A DUALITY THEOREM FOR THE REPRESENTATION RING OF A COMPACT CONNECTED LIE GROUP

BY
JACK M. SHAPIRO

Let G be a compact connected Lie group. One may define $R(G)$, the *complex representation ring* of G, as in [1], [9]. If H is a subgroup of maximal rank we may consider $R(G)$ as a subring of $R(H)$ [6], making $R(H)$ an $R(G)$-module.

An extension of the Weyl character formula found in [3] allows us to define an $R(G)$-module homomorphism from $R(H) \to \operatorname{Hom}_{R(G)}(R(H), R(G))$ in a very natural way. It is conjectured that if $\pi_1(G)$ and $\pi_1(H)$ have no 2-torsion then this map provides a *duality isomorphism* between $R(H)$ and its dual.

This result together with [8] will yield a new proof of the conjecture of Atiyah-Hirzebruch that $\alpha : R(H) \to K(G/H)$ is onto [2] (see §9). In this paper the duality isomorphism is proven for G *any classical group* and H, a suitable subgroup of maximal rank. Furthermore $R(H)$ is shown to be a free $R(G)$-module. Though this result already appears in [8] our proof will provide an explicit basis. This together with the duality isomorphism will provide a basis for the free abelian group $K(G/H)$ (see §9).

For those more familiar with the *equivariant K-theory* of [10], we know that

$$R(H) \cong K_G(G/H) \quad \text{and} \quad R(G) \cong K_G(\text{point}).$$

The duality isomorphism is then a map from

$$K_G(G/H) \to \operatorname{Hom}_{K_G(\text{point})}(K_G(G/H), K_G(\text{point})),$$

and provides a *"Poincare duality"* result for this cohomology theory.

1. Preliminaries

Let G be a compact Lie group. We consider $R(G)$, the *complex representation ring* of G, defined in the following manner. Form the free abelian group on the set of equivalence classes of finite dimensional complex representations of G. Tensor product of representations induces a ring structure and the ring thus formed is $R(G)$ (see [1], [6], [9]). If H is a closed subgroup of G we can consider $R(H)$. The restriction of representations induces a ring homomorphism $R(G) \to R(H)$.

A compact connected Lie group G always contains a maximal connected abelian subgroup (unique up to conjugation) called a *maximal torus.* Its dimension is called the *rank of* G. If T is a maximal torus we get a monomorphism $R(G) \to R(T)$ (see [6]). Let $N(T)$ denote the normalizer of T.

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The quotient group $N(T)/T$ is denoted $W(G)$ and is called the *Weyl group* of G. It is a finite group. It acts in an obvious way as a group of automorphisms of T and hence of the ring $R(T)$. It is a basic fact that via the above monomorphism *$R(G)$ can be identified with the subring of $R(T)$ left fixed by $W(G)$*. In symbols $R(G) = R(T)^{W(G)}$.

Let $L(T)$ denote the Lie algebra of T and $\exp : L(T) \to T$, the exponential homomorphism. The *integer lattice* is then $\exp^{-1}$ (identity). Since exp is onto [1] we get

$$T \cong L(T)/\exp^{-1}\ (\text{id.}).$$

If w is a linear form, $w : L(T) \to R^1$, which takes integer values on the integer lattice, we call w a *weight* of G. We will also think of a weight, w, as a 1-dimensional representation, where the representation maps $t \in T$ to multiplication by $e^{2\pi i w(t)}$.

As usual we need to consider the *adjoint representation*, Ad, of G. When G is assumed compact and connected, we may view it as a homomorphism $\mathrm{Ad} : G \to SO(n)$, $n =$ dimension of G. (See e.g. [1].) The restriction of Ad to T acts trivially on T; the quotient space g/t may be decomposed into a number (denoted m) of 2-dimensional T invariant subspaces. The complexification of each of these 2-dimensional representations of T splits into mutually dual 1-dimensional complex representations; the two corresponding weights are thus negatives of each other (a 1-dimensional representation is multiplication by $e^{2\pi i w(t)}$ for some weight w [1]). The collection of $2m$ weights thus obtained are the roots of G. The hyperplanes of t on which the roots vanish are the walls of the Stiefel Diagram [1]. Any component of the complement of the walls can be selected as the *fundamental Weyl chamber* and this, in turn, determines a set, R^+, of m *positive roots* of G.

Each element, σ, of $W(G) = N(T)/T$ gives an automorphism of T. We set $(-1)^\sigma = -1$ if σ reverses the orientation of T and $(-1)^\sigma = +1$ otherwise. An element of $x \in R(T)$ is called *symmetric* if $\sigma(x) = x$ for all $\sigma \in W(G)$, and *anti-symmetric* if $\sigma(x) = (-1)^\sigma x$ for all $\sigma \in W(G)$.

Given any $x \in R(T)$, let $A(x) = \sum_{\sigma \in W(G)} (-1)^\sigma \sigma(x)$. $A(x)$ is called the *elementary alternating sum* of x. It is clear that $A(x)$ is anti-symmetric for all $x \in R(T)$. If w is a weight then by $A(w)$ we mean $A(e^{2\pi i w})$.

If $\pi_1(G) = 0$ (i.e. G is 1-connected) then "one half the sum of the positive roots", denoted by β_G, is a weight (see [1]). In this case, $A(\beta_G)$, denoted by δ, relates the symmetric and anti-symmetric elements in the following proposition.

PROPOSITION 1 [1; 6.6, 6.16, 6.18]. *Suppose $\pi_1(G) = 0$. Then if we let δ denote $A(\beta_G)$ we get the following results.*

(a) *If $x \in R(T)$ is symmetric, then $x \to x \cdot \delta$ gives an isomorphism onto from the additive group of symmetric elements to the additive group of anti-symmetric elements.*

(b) *Let x be an irreducible representation of G, and think of $R(G) \leq R(T)$; then $x \cdot \delta = A(y)$ for some weight y.*

(c) *Conversely, every $A(y)$, y a weight, is equal to $\pm x \cdot \delta$ for some irreducible x in $R(G)$.*

Remarks. If $\pi_1(G)$ has no 2-torsion then there exists a weight β', related to β_G, such that Proposition 1 remains true if we let $\delta = A(\beta')$. For the cases considered in this paper we will be able to explicitly provide such a β' (see Proposition 2). A general proof that β' exists provided $\pi_1(G)$ has no 2-torsion appears in [11]. In [3] Bott makes a slightly less powerful assertion but does not give a proof.

PROPOSITION 2. *Suppose β_G is a weight, or if β_G is not a weight suppose there exists a one-dimensional complex representation, ρ, of G, $\rho/T = e^{2\pi i w}$, with $\beta_G + (\frac{1}{2})w \equiv \beta'$, a weight. Then the conclusions of Proposition 1 stand with δ equal to $A(\beta_G)$ and $A(\beta')$, respectively.*

Proof. The proof in [1] of Proposition 1 only uses the fact that β_G is a weight. Therefore the first case follows trivially. The second case is not much harder. We only need note that if we replace δ by $A(\beta')$ in [1] the proof is still correct line for line.

We close this section with some observations about the "algebraic" structure of $R(G)$ and $R(T)$.

If T is an n-torus and if we pick a Euclidean coordinate system for $L(T) \cong R^n$ we can represent T as $\{(y_1, \cdots, y_n)/y_i \in R^1/Z\}$. If p_i denotes projection onto the i^{th} factor, then p_i is a weight and by x_i we will mean the one-dimensional representation associated to p_i. $R(T_n)$ is then isomorphic to

$$Z[x_1, \cdots, x_n, x_1^{-1}, \cdots, x_n^{-1}]$$

(see [1]).

Comments. Since $R(G)$ is the fixed ring of $R(T)$ under the action of the finite group $W(G)$ it is trivial that each element of $R(T)$ is integral over $R(G)$, for r is a zero of the monic polynomial

$$\prod_{\sigma \in W(G)} (X - \sigma(r)) \in R(T)^{W(G)}[X] = R(G)[X].$$

Since $R(T)$ is isomorphic to $Z[x_1, \cdots, x_n, x_1^{-1}, \cdots, x_n^{-1}]$ it is a unique factorization domain (essentially [5, p. 128] plus localization). Hence it is integrally closed in K, the quotient field of $R(T)$ [5, p. 240]. It follows that $R(T)$ is the integral closure in K of $R(G)$. Clearly (since $R(T)$ is finitely generated) K is a finite extension of k, the quotient field of $R(G)$, hence algebraic. It is an easy consequence of these last two comments that each element of K can be written as the quotient of an element of $R(T)$ by an element of $R(G)$ (p. 238 of [5]). Hence k is the subfield of K left fixed by $W(G)$; thus K is a Galois extension of k with $W(G)$ its Galois group [5, p. 194].

2. The main theorem

If H is a closed connected subgroup of G then H is said to be of *maximal rank* if $\exists$ a torus T which is maximal for both G and H. In this case we get $R(G) \leq R(H) \leq R(T)$, where each inclusion is induced by the restriction of representations [6]. We also have $R(G) = R(T)^{W(G)}$, $R(H) = R(T)^{W(H)}$ where $W(G)$ and $W(H)$ are the respective Weyl groups.

Suppose $x \in R(T)$ is anti-symmetric with respect to $W(H)$. Then we can define

$$\bar{A}(x) = \sum_{\varphi} (-1)^{\varphi} \varphi(x),$$

where φ varies over a set of representatives for the left cosets of $W(G)/W(H)$. This sum is well defined.

Let us now assume that $\delta(G)$ and $\delta(H)$ exist (as in Proposition 2), generators for the respective anti-symmetric elements over the appropriate rings. (As we noted already this will occur provided $\pi_1(G)$ and $\pi_1(H)$ have no two torsion.) To keep our notation straight we will let A_G, A_H, denote "alternating with respect to" $W(G)$, $W(H)$, respectively. For $x \in R(H)$, $\delta(H) \cdot x$ is anti-symmetric under the action of $W(H)$. Therefore $\bar{A}(\delta(H) \cdot x)$ is well defined. As in §1, $\delta(H) = A_H(\beta')$. Thus $\bar{A}(\delta(H) \cdot x) = A_G(\beta' \cdot x)$ which is anti-symmetric under $W(G)$.

PROPOSITION 3. *$f : R(H) \to R(G)$ defined by $f(x) = \bar{A}(\delta(H) \cdot x)/\delta(G)$ is an $R(G)$-module homomorphism.*

Proof. Propositions 1 and 2 together with the previous assertions.

If $H = T$, a maximal torus, then $f(x) = A(x)/\delta$, and f is related to the classical *Weyl character formula* (see [1], [3]).

Given f, as in proposition 3, we can define

$$F : R(H) \to \mathrm{Hom}_{R(G)} (R(H), R(G))$$

by

$$F(x)(y) = f(x \cdot y) \quad \text{for } x, y \in R(H).$$

Conjecture. If $\pi_1(G)$ and $\pi_1(H)$ have no 2 torsion then F is an $R(G)$-module isomorphism (i.e. — we have a "duality theorem" for the $R(G)$-module, $R(H)$).

