ; t(S^n) \otimes \mathbf{Q}).$$

Proof. Let t_1 and t_2 denote the left and right hand sides, respectively, and let $\rho(S^r) : t_1(S^r) \otimes t_2(S^r)$ be the obvious isomorphism, $r \geqq 1$. Since both t_1 and t_2 are half exact, $\otimes t_i(S^r)$ is an exact functor and $G \otimes t_i(S^r) = 0$ for each finite abelian group G, it follows from Prop. 2.6 and footnote (*), page A.5, and Prop. 2.1, page A.3, of Dold [1] that $\rho(S^r)$ extends to a natural isomorphism $\rho : t_1 \cong t_2$.

Specializing (8.3) yields

$$[K \wedge K, B] \cong \mathbf{Q} \cong \textstyle\prod_{n \geqq 1} H^n(K \wedge K; \pi_n(B) \otimes \mathbf{Q}),$$

and we let $\prod c_n$ correspond to $[m_K] \otimes 1$.

(8.4) LEMMA. *Let X and K be as in* (8.1). *Then c_n is nonzero for some n satisfying*

$$2(\text{connectivity of } X) + 2 \leqq n \leqq 2 \dim X.$$

Proof. Supposing $[f][g] \in [X, B]$ has infinite order, consider the commutative diagram

$$\begin{array}{ccc}
[X, B] \otimes \mathbf{Q} & & \\
\uparrow \Delta'^* \otimes 1 & & \\
[X \wedge X, B] \otimes \mathbf{Q} & \cong & \prod H^n(X \wedge X; \pi_n(B) \otimes \mathbf{Q}) \\
\uparrow (f \wedge g)^* \otimes 1 & & \uparrow \prod (f \wedge g)^* \\
[K \wedge K, B] \otimes \mathbf{Q} & \cong & \prod H^n(K \wedge K; \pi_n(B) \otimes \mathbf{Q}).
\end{array}$$

Since $[m_K] \otimes 1$ maps vertically to $[f][g] \otimes 1$, which is nonzero by (8.2), we find that

$$\prod (f \wedge g)^*(\prod c_n) \neq 0.$$

But $H^n(X \wedge X; \pi_n(B) \otimes \mathbf{Q})$ is 0 for $n \leqq 2(\text{connectivity of } X) + 1$ by the reduced Künneth Theorem (9.2), and $H^n(K \wedge K, \pi_n(B) \otimes \mathbf{Q})$ is 0 for $n > 2 \dim X$ since $\dim X \geqq \dim K$. Hence (8.4) follows.

(8.5) LEMMA. *If $c_n \neq 0 \in H^n(K \wedge K; \pi_n(B) \otimes \mathbf{Q})$, then the secondary coproduct $\theta : A_n \to (\bar{A} \otimes \bar{A})_n$ is non-zero.*

Proof. Choose a finite connected subcomplex B_α of B containing $m(K \wedge K)$, and let $\mathcal{C}$ be the directed set of finite connected subcomplexes B_β of B containing B_α. Since $\pi_n(B) \otimes \mathbf{Q}$ is a divisible group, Theorem (9.3) implies that

(8.6) $$H^n(B; \pi_n(B) \otimes \mathbf{Q}) \cong \text{inv lim}_{\mathcal{C}} \{H^n(B_\beta; \pi_n(B) \otimes \mathbf{Q})\}.$$

Letting $i_\beta : B_\beta \to B$ denote inclusion, $[i_\beta] \otimes 1$ corresponds to an element $\prod b_{\beta,n}$ under the isomorphism of (8.3):

$$[B_\beta, B] \otimes \mathbf{Q} \cong \textstyle\prod_{n \geqq 1} H^n(B_\beta; \pi_n(B) \otimes \mathbf{Q}).$$

By naturality $\{b_{\beta,n}\}$ defines an element of

$$\text{inv lim} \{H^n(B_\beta; \pi_n(B) \otimes \mathbf{Q})\}.$$

Letting $\rho_n \in H^n(B; \pi_n(B) \otimes \mathbf{Q})$ be the corresponding element under isomorphism (8.6), we find that

$$i_\beta^* : H^n(B; \pi_n(B) \otimes \mathbf{Q}) \to H^n(B_\beta; \pi_n(B) \otimes \mathbf{Q})$$

takes ρ_n to $b_{\beta,n}$.

Let $m_K^\beta : K \wedge K \to B_\beta$ be the restriction of m. Then naturality and the observation that $(m_K^{\beta *} \otimes 1)([i_\beta] \otimes 1) = [m_K] \otimes 1$ implies that $m_K^{\beta *}(b_{\beta,n}) = c_n$ in cohomology.

Consider the following commutative diagram in which the coefficient group $\pi_n(B) \otimes \mathbf{Q}$ is suppressed from the notation.

$$\begin{tikzcd} H^n(B) \arrow[r, "m^*"] \arrow[d, "i_\beta^*"'] \arrow[dr, "m_K^*"] & H^n(B \wedge B) \arrow[d, "j^*"] \\ H^n(B_\beta) \arrow[r, "m_K^{\beta *}"] & H^n(K \wedge K) \end{tikzcd}$$

It follows that $m_K^*(\rho_n) = m_K^{\beta *}(b_{\beta,n}) = c_n$. So if c_n is nonzero then $m^*\rho_n \neq 0$.

