REGULAR PROPERLY DISCONTINUOUS Z^n-ACTIONS ON OPEN MANIFOLDS

BY

P. F. DUVALL, JR. AND L. S. HUSCH[1]

0. Introduction

Let X be a space with metric d, and let h be a homeomorphism of X onto itself. We say that h is *regular* at $x \in X$ provided that for each $\varepsilon > 0$ there is a $\delta > 0$ such that $d(x, y) < \delta$ implies that $d(h^n(x), h^n(y)) < \varepsilon$ for all integers n. Two homeomorphisms h_1 and h_2 of X are *topologically equivalent* if there exists a homeomorphism k of X such that $h_1 = k^{-1}h_2 k$. B. v. Kerékjártó [20] introduced the notion of regularity and showed that homeomorphisms of the 2-sphere which were regular except at a finite number of points were topologically equivalent to fractional linear transformations of complex numbers. S. Kinoshita [22], T. Homma and S. Kinoshita [8], and L. S. Husch [14], [15], [16], have extended these investigations to higher dimensions.

In Sections 1 and 3 of this paper, we investigate the notions of regularity and proper discontinuity for actions of infinite groups on metric spaces. In sections 2 and 4 we consider actions of Z^k, the free abelian group on k generators with the discrete topology, with the following two questions in mind: What manifolds M can support (effective) regular, properly discontinuous Z^k actions? When such actions exist, how can one classify them with respect to topological equivalence? In particular, for $k \leq n$, let the *standard* Z^k-action on $\mathbf{R}^n$ be the group whose i^{th} generator is the map

$$(x_1, \cdots, x_i, \cdots, x_n) \to (x_1, \cdots, x_i + 1, \cdots, x_n).$$

We show (Theorem 11) that if G is a regular, properly discontinuous Z^k-action on $\mathbf{R}^n$ whose extension to S^n is irregular at ∞, (definitions below), then $k \leq n$, and if $k = n > 4$, then G is topologically equivalent to the standard Z^n action, and we give examples of non-standard Z^k actions on $\mathbf{R}^n$ for $n \geq k + 3$.

We also show that Z is the only group which can have regular properly discontinuous (effective) actions on an open manifold M with two ends, (Theorem 8), and make a start on the classification problem for such actions.

1. Some preliminary definitions and results

By a *space*, we will mean a locally compact, separable metrizable space. Let X be a space with metric d and let $H(X)$ be the group of homeomorphisms of X with the compact open topology. If G is a subgroup of $H(X)$ which is a

Received March 8, 1971.

[1] The research of the second author was partially supported by a National Science Foundation grant.

topological group, we say that G *acts* on X and refer to G as an *action*. If K is a topological group which is isomorphic to G, we may also refer to G as a *K-action* on X. (Thus we consider only effective transformation groups.) If G_1 and G_2 are actions on X we say that G_1 *is topologically equivalent to* G_2 if G_1 is conjugate to G_2 in $H(X)$. We say that the action G is *regular* at $x \in X$ provided that, for each $\varepsilon > 0$, there is $\delta > 0$ such that for each $g \in G$, $d(x, y) < \delta$ implies that $d(g(x), g(y)) < \varepsilon$. If G is regular at each $x \in U \subseteq X$, we say that G is a *regular action* on U. G is *irregular at* x if for each $g \in G$, $g \neq$ identity, g fails to be regular at x. G is *properly discontinuous* at x if there is a neighborhood U of x such that $gU \cap U = \emptyset$ for each $g \in G$ such that $g \neq$ identity. G satisfies *Sperner's condition* on $U \subseteq X$ if for each compact set $C \subseteq U$, the set $\{g \in G \mid gC \cap C \neq \emptyset\}$ is finite.

Following Freudenthal [7], we define an *end* of a space X to be a collection $\mathcal{E}$ of subsets of X which is maximal with respect to the properties:

(i) each $E \in \mathcal{E}$ is a connected open non-empty set with compact frontier;
(ii) for each pair $E_1, E_2 \in \mathcal{E}$ there is an E_3 in $\mathcal{E}$ such that $E_3 \subseteq E_1 \cap E_2$; and
(iii) $\bigcap \{\mathrm{Cl}\ (E) \mid E \in \mathcal{E}\} = \emptyset$.