In this paper we will follow a computational approach and prove a slightly stronger statement for some of the classical groups. Besides being a stronger statement our theorem will give us, as a corollary, a *specific* set of generators for $K(G/H)$, for the G and H considered. ($K(X)$ represents the cohomology theory of Atiyah-Hirzebruch based on vector bundles over X; see [4].)

For our purposes we will think of F as a bilinear map from

$$R(H) \times R(H) \to R(G)$$

defined by $F(x, y) = f(x \cdot y)$.

MAIN THEOREM. *Suppose $T \leq H \leq G$ where H is a subgroup of maximal rank and T is a maximal torus for H and G. Then if $\delta(H)$ and $\delta(G)$ exist as in Proposition 2:*

(1) *$R(H)$ is a free $R(G)$-module of rank $N = |W(G)/W(H)|$. (For $H = T, W(T) = 1$.)*
(2) *There exist two sets of bases $\{a_i\}_{i=1}^N$, $\{b_j\}_{j=1}^N$ such that*

$$F(a_i, b_j) = \pm 1 \quad \text{for } i = j$$
$$= 0 \quad \text{for } i < j$$

(*i.e.* $\det[((F(a_i, b_j)))] = \pm 1$, *a unit in* $R(G)$).

We will prove this theorem for G, a classical group and H a suitable subgroup of maximal rank.

Remarks. (a) Condition (2) implies that an inverse for $((F(a_i, b_j)))$ exists in the ring of $N \times N$ matrices with coefficients in $R(G)$. Therefore this condition implies that F is an isomorphism when viewed as a map

$$R(H) \to \operatorname{Hom}_{R(G)}(R(H), R(G)).$$

(b) In [8], Pittie shows that $R(T)$ is a free $R(G)$-module provided $\pi_1(G)$ is torsion free. Inherent in our condition (1) is an explicit set of generators for the cases discussed.
(c) If condition (b) is satisfied we say that F is strongly non-singular (often abbreviated s.n.s.).

We have previously noted that both $R(T)$ and $R(G)$ are integrally closed rings and that $R(T)$ is the integral closure of $R(G)$ in K, the quotient field of $R(T)$. Furthermore k is the fixed subfield of K under the induced action of $W(G)$. It follows that (K/k) is a Galois extension of rank $|W(G)|$. Since the same follows for (K/L), where L is the quotient field of $R(H)$, we get $[K:L] = |W(H)|$ and $[L:k] = |W(G)/W(H)|$.

PROPOSITION 4. *Suppose we have $\{a_i\}_{i=1}^N$, $\{b_j\}_{j=1}^N$, $a_i, b_j \in R(T)$ such that*

$$F(a_i, b_j) = \delta_{ij}, \quad N = |W(G)|.$$

Then $R(T)$ is a free $R(G)$-module, freely generated over $R(G)$ by either $\{a_i\}_{i=1}^N$ or $\{b_j\}_{j=1}^N$.

Remark. For

$$F : R(H) \times R(H) \to R(G)$$

and $\{c_i\}_{i=1}^M$, $\{d_j\}_{j=1}^M$, $F(c_i, d_j) = \delta_{ij}$ with $M = |W(G)/W(H)|$, we get $R(H)$ as a free $R(G)$-module, analogously.

Proof. Since we have an induced action of $W(G)$ on K, with k the fixed subfield, F induces a bilinear form on the N-dimensional k-vector space, K. $F(a_i, b_j) = \delta_{ij}$ implies that $\{a_i\}_{i=1}^N$ is a basis for K over k. Let r be in $R(T)$,

$R(T) \subset K$ in the usual way; $r = \sum_{i=1}^{n} \alpha_i a_i$, $\alpha_i \epsilon k$. $F(r, b_j) = \alpha_j$ is an element of $R(G)$, which implies that $\{a_i\}_{i=1}^{N}$ generates $R(T)$, freely, over $R(G)$, and similarly for $\{b_j\}_{j=1}^{N}$.

Remarks. In view of these results we make the following observation. We need not prove part 1 of Theorem 4 explicitly, since if we have a collection of elements of the "right" number satisfying the previous proposition, then $R(H)$ is automatically a free $R(G)$-module with an explicit set of free generators. We will indicate, though, how these bases come about and also assert that using these strong indications together with some well known results of commutative algebra [12; V] it is not hard to prove directly that $R(H)$ is $R(G)$-free with these explicit elements as bases. Let us note, of course, that if $\{a_i\}_{i=1}^{N}$ and $\{b_j\}_{j=1}^{N}$ exist satisfying the second condition of the main theorem then it is a triviality to find a_i' and b_j' such that $F(a_i', b_j') = \delta_{ij}$.

3. Inductive lemma

In Section 2 we stated the main theorem of this paper. Assuming that that $R(H)$ is a free $R(G)$-module, we wish to find two sets of bases $\{a_i\}$ and $\{b_j\}$ such that

$$\begin{aligned} F(a_i, b_j) &= \pm 1 \quad \text{for } i = j \\ &= 0 \quad\;\; \text{for } i < j. \end{aligned}$$

We have previously noted that if a bilinear form has this property, it implies the "duality isomorphism" (see §2).

We also noted that if such bases exist, it implies the existence of $\{a_i'\}$ and $\{b_j'\}$, two other sets of bases such that $F(a_i', b_j') = \delta_{ij}$. Using this fact, we will be able to prove the following strong inductive tool:

Suppose we have $T \leq H \leq G$. Then we can define

$$F_1 : R(T) \times R(T) \to R(H), \quad F_2 : R(H) \times R(H) \to R(G)$$

and

$$F : R(T) \times R(T) \to R(G).$$

Although the domain of definition of these forms will not allow us to claim that $F = F_2 \cdot F_1$, in some sense this is the case. For, recall we defined $f : R(T) \to R(G)$ such that $F(x, y) = f(x \cdot y)$ for $x, y \,\epsilon\, R(T)$. Similarly, we can define

$$f_1 : R(T) \to R(H) \quad \text{and} \quad f_2 : R(H) \to R(G).$$

For these functions it follows trivially from the definitions that $f = f_2 \cdot f_1$.

THE INDUCTIVE LEMMA. *If $F_1 : R(T) \times R(T) \to R(H)$ and $F_2 : R(H) \times R(H) \to R(G)$ are both strongly non-singular, then so is $F : R(T) \times R\ (T) \to R(G)$.*

Proof. By the previous observations we may assume that

$$\text{there exist } \{x_1, \cdots, x_n\} \quad \text{and} \quad \{y_1, \cdots, y_n\},$$

two sets of bases for $R(T)$ as an $R(H)$-module, such that $f_1(x_i \cdot y_j) = \delta_{ij}$. Similarly,

$$\text{there exist } \{z_1, \cdots, z_m\} \quad \text{and} \quad \{w_1, \cdots, w_m\},$$

two sets of bases for $R(H)$ as an $R(G)$-module, such that $f_2(z_\alpha w_\beta) = \delta_{\alpha\beta}$. Clearly

$$\{x_i \cdot z_\alpha\}_{i=1}^{N} {}_{\alpha=1}^{M} \quad \text{and} \quad \{y_j \cdot w_\beta\}_{j=1}^{N} {}_{\beta=1}^{M}$$

are bases for $R(T)$ as an $R(G)$-module. Furthermore

$$\begin{aligned} F(x_i \cdot z_\alpha, y_j \cdot w_\beta) &= f(x_i \cdot y_j \cdot z_\alpha \cdot w_\beta) = f_2 \cdot f_1(x_i y_j \cdot z_\alpha \cdot w_\beta) = f_2(z_\alpha \cdot w_\beta(f_1(x_i \cdot y_j))) \\ &= f_2(z_\alpha \cdot w_\beta) \cdot \delta_{ij} = \delta_{\alpha\beta} \delta_{ij}. \end{aligned}$$

Q.E.D.

4. $SU(n)$ and $U(n)$

We can include $i : U(n) \to SU(n+1)$ in the usual way by

$$i(A) = \begin{pmatrix} A & & 0 \\ & & \vdots \\ & & 0 \\ 0 & \cdots & 0 * \end{pmatrix}$$

where $A \in U(n)$ and $* = 1/\det A$. Under this injection the subgroup of diagonal matricies, which is a maximal torus for $U(n)$, is mapped isomorphically onto a maximal torus for $SU(n+1)$. This makes $U(n)$ a subgroup of maximal rank of $SU(n+1)$. As usual this induces

$$R(SU(n+1)) \leq R(U(n)) \leq R(T_n) \cong z[x_1^{\pm}, \cdots, x_n^{\pm 1}].$$

$W(U(n))$ acts on $R(T_n)$ as the group of permutations on the set $\{x_1, \cdots, x_n\}$.

$$|W(U(n))| = n! \quad \text{and} \quad R(U(n)) \cong Z[\tau_1, \cdots, \tau_n, \tau_n^{-1}]$$

where τ_i are the elementary symmetric functions on $\{x_1, \cdots, x_n\}$ [1].

$W(SU(n+1))$ acts on $R(T_n)$ as the group of permutations of the set

$$\{x_1, \cdots, x_n, x_1^{-1} \cdots x_n^{-1}\}.$$

$$|W(SU(n+1))| = (n+1)! \quad \text{and} \quad R(SU(n+1)) \cong Z[\rho_1, \cdots, \rho_n]$$

where the ρ_i are the elementary symmetric functions on the above set [1].

Let W' be the subgroup of $W(SU(n+1))$ which fixes $x_1^{-1} \cdots x_n^{-1}$; then

$$W' \leq W(SU(n+1))$$

and $R(U(n))$ is the fixed subring of $R(T_n)$ under W' (i.e., $W' \cong W(U(n))$).

THEOREM 1. *$R(T_n)$ is a free module over $R(U(n))$ of rank $n!$. A set of generators for $R(T_n)$ over $R(U(n))$ is the set $\{x_1^{a_1} \cdots x_{n-1}^{a_{n-1}}\}$ where $0 \leq a_i \leq n - i$.*

THEOREM 2. *$R(T_n)$ is a free module over $R(SU(n+1))$ of rank $(n+1)!$. A set of generators is $\{x_1^{a_1+j} \cdots x_{n-1}^{a_{n-1}+j} x_n^{j}\}$ where $0 \leq a_i \leq n - i$ and $-n \leq j \leq 0$.*

LEMMA 1. $R(U(n))$ is a free $R(SU(n+1))$-module of rank $n+1$, with

$$\{(x_1 \cdots x_n)^j\}_{j=-n}^{0}$$

a set of generators.

Note. Lemma 1 and Theorem 1 imply Theorem 2.