It remains to show that m^* is nonzero if rational coefficients are used. The reduced Künneth Theorem (9.2) implies that $H_*(B \wedge B; \mathbf{Q})$ is of finite type, hence the cohomology Universal Coefficient Theorem [5, p. 246] applies. We conclude that

$$H^n(B; \pi_n(B) \otimes \mathbf{Q}) \cong H^n(B) \otimes \pi_n(B) \otimes \mathbf{Q} \cong H^n(B; \mathbf{Q}) \otimes \pi_n(B)$$

and similarly for $B \wedge B$. Since m^* on the left side is non-zero and corresponds to $m^* \otimes 1$ on the right, the proof of (8.5) is complete.

Theorem (7.1) follows immediately from (8.1), (8.4) and (8.5).

9. Appendix

We present here three basic algebraic topology results.

(9.1) THEOREM. *Let B be a pathwise connected H-commutative group whose singular homology $H_*(B)$ is of finite type. Then the abelian group $[K, B]$ is finitely generated for every finite CW complex K.*

Proof. First observe that each $\pi_q(B)$ is finitely generated. This follows in the same way that Cor. 16, p. 509, in [5] is derived from Theorem 15 except that we use Theorem 20 and the fact that any H-space is strongly simple (example 18, p. 510).

To prove that $[K, B]$ is finitely generated we proceed by induction. Observe that $[K, B]$ is trivial if $\dim K = 0$. Suppose $K = L \cup e^n$ where e^n is an n-cell attached to subcomplex L by a map $f : S^{n-1} \to L$. By induction we may assume that $[L, B]$ is finitely generated, as is $[S^n, B]$. Applying $[\ \ , B]$ to the Puppe sequence of f we conclude from exactness that $[K, B]$ is finitely generated.

Our next result is a reduced Künneth Theorem.

(9.2) THEOREM. *Let X, Y be pointed CW complexes, and let G, G' be abelian groups with* Tor $(G, G') = 0$. *There exists a natural split exact sequence*

$$0 \to \tilde{H}_*(X; G) \otimes \tilde{H}_*(Y; G') \xrightarrow{\alpha}$$

$$\tilde{H}_*(X \wedge Y; G \otimes G') \xrightarrow{\beta} \mathrm{Tor}\,(\tilde{H}_*(X; G), \tilde{H}_*(Y; G')) \to 0$$

where α is of degree 0 and β of degree -1. If either (a) $\tilde{H}_*(X)$ *and* $\tilde{H}_*(Y)$ *are of finite type, or* (b) $\tilde{H}_*(Y)$ *is of finite type and G' is finitely generated, then there exists a natural split exact sequence*

$$0 \to \tilde{H}^*(X; G) \otimes \tilde{H}^*(Y; G') \xrightarrow{\alpha}$$

$$\tilde{H}^*(X \wedge Y; G \otimes G') \xrightarrow{\beta} \operatorname{Tor}(\tilde{H}^*(X; G), \tilde{H}^*(Y; G')) \to 0$$

where α is of degree 0 and β is of degree $+1$.

The homology version follows from the relative Künneth Theorem [5, p. 235] applied to the couple $\{X \times \{*\}, \{*\} \times Y\}$ in $X \times Y$. This couple is excisive; indeed, the proof of Lemma 7, p. 190, in [5] generalizes if one uses local contractibility of a CW complex [9, §5, Prop M]. Moreover,

$$H_*(X \times Y, X \vee Y) \cong \tilde{H}_*(X \mathbin{\bar{\wedge}} Y) \cong \tilde{H}_*(X \wedge Y)$$

using [3], where $X \mathbin{\bar{\wedge}} Y = (X \times Y) \cup C(X \vee Y)$ has the homotopy type of $X \wedge Y$ since the base points are non-degenerate.

For the cohomology results apply Theorem 1, p. 249, of [5].

In order to state our final result let (X, A) be a CW pair and let $\mathfrak{D}$ be the directed set of finite subcomplex pairs (Y, B). Inclusion $(Y, B) \to (X, A)$ induces a homomorphism $H^*(X, A; G) \to H^*(Y, B; G)$ and in turn a homomorphism

$$\Phi : H^*(X, A; G) \to \operatorname{inv\,lim}_{\mathfrak{D}} H^*(Y, B; G).$$

Although Φ is not, in general, an isomorphism we do have

(9.3) Theorem. *If the group G is divisible then Φ is an isomorphism.*

Proof. For convenience we deal with single spaces instead of pairs. Divisibility of G and the Universal Coefficient Theorem implies that $H^n(X; G) \cong \operatorname{Hom}(H_n X, G)$ since $\operatorname{Ext}(\ \ , G) = 0$. It follows that

$$\begin{aligned} \operatorname{inv\,lim} H^n(Y; G) &\cong \operatorname{inv\,lim} \operatorname{Hom}(H_n Y, G) \\ &\cong \operatorname{Hom}(\operatorname{dir\,lim} H_n Y, G) \\ &\cong \operatorname{Hom}(H_n X, G) \\ &\cong H^n(X; G) \end{aligned}$$

where we have made use of naturality and the fact that homology behaves nicely with respect to direct limits (cf. Theorem 6, p. 175, of [5]).

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