Given a space with ends $\{\mathcal{E}_\alpha\}$, we can define a new space X^*, called the (Freudenthal) *end point compactification* of X, $X^* = X \cup \{\omega_\alpha\}$ where ω_α is a point associated with the end $\mathcal{E}_\alpha$. A topology is defined on X^* by letting a neighborhood basis for $x \in X$ be

(i) a neighborhood basis for x in X, if $x \in X$, and
(ii) the collection of sets of the form $E \cup \omega_\alpha$, where $E \in \mathcal{E}_\alpha$, if $x = \omega_\alpha$.

If X is connected, it follows from [17] that X^* is a compact metric space. Henceforth, we shall assume that X has metric induced from a metric on X^*. This choice of metric is important since the regularity of an action on a non compact space depends on the metric. For example, the dilation $x \to \frac{1}{2}x$ generates a Z-action on $\mathbf{R}^n$ which is not regular anywhere with respect to the usual metric, but which is regular at each point except 0 and ∞ with respect to the metric induced from S^n. However, the reader can easily verify the following.

PROPOSITION 1. *Suppose X is connected. If G is an action on X, then G induces a unique action G^* on X^*. The regularity of G at $x \in X$ is independent of the metric induced from X^*. If H and K are topologically equivalent G actions on X, $H = k^{-1}Kk$, H is regular at x if and only if K is regular at $k(x)$.*

We will also need the following proposition.

PROPOSITION 2. *Let X be connected with finitely many ends, and let the action G be regular at $x_0 \in X$ with respect to the metric d. Then G is regular at x_0 with respect to every metric d^* induced from X^*.*

Proof. Let $\omega_1, \cdots, \omega_n$ be the end points of X^*, and let $\varepsilon > 0$ be given. We may assume that ε is small enough that the sets $\{N(\varepsilon, \omega_i)\}_{i=1}^n$ are pairwise disjoint, where $N(\alpha, S)$ denotes the α neighborhood of the set S with respect to the metric d^*. Let $W_1 = X^* - \bigcup_{i=1}^n N(\varepsilon/6, \omega_i)$. Since W_1 is compact, there is an $\varepsilon_1 > 0$ such that if $x, y \in W_1$ and $d(x, y) < \varepsilon_1$, then $d^*(x, y) < \varepsilon$. Let $W_2 = X^* - \bigcup_{i=1}^n N(\varepsilon/3, \omega_i)$. There is an $\varepsilon_2 > 0$ such that if $x \in W_2$ and $d(x, y) < \varepsilon_2$, then $y \in W_1$. Finally, there is an $\varepsilon_3 > 0$ such that for each i, if $x \in N(\varepsilon/3, \omega_i)$ and $d(x, y) < \varepsilon_3$, then $y \in N(\varepsilon/2, \omega_i)$. There is a $\delta_1 > 0$ such that if $d(x_0, y) < \delta_1$, then

$$d(g(x_0), g(y)) < \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\} \quad \text{for all } g \in G.$$

If $\delta > 0$ is chosen so that $d^*(x_0, y) < \delta$ implies that $d(x_0, y) < \delta_1$, it is easy to check that $d^*(x_0, y) < \delta$ implies that $d^*(g(x_0), g(y)) < \varepsilon$ for all $g \in G$.

If G is an action on X and $x \in X$, the *orbit*, Gx, of x under G is the set $\{g(x) \mid g \in G\}$. The orbits of G partition X, and the resulting quotient space X/G is called the *orbit space*. We will often use the fact that if G is a properly discontinuous action on a connected space, the natural projection $X \to X/G$ is a covering map [32]. In particular, we have the following proposition.