For the proof of Lemma 1 we note that $R(SU(n+1))[\tau_n^{-1}]$, the ring generated by $R(SU(n+1))$ and τ_n^{-1}, is $R(U(n))$. τ_n^{-1} has $(n+1)$-images under the action of $W(SU(n+1))$ and therefore $(\tau_n^{-1})^{n+1}$ is in the module generated by $\{(\tau_n^{-1})^j\}_{j=0}^{n}$ over $R(SU(n+1))$. These are a free basis for $R(U(n))$ because the minimal integral polynomial for τ_n^{-1} over $R(SU(n+1))$ is of degree $n+1$.

The proof of Theorem 1 involves an induction argument on n. Let us consider $U(n) \times U(1)$ as a subgroup of $U(n+1)$ by taking matricies which have $+1$ in the upper left hand corner. Then $U(n) \times U(1)$ is a subgroup of maximal rank and

$$R(U(n) \times U(1)) \cong R(U(n)) \otimes R(U(1))$$

(see [1]) which is

$$Z[\tau_1, \cdots, \tau_n, \tau_n^{-1}, x_1, x_1^{-1}] \leq Z[x_1^{\pm 1}, \cdots, x_{n+1}^{\pm 1}] = R(T_{n+1})$$

where the τ_i are considered as elements of

$$Z[x_2^{\pm 1}, \cdots, x_{n+1}^{\pm 1}] \cong R(T_n) \supset R(U(n)).$$

It follows easily that if $\{a_1, \cdots, a_N\}$ generates $R(T_n)$ freely over $R(U(n))$ then $\{a_i\}_{i=1}^{N}$, considered as elements of $R(T_{n+1})$ in the obvious way, generate $R(T_{n+1})$ freely over $R(U(n) \times U(1))$. The induction argument therefore involves proving that $\{x_1^{a_1}\}_{a_1=0}^{n-1}$ generates $R(U(n) \times U(1))$ freely over $R(U(n))$. As in the case of Lemma 1 it is easy to see that $R(U(n))[x_1]$ is $R(U(n) \times U(1))$. (Since x_i is integral over $R(U(n))$ for all i, adjoining any x_i also adjoins x_i^{-1} [7, I, §10].) The fact that a minimal integral polynomial for x_1 over $R(U(n))$ is of degree n (e.g. $P(Y) = \prod_\varphi (Y - \varphi(x_1))$, φ ranging over $W(U(n))$) implies the rest.

To start the induction note that for $n = 2$, the first non-trivial case

$$R(U(2)) = Z[x_1 + x_2, x_1 \cdot x_2, x_1^{-1} \cdot x_2^{-1}] \leq Z[x_1^{\pm 1}, x_2^{\pm 1}] = R(T_2),$$

and $R(T_2)$ can be generated by $\{1, x_1\}$ over $R(U(2))$.

We will now show that the bilinear forms

$$F : R(T_n) \times R(T_n) \to R(U(n))$$

and

$$F : R(U(n)) \times R(U(n)) \to R(SU(n+1))$$

are s.n.s. As we have seen (§3), this will imply that

$$F : R(T_n) \times R(T_n) \to R(SU(n+1))$$

is also s.n.s. (Let us note that this theorem as well as the analogous ones which follow are much stronger than just a statement of duality, for in each case a specific pair of dual bases is exhibited.) Before we can calculate these forms, we must first know the elements $\delta(SU(n+1))$ and $\delta(U(n))$.

$\pi_1(SU(n+1)) = 0$ [1; 5.49], therefore β is a weight ($\beta = \frac{1}{2}$ the sum of the positive roots) $\delta(SU(n+1)) = A(\beta)$. In [1] it is shown that the roots of $SU(n+1)$ are the

$$(x_i - x_j), \quad i \neq j, \quad 1 \leq i, \quad j \leq n+1,$$

where by x_{n+1} we mean $-(x_1 + \cdots + x_n)$. If we choose the Fundamental Weyl Chamber to be $x_1 > x_2 \cdots > x_n > x_{n+1}$, then the positive roots are the $(x_i - x_j)$ with $i < j$. β is then just $nx_1 + (n-1)x_2 + \cdots + x_n$. We will abuse notation and let β also stand for $e^{2\pi i\beta}$, the representation of T "associated" to the weight β. $\beta = x_1^n \cdots x_n$, where, as previously noted x_i stands for the representation

$$(x_1, \cdots, x_n) \to e^{2\pi i x_i} \in S^1.$$

$\pi_1(U(n))$ is Z, (see [1]); therefore we do not expect β to be a weight. The roots of $U(n)$ are the $(x_i - x_j)$ ($i \neq j, 1 \leq i, j \leq n$). Let the Fundamental Weyl Chamber be described by $x_1 > x_2 \cdots > x_n$; then the positive roots are the $(x_i - x_j)$ with $i < j$. It is easy to see that β is equal to

$$(\tfrac{1}{2})[(n-1)x_1 + (n-3)x_2 + \cdots - (n-1)x_n].$$

Therefore β will be a weight if and only if n is odd.

Case (i) ($n = 2k+1$). $\beta = kx_1 + \cdots - kx_n$, (note that x_{k+1} has coefficient zero). β is a weight, and the associated representation is

$$\beta = x_1^k x_2^{k-1} \cdots x_n^{-k}.$$

Case (ii) ($n = 2k$). $\beta = (k - \frac{1}{2})x_1 + \cdots - (k - \frac{1}{2})x_n$, and β is not a weight. Recalling Proposition 2, we want to find a one-dimension representation of $U(n)$, ρ, such that $\rho \mid T = e^{2\pi i w}$ and such that $\beta + (\frac{1}{2})w = \beta'$ is a weight. It will then follow that $A(\beta') = \delta(U(n))$.

τ_n^{-1} is a one-dimensional representation of $U(n)$ and $\tau_n^{-1} \mid T_n = e^{2\pi i(-x_1 - \cdots - x_n)}$. Let

$$w = -x_1 - \cdots - x_n;$$

then $\beta' = \beta + (\frac{1}{2})w$ is equal to $(k-1)x_1 + (k-2)x_2 + \cdots - kx_n$, a weight. The associated representation is $\beta' = x_1^{k-1} \cdots x_n^{-k}$, and by Proposition 2, $A(\beta') = \delta(U(n))$.

THEOREM 3. *$F : R(T_n) \times R(T_n) \to R(U(n))$ is strongly non-singular.*

Proof. (induction on n). Suppose the theorem is true for $n < r$.

$R(U(r-1)) = Z[\bar{\tau}_1, \cdots, \bar{\tau}_{r-1}, \bar{\tau}_{r-1}^{-1}] \leq Z[x_2^{\pm 1}, \cdots, x_r^{\pm 1}] = R(T_{r-1})$, where the $\bar{\tau}_i$ are the elementary symmetric functions on the set $\{x_2, \cdots, x_r\}$. By the induction hypothesis we have $\{a_i\}$ and $\{b_j\}$, two bases for $R(T_{r-1})$ over $R(U(r-1))$ such that $F(a_i, b_j) = \delta_{ij}$.

We again consider $U(r-1) \times U(1) \leq U(r)$, with

$$R' = R(U(r-1) \times U(1)) = Z[x_1^{\pm 1}, \bar{\tau}_1, \cdots, \bar{\tau}_{r-1}, \bar{\tau}_{r-1}^{-1}] \leq R(T_r).$$

$R(T_{r-1})$ can be isomorphically inbedded as a subring of $R(T_r)$ by sending $x_i \to x_i$, $i \geq 2$, and we can therefore consider the elements a_i, b_j as being in $R(T_r)$. This isomorphism commutes with the action of

$$W(U(r-1)) \cong W(U(r-1) \times U(1)),$$

and it follows that for

$$F : R(T_r) \times R(T_r) \to R', \quad F(a_i, b_j) = \delta_{ij}.$$

(Under the isomorphism, $\delta(U(r-1)) = \delta(U(r-1) \times U(1))$.) The inductive lemma implies that if

$$F : R' \times R' \to R(U(r))$$

is s.n.s. then the result will follow for $F : R(T_r) \times R(T_r) \to R(U(r))$.

As previously noted R' can be generated over $R(U(r))$ by

$$\{1, x_1, \cdots, x_1^{r-1}\}.$$

Let this basis, in the order written, be denoted by $\{a_i\}$. It is equally clear that

$$\{1, x_1^{-1}, \cdots, x_1^{-(r-1)}\}$$

will generate R', and we denote this basis by $\{b_j\}$. Let us recall that

$$F : R' \times R' \to R(U(r))$$

is defined by $F(z, w) = A(\beta' zw)/A(\beta)$, where $A_H(\beta') = \delta(U(r-1))$, A_H denoting "alternating with respect to $W(U(r-1))$", and $A(\beta) = \delta(U(r))$.

Case 1 $(r = 2k+1)$. We have $\beta = x_1^k \cdots x_r^{-k}$ and $\beta' = x_2^{k-1} \cdots x_r^{-k}$. Let a_i be as described and let b_j' be $b_j x_1^k$. x_1^k is a unit in R', therefore $\{b_j x_1^k\} = \{b_j'\}$ is also a basis for R' over $R(U(r))$.

$$F(a_i, b_j') = A(\beta' a_i b_j')/A(\beta) = A(\beta a_i b_j)/A(\beta).$$

Case 2 $(r = 2k+2)$. Then, $\beta = x_1^k \cdots x_r^{-(k+1)}$ and $\beta' = x_2^k \cdots x_r^{-k}$. Let a_i be as described and let

$$b_j' \text{ be } b_j x_1^k (x_2^{-1} \cdots x_r^{-1}).$$

$x_1^k(x_2^{-1} \cdots x_r^{-1}) = x_1^k \bar{\tau}_{r-1}^{-1}$ is a unit in R', therefore $\{b_j'\}$ is a basis.

$$F(a_i, b_j') = A(\beta' a_i b_j')/A(\beta) = A(\beta a_i b_j)/A(\beta).$$

It therefore follows that in either case, r even or odd, that we must calculate

$$A(\beta a_i b_j)/A(\beta).$$

If $i = j$, then $a_i = b_j^{-1}$ and we have $A(\beta)/A(\beta) = 1$.

If $i < j$, then we have $A(\beta x_1^l)/A(\beta)$, where $-(r-1) \leq l < 0$. In either case, r even or odd, we can see that there is some x_s, $s \neq 1$, where in $\beta \cdot x_1^l$, x_1 and x_s are raised to the same power. Let φ be the element of $W(U(r))$ permuting x_1 and x_s, then sgn $(\varphi) = -1$ and $\varphi(\beta x_1^l) = \beta x_1^l$. This implies that $A(\beta x_1^l) = 0$ (see [1]). We must now *start* the induction.

