CHERN CHARACTERS REVISITED

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1. Introduction

The title of this paper refers to an earlier one [1]. Although I still feel that the questions studied in the earlier paper were indeed worth study, I have long felt that when I wrote the earlier paper I did not have a satisfactory way of stating the results. Recently I had occasion to reformulate the results of [1], for some lectures I gave at the University of Chicago. This reformulation is given as Theorem 1 below. I then found that results of a more general nature were contained in some work by Larry Smith [3]. I am grateful to Larry Smith and A. Liulevicius for letting me read a copy of [3] before publication. The object of this note, then, is to answer the question raised in the last sentence of [3], by recording a proof of Larry Smith's theorem which seems more elementary and direct than the one in [3].

2. Statement of results

Let **bu** be the connective BU-spectrum. Then $\pi_2(\mathbf{bu})$ is isomorphic to Z; let $u \in \pi_2(\mathbf{bu})$ be a generator. The homotopy ring $\pi_*(\mathbf{bu})$ is the polynomial ring $Z[u]$. We may identify $u \in \pi_2(\mathbf{bu})$ with its image in $H_2(\mathbf{bu}; Z)$ or $H_2(\mathbf{bu}; Q)$. The homology ring $H_*(\mathbf{bu}; Q)$ is the polynomial ring $Q[u]$. As in [1], let $m(r)$ be the numerical function given by

$$m(r) = \prod_p p^{[r/(p-1)]}.$$

THEOREM 1. *The image of $H_*(\mathbf{bu}; Z)$ in $H_*(\mathbf{bu}; Q)$ is the Z-submodule generated by the elements*

$$u^r/m(r), \quad r = 0, 1, 2, 3, \cdots .$$

Let $H(Q, n)$ be the Eilenberg-MacLane spectrum for the group Q of rational numbers in dimension n. The r^{th} component of the Chern character defines an element

$$ch_r \in H^{2r}(\mathbf{bu}; Q)$$

or a map of spectra

$$\mathbf{bu} \to H(Q, 2r).$$

This map of spectra induces a homomorphism of homology theories, say

$$ch_r : \mathbf{bu}_n(X) \to H_{n-2r}(X; Q).$$

This homomorphism is defined whether X is a space or a spectrum.

Received May 22, 1971.

THEOREM 2 (L. Smith). *The image of*

$$m(r)ch_r : \mathbf{bu}_n(X) \to H_{n-2r}(X; Q)$$

is integral, that is, it is contained in the image of

$$H_{n-2r}(X; Z) \to H_{n-2r}(X; Q).$$

This theorem differs only in minor details from Theorem 3.1 of [3]. That is, I have written $m(r)$ where Smith writes μ_r; the dimensional indexing is slightly different; n may be odd as well as even; and X may be a spectrum as well as a space. Theorem 2.1 of [3] follows, as is remarked at the end of [3].

3. Proof of Theorem 1

The proof proceeds by separating the primes p. Let Q_p be the localisation of Z at p, that is, the subring of fractions a/b with b prime to p. We wish to prove that the image of $H_*(\mathbf{bu}; Q_p)$ in $H_*(\mathbf{bu}; Q)$ is the Q_p-subalgebra generated by u and u^{p-1}/p. We give the proof for the case $p = 2$; the case of an odd prime is similar.

The spectrum **bu** has a (stable) cell decomposition of the form

$$\mathbf{bu} = S^0 \cup_\eta e^2 \cup \cdots$$

where η is the generator for the stable 1-stem, and the cells omitted have (stable) dimension ≥ 4. It follows that the Hurewicz homomorphism

$$Z \cong \pi_2(bu) \to H_2(\mathbf{bu}) \cong Z$$

is multiplication by 2; that is, $H_2(\mathbf{bu})$ is generated by $u/2$. It follows immediately that the image of

$$H_*(\mathbf{bu}) \to H_*(\mathbf{bu}; Q)$$

contains $(u/2)^r$. We wish to prove a result in the opposite direction.