PROPOSITION 3. *Let X be connected and locally path connected with a finite number of ends and let G and H be properly discontinuous actions on X such that there is a homeomorphism $h : X/G \to X/H$ with the property that*

$$(hp_1)_*(\pi_1(X)) = p_{2*}(\pi_1(X)),$$

where $p_1 : X \to X/G$ and $p_2 : X \to X/H$ are the natural projections. Then G and H are topologically equivalent.

Proof. By [32; p. 76] there exists a homeomorphism $k : X \to X$ such that $p_2 k = hp_1$. Let $g \in G$ and $x \in X$. Since $p_1 g(x) = p_1(x)$,

$$p_2 kg(x) = hp_1 g(x) = hp_1(x_1) = p_2 k(x),$$

so there exists $j \in H$ such that $kg(x) = jk(x)$. Let

$$Y = \{y \in X \mid g(y) = k^{-1}jk(y)\};$$

it is not difficult, using covering space theory, to show that $Y = X$, so that $g \in k^{-1}Hk$. Suppose that $g \in k^{-1}Hk$. For $x \in X$ and $g = k^{-1}jk$,

$$p_1 k^{-1}jk(x) = h^{-1}hp_1k^{-1}jk(x) = h^{-1}p_2 kk^{-1}jk(x) = h^{-1}p_2 jk(x)$$
$$= h^{-1}p_2 k(x) = p_1(x).$$

It follows as before that $g \in G$. Hence $G = k^{-1}Hk$.

Remark. If, in the above proposition, X is a smooth manifold and G a group of diffeomorphisms, we may conclude that k is a diffeomorphism. Similar remarks hold in the piecewise linear (PL) category.

In the light of Homma and Kinoshita's work [10], [11] on Z-actions, one might suspect that if X is "nice" and G is a discrete action such that G^* is regular on X and irregular on $X^* - X$, then G is properly discontinuous.

EXAMPLE 4. *There exists an action G on S^2 which is regular on $\mathbf{R}^2$ irregular at ∞ and G is algebraically isomorphic to Z^2, but which is not properly discontinuous on $\mathbf{R}^2$.*

Proof. Let $h,\ k \in H(\mathbf{R}^2)$ be defined by

$$h(x, y) = (x, y + 1), \qquad k(x, y) = (x, y + \sqrt{2}).$$

Then h and k generate an action G which extends to an action G^* on S^2. Since G is clearly regular with respect to the usual metric on $\mathbf{R}^2$, it follows from Proposition 3 that G^* is regular on $S^2 - \{\infty\}$. It is easy to check that G^* is irregular at ∞. To see that G is not properly discontinuous on $\mathbf{R}^2$, recall that the set $\{m + n\sqrt{2} \mid m,\ n \in Z\}$ is dense in $\mathbf{R}$. It follows that G is not properly discontinuous. Note, however, G is not a Z^2-action since each $g \in G$ is a limit point of G. It is unknown to the authors whether there exists a Z^2-action on S^n which is regular on $\mathbf{R}^n$ and irregular at ∞, but which is not properly discontinuous on $\mathbf{R}^n$.

We conclude this section by stating a theorem of Homma and Kinoshita and a corollary which will allow us to assume that we are working with manifolds with at most two ends.

PROPOSITION 5 (Homma and Kinoshita). *Let X be a compact metric space such that X contains no isolated points and $X - A$ is connected for each finite subset A of X. Let G be a Z-action on X which is regular on X except possibly for a finite number of points. Then the number of points at which G fails to be regular is at most two* [10].

COROLLARY 6. *Let X be connected with finitely many ends, and suppose that no finite set of points in X separates X. Let G act on X such that G is regular on X but G^* is irregular on $X^* - X$. Then X has at most two ends.*

Remark. If we assume that X is locally connected, we can omit the finiteness conditions in Corollary 6 [19].