$n = 2$ ($U(1)$ is just a torus).

$$R(U(2)) = Z[x_1 + x_2,\ x_1 x_2,\ (x_1 x_2)^{-1}] \leq Z[x_1^{\pm 1},\ x_2^{\pm 1}] = R(T_2).$$

Two bases for $R(T_2)$ over $R(U(2))$ are $\{1, x_1^{-1}\}$ and $\{1, x_1\}$. In this case β is x_2^{-1}. Let us change the first basis above by multiplying by x_2^{-1}, a unit. We then get

$$\{x_2^{-1},\ x_1^{-1}x_2^{-1}\}.$$

Let $\{1, x_1\}$ be denoted by $\{a_i\}$ and $\{x_2, x_1^{-1}x_2^{-1}\}$ by $\{b_j\}$; then

$$\begin{aligned} F(a_i, b_j) &= 1 \quad \text{for } i = j \\ &= 0 \quad \text{for } i < j. \end{aligned}$$

$F(a_i, b_i) = A(\beta)/A(\beta) = 1$, proving the diagonal part of the statement. If $i < j$, we have

$$F(x_2^{-1}, x_1^{-1}) = A(x_1^{-1}x_2^{-1})/A(\beta).$$

$(x_1 x_2)^{-1}$ is fixed under φ, the element of $W(U(2))$ permuting x_1 and x_2; therefore $A(x_1^{-1}x_2^{-1}) = 0$.

This concludes the proof of Theorem 3.

THEOREM 4. *$F : R(U(n)) \times R(U(n)) \to R(SU(n+1))$ is strongly non-singular.*

Proof. We have shown that $R(U(n))$ is freely generated over $R(SU(n+1))$ by the set

$$\{1, \tau_n^{-1}, \cdots, \tau_n^{-n}\}.$$

It can be easily seen that $\{1, \tau_n, \cdots, \tau_n^n\}$ is also a basis. Let the first basis be denoted by $\{b_j\}$ and the second by $\{a_i\}$.

The element β, for $SU(n+1)$, is $x_1^n \cdots x_n$. The corresponding β' for $U(n)$ depends on whether n is even or odd. In either case, we look for u, a unit in $R(U(n))$, such that $\beta' \cdot u = \beta$. If we can find such a u, let $b_j' = b_j u$. Then $F(a_i, b_j')$ is just

$$A(\beta' a_i b_j')/A(\beta) = A(\beta a_i b_j)/A(\beta).$$

For $n = 2k$, $\beta' = x_1^{k-1} \cdots x_n^{-k}$, so we let u be τ_n^{k+1}. For $n = 2k+1$,

$\beta' = x_1^k \cdots x_n^{-k}$ and let u be τ_n^k. In either case, to compute $F(a_i, b_j')$ we must really compute $A(\beta a_i b_j)/A(\beta)$.

If $i = j$, $a_i = b_j^{-1}$ and we get $A(\beta)/A(\beta)$ which equals $+1$.

Suppose $i < j$; then we have $A(\beta\tau_n^{-r})/A(\beta)$ where $1 \leq r \leq n$. If $r < n$,

$$\beta\tau_n^{-r} = (x_1^n \cdots x_n)(x_1 \cdots x_n)^{-r}$$

is a monomial where for some pair $(i, i+1)$, x_i appears to the first power and x_{i+1} to the zero power. Let φ in $W(SU(n+1))$ be that element permuting x_i and $x_1^{-1} \cdots x_n^{-1}$. Then in the monomial $\varphi(\beta\tau_n^{-r})$, x_i and x_{i+1} both appear raised to the power -1. Therefore $A(\beta\tau_n^{-r}) = -A(\varphi(\beta\tau_n^{-r})) = 0$, since $\varphi(\beta\tau_n^{-r})$ is fixed by α in $W(SU(n+1))$, where α permutes x_i and x_{i+1} (see [1; 6.12]). ($r = n$ follows similarly).

This concludes the proof of Theorem 4, and implies the following:

THEOREM 5. *$F : R(T) \times R(T) \to R(SU(n+1))$ is strongly non-singular.*

Proof. The theorem follows from Theorems 3, 4, and the inductive lemma.

5. $Sp(n)$ and $SO(2n+1)$

$Sp(n)$ and $SO(2n+1)$ are both of rank n. (See [1] for a description of maximal tori for these groups.) The Weyl groups of $Sp(n)$ and $SO(2n+1)$ act on $R(T_n)$ as a subgroup of the group of permutations on the set $\{x_1, \cdots, x_n, x_1^{-1}, \cdots, x_n^{-1}\}$. Since

$$R(Sp(n) = R(T_n)^w = R(SO(2n+1)),$$

both representation rings are isomorphic. (This is the reason we consider both cases simultaneously.) In either case $R(T_n)^w = Z[\tau_1, \cdots, \tau_n]$ where τ_i is the i^{th} elementary symmetric function on the set $\{(x_j + x_j^{-1})\}_{j=1}^n$ [6].

We state the following theorem for $Sp(n)$ and the result will follow for $SO(2n+1)$.

THEOREM 6. *$R(T_n)$ is a free $R(Sp(n))$-module of rank $2^n n!$. A set of generators is $\{x_1^{a_1} \cdots x_n^{a_n}\}$ where $0 \leq a_i \leq 2(n-i)+1$.*

Proof. As in the case of $U(n)$, we need consider the subgroup

$$Sp(n) \times U(1) \leq Sp(n+1)$$

by placing $+1$ in the upper left hand corner of the matrix and the induction argument follows as in the previous case.

In an analogous fashion we can copy the induction argument in $U(n)$ used to proved the non-singularity of F. This leads to the following result.

THEOREM 7. *$F : R(T_n) \times R(T_n) \to R(Sp(n))$ is strongly non-singular.*

Let us also note that the bases $\{a_i\}$ and $\{b_j\}$ of the main theorem can be chosen to be the generators of Theorem 6 together with the inverse set, i.e.

$$\{x_1^{-a_1} \cdots x_n^{-a_n}\} \; 0 \leq a_i \leq 2(n-i)+1.$$

A complete description of this case can be found in [11].

6. $SO(2n)$

We will not prove a theorem about the freeness of $R(T_n)$ over $R(SO(2n))$ because $\pi_1(SO(2n)) \cong Z_2$ and our main theorem will not apply. (For a counter-example see [11].) The only reason we introduce this case is because of its usefulness in the discussion of $Spin(2n)$.

We can include $SO(2n)$ in $SO(2n+1)$, as a subgroup of maximal rank in the following manner: If $A \in SO(2n)$ map

$$A \to \begin{pmatrix} A & & 0 \\ & & \vdots \\ 0 \cdots 0 & & 1 \end{pmatrix}.$$

If $R(T_n) \cong Z[x_1^{\pm 1}, \cdots, x_n^{\pm 1}]$ then we could have described the action of

$$\sigma \in W(SO(2n+1))$$

as follows. It first permutes the n subscripts, as an element of S_n, and then makes an arbitrary number of sign changes of the form $x_i \leftrightarrow x_i^{-1}$. We can now describe $W(SO(2n))$ as the subgroup of $W(SO(2n+1))$ where only an even number of sign changes are allowed.

Let ρ_i be the i^{th} elementary symmetric function on the set

$$\{x_1, x_1^{-1}, x_2, x_2^{-1}, \cdots, x_n, x_n^{-1}\}, \quad i = 1, 2, \cdots, n.$$

$R(SO(2n))$ is then equal to $Z[\rho_1, \cdots, \rho_{n-1}, \overset{+}{\rho_n}, \overset{-}{\rho_n}]$ where $\overset{+}{\rho_n}$ and $\overset{-}{\rho_n}$ are defined as follows. Recalling the elements τ_i of the previous section, $\rho_1 = \tau_1$ and ρ_i, $i > 1$, can be written as $\tau_i + n_1 \tau_{i-1} + \cdots + n_i$ where the $n_j \in Z$. If we write τ_n as

$$\textstyle\sum_{\varepsilon_1 = \pm 1} x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n}$$

then we call $\overset{+}{\tau_n}$ the subsum where $\prod \varepsilon_i = +1$ and analogously $\overset{-}{\tau_n}$ the subsum where $\prod \varepsilon_i = -1$. With this notation

$$\overset{+}{\rho_n} = \overset{+}{\tau_n} + \sigma, \quad \overset{-}{\rho_n} = \overset{-}{\tau_n} + \sigma, \quad \sigma \in R(SO(2n)) \quad \text{and} \quad \overset{+}{\rho_n} + \overset{-}{\rho_n} = \rho_n$$

(i.e. $2\sigma = \rho_n - \tau_n$). See [1] for the entire assertion.

Note. If we let ρ_i be i^{th} elementary symmetric function on the set

$$\{x_1, x_1^{-1}, x_2, x_2^{-1}, \cdots, x_n, x_n^{-1}, 1\}$$

then analogous to $R(SO(2n))$, $R(SO(2n+1)$ can be described by $Z[\rho_1, \cdots, \rho_n]$. As in the previous assertion each ρ_i can be written as a linear combinations of the τ_j, $1 \leq j \leq i$ and 1. This fact will be useful in the discussion of $Spin(2n + 1)$.

7. $Spin(n)$

As we have just mentioned, $\pi_1(SO(n)) \cong Z_2$, $n > 2$, and therefore $SO(n)$ has a simply connected double cover. This double cover is called $Spin(n)$, the Spinor group.

We will take as a maximal torus for $Spin(n)$ the cover of a maximal torus for $SO(n)$. $L(\tilde{T})$ and $L(T)$ are isomorphic as vector spaces, where $\tilde{T}$ and T are the maximal tori of $Spin(n)$ and $SO(n)$, respectively, but the integer lattices of each differ (see [1]). The integer lattice of $\tilde{T}$ is the subinteger lattice of T made up of $(y_1, \cdots, y_k)$ such that $y_1 + \cdots + y_k = 2N$, for some integer N.

Let $\pi : Spin(n) \to SO(n)$ be the double cover and let ε be the non-identity element in the kernel of π. Then with the previous interpretation, ε can be represented by any

$$(y_1, \cdots, y_k) \in L(\tilde{T}) \quad \text{with} \quad y_1 + \cdots + y_k = 2N + 1.$$

The walls of the Weyl Chambers of $L(T)$, considered as hyperplanes in $L(\tilde{T})$, yield the Weyl Chambers of $Spin(n)$. In particular, $W(Spin(n)) \cong W(SO(n))$, because both groups are generated by reflections in the same hyperplanes.

Let $x_1^{1/2} \cdots x_k^{1/2}$ be the element of $R(\tilde{T}^k)$ defined as follows: send

$$(y_1, \cdots, y_k) \to e^{\pi i(y_1 + \dots + y_k)}.$$

Because $y_1 + \cdots + y_k = 2N$ whenever $(y_1, \cdots, y_k)$ is in the integer lattice,

$$x_1^{1/2} \cdots x_k^{1/2}$$

is well defined. Furthermore, $R(\tilde{T}_k) \cong Z[x_1^{\pm 1}, \cdots, x_k^{\pm 1}, x_1^{1/2} \cdots x_k^{1/2}]$. (See [6].)