Recall from [1] that we have

$$H^*(\mathbf{bu}; Z_2) = A/(ASq^1 + ASq^{01}),$$

where A is the mod 2 Steenrod algebra. Equivalently, let $\mathbf{HZ}$, $\mathbf{HZ}_2$ be the Eilenberg-MacLane spectra for the groups Z, Z_2 in dimension 0; then the generator in $H^0(\mathbf{bu}; Z_2)$ gives a map of spectra $\mathbf{bu} \to \mathbf{HZ}_2$, which induces a monomorphism

$$H_*(\mathbf{bu}; Z_2) \to H_*(\mathbf{HZ}_2; Z_2).$$

Here $H_*(\mathbf{HZ}_2; Z_2)$ is A_*, the dual of the mod 2 Steenrod algebra [2]. We use this monomorphism to identify $H_*(\mathbf{bu}; Z_2)$ with a subalgebra of A_*; we write ξ_r for the Milnor generators in A_* [2]. Then the image of $u/2 \in H_2(\mathbf{bu}; Z)$ in $H_2(\mathbf{bu}; Z_2)$ is ξ_1^2. The E_2-term of the Bockstein spectral sequence, namely

$$\operatorname{Ker} Sq^1/\operatorname{Im} Sq^1 = \operatorname{Ker} \beta_2/\operatorname{Im} \beta_2,$$

is the polynomial algebra $Z_2[\xi_1^2]$; this fact is essentially in [1], and is easily proved using A_*. The remainder of the argument is obvious from the Bockstein spectral sequence, but I give it in full.

The image of

$$H_{2r}(\mathbf{bu}) \to H_{2r}(\mathbf{bu}; Q)$$

is a finitely-generated abelian group, and since it is non-zero, it is isomorphic to Z. Let $h \in H_{2r}(\mathbf{bu})$ map to a generator. Let $v = u/2$, and let us write $\bar{h}$, $\bar{v}$ for the images of these elements in $H_*(\mathbf{bu}; Z_2)$. Then we have

$$\beta_2 \bar{h} = 0;$$

therefore

$$\bar{h} = \lambda \xi_1^{2r} + \beta_2 k$$

where $\lambda \in Z$ and $k \in H_{2r-1}(\mathbf{bu}; Z_2)$. That is,

$$\bar{h} = \lambda \bar{v}^r + (\delta_2 k)^-,$$

where $\delta_2 : H_{2r-1}(\mathbf{bu}; Z_2) \to H_{2r}(\mathbf{bu}; Z)$ is the integral Bockstein. This gives

$$h = \lambda v^r + \delta_2 k + 2l,$$

where $l \in H_{2r}(\mathbf{bu})$. For the images in $H_{2r}(\mathbf{bu}; Q)$ we have

$$h = \lambda (u/2)^r + 2\mu h$$

where $\mu \in Z$, that is,

$$h = \frac{\lambda}{1 - 2\mu} (u/2)^r$$

where $\lambda/(1 - 2\mu) \in Q_2$. This proves Theorem 1.

4. Proof of Theorem 2

By definition, we have

$$\mathbf{bu}_n(X) = \pi_n(\mathbf{bu} \wedge X).$$

We have therefore to consider the map of homotopy induced by

$$\mathbf{bu} \wedge X \xrightarrow{ch_r \wedge 1} H(Q, 2r) \wedge X.$$

This map evidently factors through

$$\mathbf{HZ} \wedge \mathbf{bu} \wedge X \xrightarrow{1 \wedge ch_r \wedge 1} \mathbf{HZ} \wedge H(Q, 2r) \wedge X \xrightarrow{\mu \wedge 1} H(Q, 2r) \wedge X,$$

where μ is the obvious pairing of Eilenberg-MacLane spectra. Now

$$\pi_n(\mathbf{HZ} \wedge \mathbf{bu} \wedge X)$$

may be interpreted as $H_n(\mathbf{bu} \wedge X)$ and calculated by the ordinary Künneth formula. The terms $\mathrm{Tor}_1^Z\ (H_i(\mathbf{bu}),\ H_j(X))$ evidently map to zero in

$$\pi_n(H(Q,\ 2r) \wedge X) = H_{n-2r}(X;\ Q),$$

since $H_{n-2r}(X;Q)$ is torsion-free. If we consider the term $H_i(\mathbf{bu}) \otimes H_j(X)$, we see that $H_i(\mathbf{bu})$ maps into $\pi_i(H(Q,\ 2r))$, which is zero unless $i = 2r$. There remains the term $H_{2r}(\mathbf{bu}) \otimes H_{n-2r}(X)$. Here $H_{n-2r}(X)$ maps to $H_{n-2r}(X;Q)$ by the canonical map, and $u^r \in H_{2r}(\mathbf{bu})$ maps to $1 \in H_{2r}(Q;2r) = Q$ under ch_r. Using Theorem 1, we see that the image of $H_{2r}(\mathbf{bu}) \otimes H_{n-2r}(X)$ in $H_{n-2r}(X;\ Q)$ is $1/m(r)$ times the image of $H_{n-2r}(X;\ Z)$. This proves Theorem 2.

References

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