2. Manifolds with two ends

PROPOSITION 7. *Let X be connected with two ends and let G be a regular action on X such that G^* is irregular on $X^* - X$. Then the orbit space X/G is compact.*

Proof. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be the two ends of X, let $V_0 \in \mathcal{E}_1 - \mathcal{E}_2$, and let V be the closure of V_0 in X^*. If $g \in G$, there exists an integer n such that $g^n(V) \subseteq \operatorname{int} V$ [22], [11]. Let $W = \operatorname{cl}(V - g^n(V))$; we claim that

$$\bigcup_{i=0}^{\infty} g^{ni}(W) = V - \omega(\mathcal{E}_1).$$

Let $v \in V - \omega(\mathcal{E}_1)$; since $\limsup_{i\to+\infty} g^{ni}(V) = \omega(\mathcal{E}_1)$, there exists only finitely many j's such that $v \in g^{nj}(V)$. Hence

$$v \in g^{nj}(V) - g^{n(j+1)}(V) = g^{nj}(V - g^{n}(V)) = g^{nj}(W)$$

for some j. Therefore $\bigcup_{i=0}^{\infty} g^{ni}(W) = V - \omega(\mathcal{E}_1)$.

Suppose $x \in X$; since $\lim_{i\to+\infty} g^{ni}(x) = \omega(\mathcal{E}_1)$ [22], for some i, $g^{ni}(x) \in V$. It follows that $X = \bigcup_{i=-\infty}^{\infty} g^{ni}(W)$. Consider the natural projection $p : X \to X/G$. Note that $p(W) = X/G$ and since W is compact, X/S is compact.

The following theorem is a partial generalization of a theorem of Kinoshita [23].

THEOREM 8. *Let X be connected with two ends and let G be a properly discontinuous regular action on X such that G^* is irregular on $X^* - X$. Then G is a Z-action.*

Proof. By Theorem 3 of [23], G satisfies Sperner's condition on X. Since X/G is compact, by Theorem 12 of [5], G contains an infinite cyclic subgroup H of finite index, say r. (Although Theorem 12 of [5] is stated for complexes, the proof generalizes to the case under consideration.)

Suppose $G = g_1H \cup g_2H \cup \cdots \cup g_rH$, where $g_1 =$ identity. Let h be a generator of H; then since $g_iHg_i^{-1}$, $i = 1, 2, \cdots, r$, also has index r in G, some power of h lies in $g_iHg_i^{-1}$. Hence $H \cap g_iHg_i^{-1}$ is a nontrivial subgroup of H. Since the intersection of a finite number of nontrivial subgroups of H is also nontrivial, $\bigcap_{i=1}^{r} g_iHg_i^{-1}$ is nontrivial. But $\bigcap_{g\in G} gHg^{-1} = \bigcap_{i=1} g_iHg_i^{-1}$ is therefore a normal infinite cyclic subgroup of G of finite index. Hence there is no loss of generality in assuming that H is normal in G.

Suppose that there exists $g \in G$ such that g does not commute with h, the generator of H. Since the inner automorphism defined on G by g maps H onto H, we have $gh^{-1} = hg$. Since G/H is finite, there exist integers n and m such that $g^n = h^m$. Hence $g^{n-1}hg = h^{m-1}$ and we have

$$h^{2m+1} = g^nhg^n = gh^{m-1}g^{n-1} = h^{1-m}g^n = h.$$

It follows that $m = 0$ and G has an element of finite order contradicting [22]. Hence H lies in the center of G; this implies that the center has finite index, say n, in G. By [4], each commutator in G has order dividing n and hence must be the identity. Therefore G is abelian and therefore $G = Z$ by [23].

THEOREM 9. *Let M be an open connected n-manifold with two ends which has the homotopy type of a finite complex, $n \neq 4, 5$. If $n = 3$, suppose that M contains no fake 3-cells;*—i.e. *if Σ is a locally flat contractible 2-sphere in M, then Σ bounds a 3-cell in M and if $n > 5$, suppose that the Whitehead group of $\pi_1(M)$ is trivial. If G is a regular Z-action on M such that G^* is irregular on $M^* - M$, then there exists a closed submanifold N of M and homeomorphisms*

$$\lambda : M \to N \times \mathbf{R} \quad \textit{and} \quad \eta : N \to N$$

such that, if H is the action of $N \times \mathbf{R}$ generated by $(x, t) \to (\eta(x), t + 1)$, then $\lambda^{-1}H\lambda$ is topologically equivalent to G.