Let $\rho : SO(n) \to \mathrm{Aut}(V)$, be a representation of $SO(n)$; then

$$\rho \circ \pi : Spin(n) \to \mathrm{Aut}(V)$$

is a Spinor representation. In particular, the ρ_i,

$$(\text{recall} : R(SO(2k + 1)) = Z[\rho_1, \cdots, \rho_k])$$

are elements of $R(Spin(n)$. Because of the projection map, in any representation induced from $SO(n)$ ε acts as the identity. There is a special Spinor representation, Δ, under which ε does not act as the identity:

$$\Delta \mid \tilde{T}^k = \prod_{i=1}^k (x_i^{1/2} + x_i^{-1/2}) = \sum_{\varepsilon_i = \pm 1} x_1^{\varepsilon_1/2} \cdots x_k^{\varepsilon_k/2}.$$

(See [19].)

As in the case of $SO(n)$ it makes a difference whether n is of the form $2k$ or $2k + 1$.

$$R(Spin(2k + 1)) \cong Z[\rho_1, \cdots, \rho_{k-1}, \Delta]$$

where the generators are algebraically independent. For the case $Spin(2k)$, Δ is not irreducible, but it splits into two irreducible representations Δ^+ and Δ^-.

$$\Delta^+ \mid \tilde{T}_k = \sum_{\pi\varepsilon_i=+1} x_1^{\varepsilon_1/2} \cdots x_k^{\varepsilon_k/2} \text{ and } \Delta^- \mid \tilde{T}_k = \sum_{\pi\varepsilon_i=-1} x_1^{\varepsilon_1/2} \cdots x_k^{\varepsilon_k/2}.$$

$R(Spin(2k))$ is the polynomial ring $Z[\rho_1, \cdots, \rho_{k-2}, \Delta^+, \Delta^-]$.

In either case, we have the following important relations:

(a) $\Delta^+\Delta^+ = \overset{+}{\rho_k} + \rho_{k-2} + \cdots$
(b) $\Delta^+\Delta^- = \rho_{k-1} + \rho_{k-3} + \cdots$
(c) $\Delta^-\Delta^- = \overset{-}{\rho_k} + \rho_{k-2} + \cdots$
(*d*) $\Delta\Delta = \rho_k + \rho_{k-1} + \cdots + \rho_1 + 1$

Note. For (a), (b) and (c), let us understand ρ_0 as equal to $+1$.

Since $SO(2k)$ is contained in $SO(2k + 1)$ as a subgroup of maximal rank' $Spin(2k)$ is of maximal rank in $Spin(2k + 1)$. This yields

$$R(Spin(2k + 1)) \leq R(Spin(2k)) \leq R(\tilde{T}_k).$$

Our approach will be to first prove that $R(\tilde{T}_k)$ is free over $R(Spin(2k))$ and then prove that $R(Spin(2k))$ is free over $R(Spin(2k + 1))$. (See [6] for entire previous assertion.)

It is the presentation of $R(\tilde{T}_k)$ in this complicated form (the only form which will allow a reasonable description of the action of $W(Spin(n))$) that makes the computations in this case highly non-trivial. Although the generators will look familiar (see [2]) one may rightfully ask why they show up in just the form they do.

Discussion. In the ring $R(\tilde{T}_k)$ (described above) there are two types of monomials. One type is a monomial consisting of a product of x_i with integral powers (e.g., $x_1 x_2$); in the second type all the x_i have $\frac{1}{2}$ integral powers (e.g., $x_1^{1/2} \cdots x_k^{1/2}$). We will call these the "integral" and "non-integral" monomials, respectively. A polynomial in $R(\tilde{T}_k)$ is said to be homogeneous if it is a sum of monomials, all of which are either integral or non-integral. It is clear that as an abelian group $R(\tilde{T}_k) = I \oplus N$, where I and N represent the homogeneous integral and homogeneous non-integral polynomials, respectively. This also gives us a Z_2 graded ring related to the ring $R(\tilde{T}_k)$. Since, if p_i is in I and q_j is in N then $p_i q_j \in N$, $p_{i_1} p_{i_2} \in I$ and $q_{j_1} q_{j_2} \in I$.

THEOREM 8. *$R(\tilde{T}_k)$ is a free module over $R(Spin(2k))$, freely generated by $|W| = 2^{k-1}k!$ homogeneous polynomials.*

The proof will be by induction on k.

LEMMA 2. *Suppose $R(\tilde{T}_{k-1}) = Z[x_1^{\pm 1}, \cdots, x_{k-1}^{\pm 1}, x_1^{1/2} \cdots x_{k-1}^{1/2}]$ is generated over*

$$R(Spin(2k - 2)) = Z[\rho_1', \cdots, \rho_{k-3}', \Delta_{k-1}^+, \Delta_{k-1}^-]$$

by the set $\{a_1, \cdots, a_s\}$ where the a_i are homogeneous polynomials in $R(T_{k-1})$.

Let $R' < R(\tilde{T}_k)$ be the following subring:

$$R' = Z[x_k^{\pm 1}, \rho_1', \cdots, \rho_{k-3}', x_k^{1/2}\Delta_{k-1}^+, x_k^{1/2}\Delta_{k-1}^-].$$

Let $a_i' \in R(\tilde{T}^k)$ be defined by $a_i' = a_i$ if a_i is integral and $a_i' = x_k^{1/2}a_i$ if a_i is non-integral. Then $R(\tilde{T}_k)$ is generated by $\{a_i'\}_{i=1}^s$ over R'.

Explanation. The notation ρ_i' is used to differentiate these elements from the ρ_i, where

$$R(Spin(2k)) = Z[\rho_1, \cdots, \rho_{k-2}, \Delta_k^+, \Delta_k^-].$$

Secondly, any homogeneous integral element in $R(\tilde{T}_{k-1})$ can be considered as an element of $R(\tilde{T}_k)$. This explains why the ρ_i' are elements of $R(\tilde{T}_k)$- and the definition of a_i'.

Proof. We will show that any monomial, m, of $R(\tilde{T}_k)$ can be generated by the a_i', and this will imply the lemma.

Case 1 (M is integral and has no factor of x_k). In this case m is in $R(\tilde{T}_{k-1})$ and $m = \sum r_i a_i$, where $r_i \in R(Spin(2k-2))$. By homogeneity we may assume that each $r_i a_i$ must be integral. This implies that both r_i and a_i are in I or they both are in N. Let $r_i' \in R(Spin(2k))$ be equal to r_i if r_i is integral and let it be equal to $r_i \cdot x_k^{-1/2}$ if r_i is non-integral. It is then clear that $m = \sum_{i=1}^s r_i' a_i'$.

Case 2 (m is integral and contains a factor of x_k). $m = \bar{m}x_k^p$ where $p \in Z$, $p \neq 0$. By Case 1, $\bar{m} = \sum r_i' a_i'$. But x_k^p is in R', therefore $m = \sum_{i=1}^s (r_i' x_k^p) \cdot a_i'$.

Case 3 (m is non-integral). By the reasoning of Case 2, we need only consider m, where x_k appears to the $\frac{1}{2}$ power.

$m = \bar{m}x_k^{1/2}$, where $\bar{m} = \sum r_i a_i$ is a non-integral monomial in $R(\tilde{T}_{k-1})$.

By homogeneity we may assume that each $r_i a_i$ is in N. This implies that for each i, r_i and a_i are in different parts of the Z_2 grading. Let $r_i'' = r_i$ if r_i is in I, and let $r_i'' = x_k^{1/2} r_i$ if $r_i \in N$. Then $m = \sum_{i=1}^s r_i'' a_i'$.

LEMMA 3. *Let W' be the subgroup of $W(Spin(2k))$ which fixes x_k. Then $R' = R(\tilde{T}_k)^{W'}$ is the ring R' described in the previous lemma.*

Proof. Let us first note that $W' \cong W(Spin(2k-2))$, and both have the same action on the set

$$\{x_1, \cdots, x_{k-1}, x_1^{-1}, \cdots, x_{k-1}^{-1}\}.$$

It is therefore clear that $R' < R(\tilde{T}_k)^{W'}$.

Let $p \in R(\tilde{T}_k)$ and suppose that $\phi(p) = p$ for all $\phi \in W'$. $p = \sum x_k^j \bar{p}_j$, where the $\bar{p}_j$ are elements of $R(\tilde{T}_{k-1})$ and $-N/2 \leq j \leq M/2$, N, $M \in Z$. Since W' fixes x_k, this implies that $\bar{p}_j \in R(\tilde{T}_{k-1})^{W'}$ for all j. Therefore $p \in R'$.

Proof of Theorem 8 (induction on k). (1) $k = 2$.

$$R(Spin\ 4) = Z[\Delta_2^+, \Delta_2^-] \leq Z[x_1^{\pm 1}, x_2^{\pm 1}, x_1^{1/2}x_2^{1/2}] = R(\tilde{T}_2).$$

$x_1^{1/2}x_2^{1/2}$ has 2 images under $W(Spin\ 4)$; therefore, $R'' = R(Spin\ 4)\ [x_1^{1/2} \cdot x_2^{1/2}]$ is free over $R(Spin\ 4)$, generated by 1, $x_1^{1/2}x_2^{1/2}$. σ_1, σ_2 are elements of R'', where σ_1, σ_2 are the elementary symmetric functions in $\{x_1, x_2\}$. This implies that $\{1, x_1\}$ generate $R(\tilde{T}_2)$ over R''. Putting these steps together gives us $k = 2$.

(2) Assume the theorem is true for $k < s$. Let $a_1, \cdots, a_r$ be the homogeneous elements which generate $R(\tilde{T}_{s-1})$ over $R(Spin(2s - 2))$, where

$$r = |\, W(Spin(2s - 2))\,| = 2^{s-2}(s - 1)!.$$

If W' is the subgroup of $W(Spin(2s))$ fixing $\{x_s\}$ and R' is $R(\tilde{T}_s)^{W'}$, then by Lemmas 2 and 3, we have $R(Spin(2s)) \leq R' \leq R(\tilde{T}_s)$ and $R(\tilde{T}_s)$ is generated over R' by $\{a_i'\}_{i=1}^r$. The proof will be complete provided R' can be generated over $R(Spin(2s))$ by $2 \cdot s$ homogeneous elements.

LEMMA 4. *R' is generated over $R(Spin(2s))$ by the set*

$$\{1, x_s, \cdots, x_s^{s-1}, x_s^{-1}, \cdots, x_s^{-(s-2)}), x_s^{1/2}\Delta_{s-1}^+, x_s^{1/2}\Delta_{s-1}^-\} \equiv S.$$

(*Note*: *these are exactly* $2 \cdot s$ *homogeneous elements.*)

Proof. Let us first note that $R(Spin(2s))[S]$, the smallest ring containing $R(Spin(2s))$ and the set S, is equal to R'. Let us recall that

$$R' = Z[x_s^{\pm 1}, \rho_1', \cdots, \rho_{s-3}', x_s^{1/2}\Delta_{s-1}^+, x_s^{1/2}\Delta_{s-1}^-]$$

and

$$R(Spin(2s)) = Z[\rho_1, \cdots, \rho_{s-2}, \Delta_s^+, \Delta_s^-].$$

This follows because ρ_i' can be generated over $Z[\rho_1, \cdots, \rho_i]$ by the set

$$\{(x_s + x_s^{-1})^j\}_{j=1}^{i-1}.$$

Since $R(Spin(2s))[S]$ contains $\{(x_s + x_s^{-1})^j\}_{j=1}^{s-2}$, we get that $\rho_1', \cdots, \rho_{s-3}'$ are in $R(Spin(2s)[S]$. The other generators of R' over Z are already in S; therefore $R(Spin(2s))[S] = R'$. To complete the proof of the lemma, we have to show that the module generated by S over $R(Spin(2s))$ is closed under ring multiplication. A proof of this fact involves cumbersome combinatorics and can be found in [11]. In this paper we will rely upon Proposition 4 and note that the result will follow as soon as we prove the non-singularity condition using this basis (see proof of Theorem 11).

This then completes the proof that $R(\tilde{T}_n)$ is free over $R(Spin(2n))$ of rank

$$|\, W(Spin(2n))\,| = 2^{n-1} \cdot n!$$

THEOREM 9. *$R(Spin(2n))$ is freely generated over $R(Spin(2n+1))$ by the set $\{1, \Delta^+\}$.*

Proof.

$$R(Spin(2n+1)) = Z[\rho_1, \cdots, \rho_{n-1}, \Delta] \leq Z[\rho_1, \cdots, \rho_{n-2}, \Delta^+, \Delta^-] = R(Spin(2n)).$$

It is therefore clear that $R(Spin(2n+1))[\Delta^+] = R(Spin(2n))$. Δ^+ has two images under $W(Spin(2n+1))$; therefore $\{1, \Delta^+\}$ is a set of generators.

THEOREM 10. *$R(\tilde{T}_n)$ is free over $R(Spin(2n+1))$ of rank*

$$2^n \cdot n! = |W(Spin(2n+1))|.$$

Proof. This follows from Theorems 8 and 9. It is furthermore clear exactly what will constitute a basis.

8. Duality for the cases $Spin(n)$ and $Spin^c(n)$

Let us recall that we have $R(Spin(2k+1) \leq R(Spin(2k)) \leq R(\tilde{T}^k)$. We noted in §7 that $R(Spin(2k))$ is freely generated over $R(Spin(2k+1))$ by the set $\{1, \Delta^+\}$. To compute the matrix of

$$F : R(Spin(2k)) \times R(Spin(2k)) \to R(Spin(2k+1)$$

will therefore only involve a 2 by 2 matrix, and will be quite easy. Our main task will therefore be to prove the strong non-singularity of

$$F : R(\tilde{T}_k) \times R(\tilde{T}_k) \to R(Spin(2k)).$$

$Spin(2k)$ is the simply connected double cover of $SO(2k)$ and we have previously noted that $W(Spin(2k)) \cong W(SO(2k))$. (See §7.) Furthermore $L(T) \cong L(\tilde{T})$, as Euclidean spaces, and the walls of the Weyl Chambers are defined by the "same" linear forms. If the Fundamental Weyl Chamber is chosen such that $x_k > x_{k-1} > \cdots > x_1$, then the positive roots are

$$\{(x_i - x_j)\}, i > j \quad \text{and} \quad \{(x_i + x_j)\}, \quad i \neq j.$$

β is a weight $(\pi_1(Spin(2k)) = 0)$ and is equal to $x_2 \cdots x_k^{k-1}$. (See [1].)

If we view $SO(2k-2) \times SO(2)$ as a subgroup of $SO(2k)$ in the obvious way, then we can think of $Spin^c(2k-2)$ as the subgroup of $Spin(2k)$ double covering $SO(2k-2) \times SO(2)$. It then follows that $Spin^c(2k)$ is a subgroup of maximal rank and that its Weyl group, as a subgroup of $W(Spin(2k))$, is the subgroup fixing x_k. In §7 we called this subgroup W' and denoted $R(Spin(2k))^{W'}$ by R'. We now see that

$$R' = R(Spin^c(2k-2)).$$

The roots of $Spin^c(2k-2)$ are just the roots of $SO(2k-2)$. In particular, one-half the sum of the positive roots, β, is equal to $(k-2)x_{k-1} + \cdots + x_2$, a weight.

In §7 we proved that if $a_1, \cdots, a_N$ was a basis for $R(\tilde{T}_{k-1})$ over

$$R(Spin(2k-2)), \quad N = |W(Spin(2k-2))|,$$

then the set $\{a_i'\}$ generates $R(T_k)$ over R'. (Recall that the a_i' are the images of a_i under the map fixing an integral monomial and sending a non-integral monomial m, to $mx_k^{1/2}$.)

We will now show that information about the bilinear form for the case $G = Spin(2k-2)$, will yield information for the case $Spin^c(2k-2)$.

Let us suppose that $F : R(\tilde{T}_{k-1}) \times R(\tilde{T}_{k-1}) \rightarrow R(Spin(2k-2))$ is strongly non-singular. Then there exist $\{a_i\}$ and $\{b_j\}$ such that

$$\begin{aligned} F(a_i, b_j) &= \pm 1 \quad \text{for } i = j \\ &= 0 \quad \text{for } i < j. \end{aligned}$$

Let us furthermore assume that these a_i and b_j are homogeneous elements. By Lemma 2, the sets $\{a_i'\}$ and $\{b_j'\}$ are a basis for $R(\tilde{T}_k)$ over $R(\mathrm{Spin}^c(2k-2))$. By inspecting the proof of Lemma 2, we can see that if we multiply the non-integral b_j by $x_k^{-1/2}$, the proof works just as well. Let us call this new basis $\{b_j''\}$.

Let us now inspect $\bar{F} : R(\tilde{T}_k) \times R(\tilde{T}_k) \rightarrow R(Spin^c(2k-2))$.

$$\bar{F}(a_i, b_j) = A'(a_i' \cdot b_j'')/A'(\beta'), \quad \beta' = x_2 \cdots x_{k-1}^{k-2}.$$

Notice that $a_i' \cdot b_i'' = a_i \cdot b_i$, since we multiplied one by $x_k^{1/2}$ and the other by $x_k^{-1/2}$. Since $a_i' \cdot b_i'' = a_i \cdot b_i$ is an element of $R(\tilde{T}_k)$, and since no factor of x_k appears, this implies that $a_i' \cdot b_i'' = a_i \cdot b_i$ is a homogeneous integral element of $R(\tilde{T}_k)$. Let us note that there is an isomorphism onto from

$$S = Z[x_1^{\pm 1}, \cdots, x_{k-1}^{\pm 1}] < R(\tilde{T}_{k-1}) \quad \text{to} \quad \bar{S} = Z[x_1^{\pm 1}, \cdots, x_{k-1}^{\pm 1}] \leq R(\tilde{T}_k)$$

described by sending $x_i \rightarrow x_i$, $i \leq k-1$, and that this isomorphism commutes with the action of $W(Spin(2k-2))$ on S and the action of W' on $\bar{S}$. By the assumption on F, $A(a_i \cdot b_i) = \pm A(\beta')$, where A is alternation under $W(Spin(2k-2))$ and $a_i b_i$ and β' are in S. Under the above isomorphism, we then have

$$A'(a_i' \cdot b_i'') = A'(a_i \cdot b_i) = \pm A'(\beta'),$$

where these elements lie in S. This implies that $F(a_i', b_i'') = \pm 1$.

Now let us consider $F(a_i', b_j'')$ for $i < j$, where

$$F(a_i', b_j'') = A'(a_i' \cdot b_j'')/A'(\beta').$$

We pass to $\bar{R}$, the ring extension of $R(\tilde{T}_k)$ with $x_k^{1/2}$ adjoined, and let W' act on $\bar{R}$ by its usual action on $\{x_1^{\pm 1}, \cdots, x_k^{\pm 1}\}$. Then $A'(a_i' \cdot b_j'')$ is an element of $\bar{R}$ and in $\bar{R}$ we can factor out the x_k term.

$$A'(a_i' \cdot b_j'') = x_k^{\varepsilon/2} \cdot A'(a_i \cdot b_j), \quad \varepsilon = \pm 1,$$

since W' fixes $x_k^{\varepsilon/2}$. Since $\bar{R}$ contains $x_k^{\pm 1/2}$, this implies that $R(\tilde{T}_{k-1})$ can be isomorphically imbedded in $\bar{R}$ and furthermore this isomorphism commutes with the actions of $W(Spin(2k-2))$ and W', respectively. By the assumption on F, $A(a_i \cdot b_j) = 0$, $i < j$. By the above identification of $R(\tilde{T}_{k-1})$ with a subring of $\bar{R}$, we have that $A'(a_i \cdot b_j) = 0$ in $\bar{R}$. Therefore

$$A'(a_i' \cdot b_j'') = x_k^{\varepsilon/2} A'(a_i \cdot b_j) = 0 \quad \text{in } \bar{R}.$$

But $A'(a_i' \cdot b_j'')$ is an element of $R(\tilde{T}_k) \leq \bar{R}$ and therefore $A'(a_i' \cdot b_j'') = 0$ in $R(\tilde{T}_k)$. This finally implies that $\bar{F}(a_i', b_j'') = 0$ and yields the following lemma.

LEMMA 5. *Suppose* $F : R(\tilde{T}_{k-1}) \times R(\tilde{T}_{k-1}) \to R(Spin(2k-2))$ *is strongly non-singular. Then the same is true for*

$$\bar{F} : R(\tilde{T}_k) \times R(\tilde{T}_k) \to R(Spin^c(2k-2)),$$

where, as usual, we assume that there exists homogeneous bases $\{a_i\}$ *and* $\{b_j\}$ *such that*

$$\begin{aligned} F(a_i, b_j) &= \pm 1 \quad \text{for } i = j \\ &= 0 \quad \text{for } i < j. \end{aligned}$$

THEOREM 11. $F : R(\tilde{T}_n) \times R(\tilde{T}_n) \to R(Spin(2n))$ *is strongly non-singular for all* $n \in Z$, $n \geq 2$. *Furthermore, the bases* $\{a_i\}$ *and* $\{b_j\}$ *which make* F *s.n.s. are homogeneous elements of* $R(\tilde{T}_n)$.

Proof (induction on n). *Inductive Step.* Suppose the theorem is true for $n < k$. By Lemma 7, this implies that

$$\bar{F} : R(\tilde{T}_k) \times R(\tilde{T}_k) \to R(Spin^c(2k-2))$$

is strongly non-singular. To complete the inductive step we must show that

$$F : R(Spin^c(2k-2)) \times R(Spin^c(2k-2)) \to R(Spin(2k))$$

is strongly non-singular.