Proof. By Proposition 7 and [21], M/G is a closed connected n-manifold. By [33], if $n = 3$, and by [30], if $n > 5$, there exists a closed $(n - 1)$ submanifold N of M/G such that M/G fibers over the circle with fiber N. (Although Theorem 4.1 of [30] is stated in the differential category, it is also valid in the topological category; see [30; p. 2].) Hence there exists a homeomorphism $\lambda : M \to N \times \mathbf{R}$ such that if $p : M \to M/G$ is the natural projection, then $\lambda p^{-1}(N) = \bigcup_{r \in N} N \times \{r\}$.

Let $N_r = \lambda^{-1}(N \times \{r\})$ and let $g \in G$ such that $g(N_0) = N_1$. Let

$$\eta : N \to N$$

be the homeomorphism defined by $\lambda g \lambda^{-1}(x, 0) = (\eta(x), 1)$ and let H be the action of $N \times \mathbf{R}$ generated by $(x, t) \to (\eta(x), t + 1)$.

Let T be the compact submanifold of M whose boundary is $N_0 \cup N_1$ and let $q : M \to M/\lambda H \lambda^{-1}$ be the natural projection. Note that

$$q(T) = M/\lambda H \lambda^{-1} \quad \text{and} \quad p(T) = M/G.$$

Define $\alpha : M/\lambda H \lambda^{-1} \to M/G$ by $\alpha(q(x)) = p(x)$ for each $x \in T$. It is easily seen that α is a homeomorphism such that $\alpha(q(N_0)) = p(N_0)$. We have the following commutative diagram

$$\begin{array}{ccccccc}
\pi_1 N_0 & \xrightarrow{q_*} & \pi_1 q(N_0) & \xrightarrow{\alpha_*} & \pi_1 p(N_0) & \xleftarrow{p_*} & \pi_1 N_0 \\
\Big\downarrow{\scriptstyle i_*} & & \Big\downarrow{\scriptstyle j_*} & & \Big\downarrow{\scriptstyle k_*} & & \Big\downarrow{\scriptstyle i_*} \\
\pi_1 M & \xrightarrow{q_*} & \pi_1(M/\lambda H \lambda)^{-1} & \xrightarrow{\alpha_*} & \pi_1(M/G) & \xleftarrow{p_*} & \pi_1 M
\end{array}$$

where i, j, k are inclusion maps. Note that

$$\begin{aligned}(\alpha q)_*(\pi_1 M) = (\alpha q i)_*(\pi_1 N_0) = (k \alpha q)_*(\pi_1 N_0) &= k_*(\pi_1 p(N_0)) \\ &= (kp)_*(\pi_1 N_0) = (pi)_*(\pi_1 N_0) = p_*(M).\end{aligned}$$

Apply Proposition 3.

Remarks. (1) If we assume that G is either a differentiable or piecewise linear action, then G is differentiably or piecewise linearly equivalent to $\lambda^{-1} H \lambda$.

(2) If we assume that the projective class group of $\pi_1(M)$, $\tilde{K}_0(Z\pi_1(M))$, is zero instead of the Whitehead group, it is possible to show that M is homeomorphic to $N \times R$ at least in the piecewise linear and differential case (and probably in the topological case) [29], [6]. If $\tilde{K}_0(Z\pi_1(M)) \neq 0$, it may be possible to construct a counterexample (see [29]).

(3) If M is homeomorphic to $N \times R$ but the Whitehead group of $\pi_1(M)$ is not trivial, then G need not be topologically equivalent to a product action since there exist nontrivial h-cobordisms whose boundary components are homeomorphic [26; p. 400].

3. Some equivalent conditions

The following theorem is known when $G = Z$ [22]. The implication (10.1) $\Rightarrow$ (10.3) has also been shown in [23] and [18].

THEOREM 10. *Let X be connected with a finite number of ends, suppose no finite set of points in X separates X and let G act on X. The following conditions are equivalent.*

(10.1) *G is a properly discontinuous regular action on X but G^* is irregular on $X^* - X$.*

(10.2) *G has no elements of finite order and satisfies Terasaka's condition* [34]: *$\limsup_{g \in G}\{g(C)\} = X^* - X$ for each compact set $C \subseteq X$.*

(10.3) *G has no elements of finite order and satisfies Sperner's condition on X.*

Proof. (10.1) $\Rightarrow$ (10.2). Suppose $y \in \limsup_{g \in G}\{g(C)\}$ for some compact subset C of X. There exist sequences $\{c_i\}_{i=1}^{\infty} \subseteq C$ and $\{g_i\}_{i=1}^{\infty} \subseteq G$ and $c \in C$ such that $\lim_{i \to +\infty} g_i(c_i) = y$ and $\lim_{i \to +\infty} c_i = c$. By Lemma 2.3 of [18], $\lim_{i \to +\infty} g_i(c) = y$ and by Theorem 2.2 of [18], $y \in X^* - X$. It follows from [22] that

$$X^* - X \subseteq \limsup_{g \in G}\{g(C)\}.$$

(10.2) $\Rightarrow$ (10.3). It is easily seen that (10.2) implies that for each $\varepsilon > 0$, the set $\{g \in G \mid g(C)$ does not lie in the ε-neighborhood of $X^* - X\}$ has at most finitely many elements. (10.3) follows easily.

(10.3) $\Rightarrow$ (10.1). It is easily seen that if G satisfies Sperner's condition, then G is properly discontinuous. Let G_1 be an infinite cyclic subgroup of G; then G_1 also satisfies Sperner's condition on X. As remarked above, Theorem 10 is known in the case when $G = Z$ and hence G_1 is regular on X and is irregular on $X^* - X$. By Corollary 6, X has at most two ends. If X has two ends, then the proof of Theorem 8 shows that $G = Z$ and the implication follows from [23].

Suppose X has one end $\mathcal{E}$ and let $x \in X$ and $\varepsilon > 0$. Let $\delta_0 = d(x, \omega(\mathcal{E}))$ and consider

$$G_0 = \{g \in G \mid d(x, y) < \delta_0 \text{ implies } d(\omega(\mathcal{E}), g(y)) < \varepsilon/2\};$$

we claim that G_0 is finite. Suppose to the contrary that there exist sequences

$$\{g_i\}_{i=1}^{\infty} \subseteq G_0 \quad \text{and} \quad \{x_i\}_{i=1}^{\infty} \subseteq X$$

such that $d(g_i(x_i), \omega(\mathcal{E})) \geq \varepsilon/2$ and $d(x_i, x) < \delta_0$. We may assume that

$$\lim_{i \to +\infty} x_i = y \quad \text{and} \quad \lim_{i \to +\infty} g_i(x_i) = w.$$

Let $C = \{x_i, y, g_i(x_i), w\}$; note that C is a compact subset of X such that $g_i C \cap C \neq \emptyset$ for each i. This contradicts (10.3); hence G_0 is finite.

Let $G_0 = \{g_1, g_2, \cdots, g_n\}$ and choose $\delta_i > 0$ such that $d(x, y) < \delta_i$ implies $d(g_i(x), g_i(y)) < \varepsilon$. Let $\delta = \text{Minimum}\ \{\delta_0, \delta_1, \cdots, \delta_n\}$; this is the desired δ to show that G is regular at x.

4. Manifolds with one end

THEOREM 11. *Let U be an open contractible n-dimensional manifold and let G be a properly discontinuous regular Z^k action on U such that G^* is irregular on $U^* - U$; then $k \leq n$. If $k = n > 4$ or if $k = n = 3$ and U contains no fake 3-cells, then U is homeomorphic to $\mathbf{R}^n$ and G is topologically equivalent to the standard Z^n-action.*

Proof. By [21], the orbit space U/G is an n-dimensional manifold. Note that U/G is an Eilenberg-MacLane $K(Z^k, 1)$-space [32]. Since the product of k 1-spheres, T^k, is also a $K(Z^k, 1)$-space and both T^k and U/G have the homotopy type of a CW-complex, then T^k and U/G are homotopy equivalent. Since $H_k(T^k) \neq 0$, $k \leq n$.