Let us recall that a basis for $R(Spin(2k))$ over $R(Spin^c(2k-2))$ is the set

$$\{x_k^{-(k-2)}, \cdots, x_k^{-1}, 1, x_k^{1/2}\Delta_{k-1}^{+}, x_k^{1/2}\Delta_{k-1}^{-}, x_k, \cdots, x_k^{-(k-1)}\}.$$

Let us refer to this basis as $\{a_i\}$ where the ordering is as written. These rings are completely symmetric in the x_i, x_i^{-1}. Therefore if we take the image of this basis under

$$\varphi : x_i \to x_i^{-1} \quad \text{for all } i,$$

we will get a second basis $\{b_j\}$. To be explicit

$$\{b_j\} = \{x^{(k}{}_k{}^{-2)}, \cdots, x_k, 1, x_k^{-1/2}\bar{\Delta}_{k-1}^{+}, x_k^{-1/2}\bar{\Delta}_{k-1}^{-}, x_k^{-1}, \cdots, x_k^{-(k-1)}\}$$

in the order written. $\bar{\Delta}^{\pm}$ represents the image of $\Delta^{\pm}$ under the action of φ. Unless otherwise stated $\Delta^{\pm}$ will stand for $\Delta_{k-1}^{\pm}$.

The β associated to $Spin(2k)$ is $\beta = x_2 \cdots x_k^{k-1}$ and the β' associated to $Spin^c(2k-2)$ is $\beta' = x_2 \cdots x_{k-1}^{k-2}$. As in the previous cases, there exists a unit in $R(Spin^c(2k-2))$, x_k^{k-1}, such that $\beta' x_k^{k-1} = \beta$. Therefore, let $\{b_j'\}$ be a new basis, where $b_j' = b_j x_k^{k-1}$. We will show that

$$\begin{aligned} F(a_i, b_j') &= \pm 1 \quad \text{for } i = j \\ &= 0 \quad \text{for } i < j. \end{aligned}$$

$F(a_i, b_j') = A(\beta' a_i b_j')/A(\beta) = A(\beta' a_i b_j x_k^{k-1})/A(\beta) = A(\beta a_i b_j)/A(\beta)$, because $\beta' x_k^{k-1} = \beta$. So we have to compute $A(\beta a_i b_j)/A(\beta)$, where A is alternation over $W(Spin(2k))$. By the choice of the two bases, we can see that for almost all i, $a_i b_i = 1$. This will imply that for all those i's,

$$A(\beta a_i b_i)/A(\beta) = +1.$$

The only two diagonal terms which give any trouble are the Δ terms.

Let us recall that

$$\Delta^+ = \sum_{\epsilon_i = \pm 1, \pi\epsilon_i = 1} x_1^{\epsilon_1/2} \cdots x_{k-1}^{\epsilon_{k-1}/2} \text{ and } \Delta^- = \sum_{\epsilon_i = \pm 1, \pi\epsilon_i = -1} x_1^{\epsilon_1/2} \cdots x_{k-1}^{\epsilon_{k-1}/2}.$$

We must now differentiate between two cases.

(i) *$k-1$ is an even integer.* Then

$$\bar\Delta^+ = \Delta^+ \quad \text{and} \quad \bar\Delta^- = \Delta^-$$

(ii) *$k-1$ is an odd integer.* Then

$$\bar\Delta^+ = \Delta^- \quad \text{and} \quad \bar\Delta^- = \Delta^+.$$

In Section 7 we introduced the following formulas:

$$\Delta^+\Delta^+ = \overset{+}{\rho}_{k-1} + \rho_{k-3} + \cdots, \quad \Delta^-\Delta^- = \bar\rho_{k-1} + \rho_{k-3} + \cdots, \quad \Delta^+\Delta^- = \rho_{k-2} + \cdots.$$

In these relations, determining which sums end with the term $+1$ depends on whether $(k-1)$ is an even or odd integer. In either case, the terms $\bar\Delta^+\bar\Delta^+$ and $\bar\Delta^-\bar\Delta^-$ will be of the form $1 + \rho_2 + \rho_4 + \cdots$.

To conclude the proof that the diagonal is ± 1, we will show that $A(\beta\rho_i) = A(\beta\rho_{k-1}^{\pm})$ is zero. This will imply that the diagonal terms,

$$F(x_k^{1/2}\Delta^+, x_k^{-1/2}\bar\Delta^+) \quad \text{and} \quad F(x_k^{1/2}\Delta^-, x_k^{-1/2}\bar\Delta^-),$$

are equal to 1 since

$$\begin{aligned} &A(\beta x_k^{1/2}\Delta^+ x_k^{-1/2}\bar\Delta^+) \\ &\quad = A(\beta\Delta^+\bar\Delta^+) = A[\beta(1 + \rho_2 + \cdots)] = A(\beta) + \textstyle\sum_i A(\beta\rho_i) = A(\beta). \end{aligned}$$

A similar computation shows that $A(\beta x_k^{1/2}\Delta^- x_k^{-1/2}\bar\Delta^-) = A(\beta)$. (Note: As we have done previously, the factoring actually takes place in an extension ring which contains $x_k^{1/2}$, and in which we can imbed $R(\tilde T_{k-1})$.)

LEMMA 6. *$A(\beta\rho_{k-1}^{\pm})$ and $A(\beta\rho_i)$ for $1 \le i \le k-2$ both equal zero.*

Proof. Suppose m is a monomial in $Z[x_1^{\pm 1}, \cdots, x_k^{\pm 1}, x_1^{1/2} \cdots x_k^{1/2}]$, $m = x_1^{a_1} \cdots x_k^{a_k}$. Suppose $a_i = \pm a_j$ for some i, j. Then m is fixed under some φ in $W(Spin(2k))$. For suppose $a_i = a_j$; then let φ be the element permuting x_i, x_j and fixing the other variables. If $a_i = -a_j$, let φ be the element permuting x_i and x_j^{-1}. This has an even number of sign changes and is therefore in $W(Spin(2k))$. Call such a monomial *symmetric in* (i, j) (i.e., either $a_i = a_j$ or $a_i = -a_j$). As we have noted before, $A(m) = 0$ if m is "symmetric in (i, j)".

Through an abuse of language, we will call the following m a "monomial". m is of the form $x_{j_1}^{\varepsilon_1} \cdots x_{j_i}^{\varepsilon_i}$, $\varepsilon_i = \pm 1$ and $1 \leq j_1 \leq j_2 \cdots \leq j_i \leq k - 1$ with the following understanding. If $j_s = j_{s+1}$, then $\varepsilon_s = +1$, $\varepsilon_{s+1} = -1$, and no other j_r is equal to j_s (i.e., we allow terms like $x_i x_i^{-1}$ to appear and we do not cancel).

$\beta \cdot m = (x_2 \cdots x_k^{k-1})(m)$ is therefore a monomial of the form $x_1^{a_1} \cdots x_k^{a_k}$ where the $0 \leq |a_j| \leq k - 1$. (Just note that the ε_i can only raise or lower the exponent by (1).) By the Pigeon Hole principle we have one of two possibilities.

(1) Each a_i is different, and the $\{|a_i|\}$ fill the range between zero and $k - 1$.

(2) At least two of the a_i agree, in absolute value, in which case βm is symmetric in some (i, j)

Let us write ρ_i as $\rho_i' + \rho_i''$ where ρ_i' is made up of monomials of the first type and ρ_i'' is made up of monomials of the second type. $A(\beta\rho_i'') = 0$ because each m in ρ_i'' is of type 2. Therefore $A(\beta\rho_i) = A(\beta\rho_i')$.

Since for each m in ρ_i', βm is equal to $x_1^{a_1} \cdots x_k^{a_k}$ where the $\{|a_j|\}$ fill out the range zero to $k - 1$, it follows that $\beta m = \varphi(\beta)$ for some φ in $W(Spin(2k))$. (Just permute the indices and change signs, until you get $x_2 \cdots x_k^{k-1}$. If, in the process, you make an odd number of sign changes, just follow the process by $x_1 \leftrightarrow x_1^{-1}$ and we have added an extra sign change which does not alter the term β.) It follows that for any m in ρ_i', $A(m\beta) = A(\varphi(\beta)) = \pm A(\beta)$, the plus or minus depending on whether sign φ is positive or negative. We will show that ρ_i' has an even number of terms,

$$\rho_i' = m_1 + \cdots + m_s + \bar{m}_1 + \cdots + \bar{m}_s,$$

such that $\beta m_i = \varphi(\beta)$ with sgn $(\varphi) = +1$ and $\beta\bar{m}_j = \varphi(\beta)$ with sgn $(\varphi) = -1$. This will imply that $A(\beta\rho_i') = 0$, completing the proof.

Let us suppose that m is a monomial in ρ_i' (i.e., $A(\beta m) \neq 0$). $m = x_{j_1}^{\varepsilon_1} \cdots x_{j_i}^{\varepsilon_i}$ with the previous conditions on the x_{j_r}. Suppose $j_1 \neq j_2$ and furthermore suppose that either $j_2 \neq j_1 + 1$ or that $j_2 = j_1 + 1$ but $\varepsilon_1 = \varepsilon_2$. (We will not consider the case $j_1 = 1$ and $\varepsilon_1 = \varepsilon_2 = -1$.) Then (βm) is symmetric in $(j_1 - 1, j_1)$ if $\varepsilon_1 = -1$, and is symmetric in $(j_1, j_1 + 1)$ if $\varepsilon_1 = +1$. This would imply $A(\beta m) = 0$ contradicting our assumption. Therefore we must have $j_1 = j_2$ (implying $\varepsilon_1 = -\varepsilon_2$), or $j_2 = j_1 + 1$ and $\varepsilon_1 = 1$, $\varepsilon_2 = -1$. (The only exception is that if $j_1 = 1$ we can have $\varepsilon_1 = -1$, $\varepsilon_2 = -1$.) Now, if $j_2 = j_1 + 1$, we must have $j_3 \neq j_2$. For, if $j_3 = j_2$ (implying $\varepsilon_3 = -\varepsilon_2$), we again have symmetry in $(j_1, j_1 + 1)$ (except in the case $j_1 = 1$ and $\varepsilon_1 = \varepsilon_2 = -1$, in which case we would get symmetry in $(1, 2)$). The same line of reasoning applied to (j_1, j_2) applies to (j_3, j_4) and we must have either $j_3 = j_4$ or $j_4 = j_3 + 1$ and $\varepsilon_3 = 1$, $\varepsilon_4 = -1$.

Following this line of reasoning we see that m must be a monomial of the following type. m has an even number of terms (i.e., $i = 2r$), and they come in "pairs" (i.e., for any $1 \leq j \leq r$, $x_{2j-1}^{\varepsilon_{2j-1}} \cdot x_{2j}^{\varepsilon_{2j}}$ is of the form $x_s x_s^{-1}$ or of the form $x_s x_{s+1}^{-1}$, with the usual exception regarding $x_1^{-1} x_2^{-1}$). The first conclusion of this assertion is that for ρ_i, $i = 2r + 1$, ρ_i' is empty and therefore $A(\beta\rho_i) = 0$.