Suppose $k = n$; since $H_k(T^k) \neq 0$, U/G is compact. By [12], U/G is homeomorphic to T^n if $n > 4$. If $n = 3$, U/G contains no fake 3-cells [1] and is homeomorphic to T^3 by [35]. By uniqueness of universal covering spaces, U is homeomorphic to $\mathbf{R}^n$ and by Proposition 3, G is equivalent to the standard Z^n-action.

EXAMPLE 12. *For each $k > 0$ and $n \geq 4$, there exists an n-manifold M and a regular properly discontinuous Z^k-action on M whose extension to M^* is irregular on $M^* - M$.*

Proof. Let K be a finite 2-complex such that $\pi_1(K) = Z^k$ and let N be a regular neighborhood of some piecewise linear embedding of K in the $(n + 1)$-sphere [13]. Note that $\pi_1(\text{bdry } N) = Z^k$. Let M be the universal covering space of bdry N and let G be the covering transformation group. By [21], G satisfies Sperner's condition and the conclusion follows from Theroem 10.

If K is formed by using the stanadard presentation for Z^k, $k \geq 2$, it is not difficult to see that M does not have the homotopy type of a finite complex.

CONJECTURE. *If U is an open connected n-manifold with the homotopy type of a finite complex and if G is a regular properly discontinuous Z^k-action on U such that G^* is irregular on $U^* - U$, then $k \leq n$.*

THEOREM 13. *Let U be an open simply connected n-manifold with the homotopy type of a finite complex and let G be a regular properly discontinuous Z^k-action on U such that G^* is irregular on $U^* - U$ and U/G is compact. Then U is homeomorphic to $V \times \mathbf{R}^k$, provided $n - k \geq 6$.*

Proof. Let $G = G_k \supset G_{k-1} \supset \cdots \supset G_1$ be a sequence of subgroups such that G_i and G_{i+1}/G_i are isomorphic to Z^i and Z respectively. Let $U_i = U/G_{k-i}$ and note that we get a sequence of covering maps

$$U \xrightarrow{p_1} U_1 \xrightarrow{p_2} U_2 \rightarrow \cdots \xrightarrow{p_k} U/G.$$

Since U is the universal covering space of each U_i, U has the homotopy type of a finite complex, and $\tilde{K}_0(Z^i) = 0$, it follows from [37] that each U_i has the homotopy type of a finite complex.

Consider $p_k : U_{k-1} \rightarrow U/G$ which induces a map $f_k : U/G \rightarrow S^1$ such that

$(f_k)_*$ is an epimorphism on the fundamental groups. Since the Whitehead group of $\pi_1(U/G) = Z^k$ is zero, by [30] U/G fibers over the circle and U_{k-1} is homeomorphic to $N_1 \times \mathbf{R}$ for some closed $(n-1)$-manifold. Suppose $U_{k-1} = N_1 \times \mathbf{R}$ and $N_1 = N_1 \times \{0\}$.

Hence U_{k-2} is homeomorphic to $p_{k-1}^{-1}(N_1) \times \mathbf{R}$. In particular, $p_{k-1}^{-1}(N_1)$ has the homotopy type of a finite complex. We proceed as before to show that $p_{k-1}^{-1}(N_1)$ is homeomorphic to $N_2 \times \mathbf{R}$ for some closed $(n-2)$-manifold and hence U_{k-2} is homeomorphic to $N_2 \times \mathbf{R}^2$. The proof is completed by induction.

THEOREM 14. *Let M be homeomorphic to the interior of a compact connected manifold N with connected boundary and let G be a regular action on M such that G^* is irregular on $M^* - M$; then $\pi_1(N, \text{bdry } N)$ is trivial.*

Proof. Note that M^* is semilocally 1-connected at each point [32], let $p : M' \to M^*$ be the universal covering of M^*. Since G^* has a fixed point, G^* can be lifted to an action G' of M' [2; p. 231];—i.e. G^* and G' are algebraically isomorphic and $pG' = G^*p$.

Let $x \in M^* - M$ and $x' \in p^{-1}(x)$. There exists a compact neighborhood U of x' in M' such that $p \mid U$ is a homeomorphism. Let $g \in G$, $g \neq$ identity and let $h \in G'$ such that $ph = gp$. There exists an integer n such that

$$g^n(\text{Cl}(M - pU)) \subseteq \text{int } pU.$$

Let $V = p^{-1}g^n(\text{Cl } (M - U)) \cap U$; note that $p \mid h^{-n}V \cup U$ is a homeomorphism of $h^{-n}V \cup U$ onto M^* and hence $M' = M^*$. Therefore $\pi_1(N, \text{bdry } N) = \pi_1(M^*, X)$ is trivial.

COROLLARY 15. *If dimension $M = 2$, $M = \mathbf{R}^2$.*

COROLLARY 16. *If dimension $M = 3$, then N is either a 3-cell or a solid torus (—i.e. N is homeomorphic to a regular neighborhood of a tamely embedded wedge of 1-spheres).*

Proof. Note that if Σ is a locally flat 2-sphere in M which bounds a contractible manifold and $g \in G$, $g \neq$ identity, then for some n, $g^n(\Sigma)$ lies in a collar of bdry N in N. Hence Σ bounds a 3-cell in M. We now apply [27].

EXAMPLE 17. *For each $n \geq 4$ and $r \leq n - 3$, there exists a regular and properly discontinuous Z^r-action on $\mathbf{R}^n$ whose extension to S^n is irregular at ∞ but which is not topologically equivalent to the standard Z^r-action.*

Proof. This is a generalization of results from [15]. Since the techniques of proof are similar in the light of the results of this paper, we sketch a proof.

If $r = n - 3$, let X be Whitehead's example of a contractible 3-manifold which is not homeomorphic to $\mathbf{R}^3$ [39] and if $r < n - 3$, let X be the interior of a compact contractible $(n - r)$-manifold whose boundary is not simply connected [24] [28] [3]. Note that $X \times \mathbf{R}^r$ is homeomorphic to $\mathbf{R}^n$ [25].

Consider $T^r \times X$; if $T^r \times X$ were homeomorphic to $T^r \times \mathbf{R}^{n-r}$, then by Proposition 1.3 of [15], X would be properly homotopically equivalent to $\mathbf{R}^{n-r}$.

In particular, X would be "simply-connected at infinity" [31]; this would be a contradiction on the choice of X.

Let U be the universal cover of $T^r \times X$ and let G be the covering transformation group. Note that U is homeomorphic to $\mathbf{R}^r \times X = \mathbf{R}^n$ and G is a Z^k-action which satisfies Sperner's condition. The result follows from Proposition 3 and Theorem 10.

Remarks. (1) In Theorem 11, if $k = n = 3$, the result is valid in both the differentiable and piecewise linear category. However, if $k = n > 4$, the results are not valid [36] in the differentiable and piecewise linear category. For example, the piecewise linear equivalence classes of Z^n actions on $\mathbf{R}^n$ are classified by $H^3(T^n; Z_2)$.

(2) The results of 12, 13, and 17 are valid in both the differentiable and piecewise linear categories.

(3) C. T. C. Wall [38] defines a P-group of rank n inductively as follows. Z is the only P-group of rank 1. A P-group of rank n is any group which is the extension of a P-group of rank $(n - 1)$ by Z. Note that Z^n is a P-group of rank n. All the theorems and examples of this section on Z^n actions remain valid when Z^n is replaced by P-group actions.

(4) One can similarly define a standard Z^k-action on an infinite dimensional separable Frechet space E. The notion of regularity is no longer useful in characterizing actions on E; however, it can be shown that any Z^k-action on E which satisfies either Sperner's condition or Terasaka's condition is topologically equivalent to the standard Z^k-action. This is a straightforward generalization of [14].

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VIRGINIA POLYTECHNIC INSTITUTE AND STATE UNIVERSITY
BLACKSBURG, VIRGINIA