For the case $i = 2r$, we will set up a one-to-one correspondence among the m_j's in ρ_i as previously described. In each case, note that multiplying β by m_j and $\bar{m}_j$, respectively, give images of β under elements of $W(Spin(2k))$ of opposite sign.

Correspondence 1. Let m be a monomial of the form $x_j x_j^{-1} \cdots$ where j is ≥ 2. Let $\bar{m}$ be equal to $x_{j-1} x_j^{-1} \cdots$ where the rest of the monomial is exactly the same as the rest of m. (Note: For any possible end for m, we have the same possibility for $\bar{m}$.) If we take $m\beta$ and add the extra reflection $x_{j-1} \leftrightarrow x_j$ we get $\bar{m}\beta$. Therefore $m\beta = \varphi(\bar{m}\beta)$ where $\operatorname{sgn}(\varphi) = -1$. This sets up our first correspondence. This case covers all possibilities except monomials beginning with $x_1 x_1^{-1}$ or with $x_1^{-1} x_2^{-1}$.

Correspondence 2 (Monomials beginning with $x_1 x_1^{-1}$). Suppose m is of the form

$$x_1 x_1^{-1} x_j x_{j+1}^{-1} \cdots \quad (j \geq 2).$$

Then let $\bar{m}$ be $x_1 x_1^{-1} x_{j+1} x_{j+1}^{-1} \cdots$, where, as in Case 1, the dots imply, that otherwise, the corresponding monomials have the same terms. (Note that βm and $\beta\bar{m}$ yield images of β with "opposite signs".) This will cover all cases except monimials beginning with $x_1 x_1^{-1} x_2 x_2^{-1}$. For this set use the following correspondence.

$$x_1 x_1^{-1} x_2 x_2^{-1} x_j x_{j+1}^{-1} \cdots \quad (j \geq 3) \quad \leftrightarrow \quad x_1 x_1^{-1} x_2 x_2^{-1} x_{j+1} x_{j+1}^{-1} \cdots.$$

This will cover all cases except monomials beginning with

$$x_1 x_1^{-1} x_2 x_2^{-1} x_3 x_3^{-1}.$$

Following this line of reasoning we can "match up" all monomials beginning with $x_1 x_1^{-1}$, except $x_1 x_1^{-1} x_2 x_2^{-1} \cdots x_r x_r^{-1}$. We will deal with this "exception" in Case 4.

Correspondence 3 (Monomials beginning with $x_1^{-1} x_2^{-1}$). We use analogous correspondences to those used in Case 2, but we substitute $x_1^{-1} x_2^{-1}$ in place of $x_1 x_1^{-1}$ e.g.,

(a) $x_1^{-1} x_2^{-1} x_j x_{j+1}^{-1} \cdots \ (j \geq 3) \leftrightarrow x_1^{-1} x_2^{-1} x_{j+1} x_{j+1}^{-1} \cdots$
(b) $x_1 x_1^{-1} x_3 x_3^{-1} x_j x_{j+1}^{-1} \cdots \ (j \geq 4) \leftrightarrow x_1^{-1} x_2^{-1} x_3 x_3^{-1} x_{j+1} x_{j+1}^{-1} \cdots$, etc.

This will cover all cases except $x_1^{-1}x_2^{-1}x_3\,x_3^{-1}\ \cdots\ x_{r+1}\,x_{r+1}^{-1}$, which will be covered in Case 4.

Correspondence 4.

$$x_1\,x_1^{-1}x_2\,x_2^{-1}\ \cdots\ x_r\,x_r^{-1} \leftrightarrow x_1^{-1}x_2^{-1}x_3\,x_3^{-1}\ \cdots\ x_{r+1}\,x_{r+1}^{-1}.$$

(Note: In all cases, the corresponding monomials yield images of β of "different signs".) For the case $A(\beta\rho_{k-1}^{\pm}) = 0$, let us first note that by our previous arguments, if $k - 1$ is odd, the result is trivial.

To check the proof for $\rho_{k-1}^{\pm}$, where $k - 1 = 2r$, just note the following. $A(\beta\rho_{k-1}) = 0$ by the preceding argument.

$$\overset{+}{\rho}_{k-1} = \sum\nolimits_{\pi\varepsilon_i=+1} x_1^{\varepsilon_1}\ \cdots\ x_{k-1}^{\varepsilon_{k-1}} + \sigma,$$

where σ is a sum of monomials, in the usual sense, with $k - 3$ or fewer $x_i^{\pm}$'s. Similarly,

$$\overset{-}{\rho}_{k-1} = \sum\nolimits_{\pi\varepsilon_i=-1} x_1^{\varepsilon_1}\ \cdots\ x_{k-1}^{\varepsilon_{k-1}} + \sigma,$$

where σ is the same polynomial. (See Section 6.) Furthermore,

$$\rho_{k-1} = \overset{+}{\rho}_{k-1} + \overset{-}{\rho}_{k-1}.$$

Let us also note that in $\rho_{k-1}^{\pm}$ there are only two monomials which appear in ρ_{k-1}'. They are

$$m_1 = x_1\,x_2^{-1}x_3\,x_4^{-1}\ \cdots\ x_{k-2}\,x_{k-1}^{-1},\quad \text{and}\quad m_2 = x_1^{-1}x_2^{-1}x_3\,x_4^{-1}\ \cdots\ x_{k-2}\,x_{k-1}^{-1},$$

one appearing $\overset{+}{\rho}_{k-1}$ and the other in $\overset{-}{\rho}_{k-1}$ (depending on whether $(k - 1)/2$ is even or odd).

We now have that

$$\begin{aligned} 0 &= A(\beta\rho_k) = A(\beta[\overset{+}{\rho}_k + \overset{-}{\rho}_k]) = A(\beta\overset{+}{\rho}_k) + A(\beta\overset{-}{\rho}_k) \\ &= A(\beta m_1) + A(\beta m_2) + 2A(\beta\sigma). \end{aligned}$$

Note that if $\beta m_1 = \varphi_1(\beta)$ and $\beta m_2 = \varphi_2(\beta)$, then $\operatorname{sgn}(\varphi_1) = \operatorname{sgn}(\varphi_2)$. Therefore

$$0 = A(\beta\rho_k) = 2\operatorname{sgn}(\varphi_1)A(\beta) + 2A(\beta\cdot\sigma) = 2[\operatorname{sgn}(\varphi_1)A(\beta) + A(\beta\cdot\sigma)] = 0.$$

Therefore $A(\beta\overset{+}{\rho}_{k-1}) = A(\beta\overset{-}{\rho}_{k-1}) = \operatorname{sgn}(\varphi_1)A(\beta) + A(\beta\sigma) = 0$. This completes the proof that $A(\beta\rho_{k-1}^{\pm}) = 0$.

This completes the proof of the Lemma 6, and as we noted previously, this implies that $F(a_i, b_i') = 1$. Therefore we now know that the diagonal terms are 1, and we will proceed to show that the terms above the diagonal are zero.

Case 1 $(a_i = x_k^s,\ b_j = x_k^{-r}\ s < r)$. Therefore

$$\beta a_i\,b_j = \beta x_k^{s-r} \quad \text{where}\quad -2k + 3 \le s - r \le -1.$$

For some l, βx_k^{s-r} is symmetric in (l, k). Therefore $A(\beta a_i\,b_j) = 0$.

Case 2 ($a_i = x_k^s$, $-(k-2) \leq s \leq 0$, $b_j = x_k^{-1/2}\bar{\Delta}^{\pm}$). Therefore $\beta a_i b_j$ is equal to

$$\beta x_k^{s-1/2}\bar{\Delta}^{\pm}.$$

The terms of $\bar{\Delta}^{\pm}$ are of the form $x_1^{\varepsilon_1/2} \cdots x_{k-1}^{\varepsilon_{k-1}/2}$. We then have that $\beta a_i b_j$ is a sum of terms of the form $x_1^{\varepsilon_1/2} x_2^{1+\varepsilon_2/2} \cdots x_k^{(k-1)+(2s-1)/2}$. The range of the absolute value of the exponents are $\frac{1}{2}$ integers between $\frac{1}{2}$ and $(2k-3)/2$. By the Pigeon Hole principle we have that each term is symmetric in some (i, j). This implies $A(\beta a_i b_j) = 0$ for Case 2.

Case 3 ($a_i = x_k^{1/2}\Delta^{\pm}$, $b_j = x_k^s$, $-1 \leq s \leq -(k-1)$). Exactly as in the previous case, we use the Pigeon Hole principle on terms of the form $x_1^{\varepsilon_1/2} \cdots x_k^{k+(2s+1)/2}$, to get $A(\beta a_i b_j) = 0$.

Case 4 ($a_i = x_k^{1/2}\Delta^{+}$, $b_j = x_k^{-1/2}\bar{\Delta}^{-}$). $A(\beta a_i b_j) = A(\beta \Delta^{+}\bar{\Delta}^{-})$. As in the case of the diagonal element, whether $k - 1$ is even or odd,

$$\Delta^{+}\bar{\Delta}^{-} = \rho_1 + \rho_3 + \cdots.$$

Therefore by the previous lemma, $A(\beta\Delta^{+}\bar{\Delta}^{-}) = 0$.

This completes the inductive step of the theorem. We must now start our induction with the case $n = 2$.

Case. $G = Spin(4)$. $\beta = x_2 \in R(\tilde{T}_2)$ and two bases for $R(\tilde{T}_2)$ over $R(Spin(4))$ are

$$a_i = \{1,\ x_2^{1/2}(x_1^{-1/2}),\ x_2^{1/2}(x_1^{1/2}),\ x_2\},\quad b_j = \{1,\ x_2^{-1/2}(x_1^{1/2}),\ x_2^{-1/2}(x_1^{-1/2}),\ x_2^{-1}\}.$$

The terms in the bracket take on the role of $\Delta_1^{\pm}$. Let $b_j' = b_j x_2$; then

$$F(a_i, b_j') = A(\beta a_i b_j)/A(\beta).$$

Suppose $i = j$; then $a_i = b_i^{-1}$ and

$$F(a_i, b_i) = A(\beta)/A(\beta) = +1.$$

Suppose $i < j$; then each of the six terms above the diagonal is symmetric in some (r, s) and are therefore zero.

This completes the proof of Theorem 11. Furthermore, as a consequence of the proof, we get the following additional theorem.

THEOREM 12. *$F : R(\tilde{T}_k) \times R(\tilde{T}_k) \to R(Spin^c(2k-2))$ is strongly non-singular for $2k - 2 \geq 4$.*

We remarked at the beginning of this section that

$$R(Spin(2k+1)) \leq R(Spin(2k))$$

and that $R(Spin(2k))$ is generated by 1, Δ^{+} over $R(Spin(2k+1))$.

THEOREM 13. *$F : R(Spin(2k)) \times R(Spin(2k)) \to R(Spin(2k+1))$ is strongly non-singular.*

Proof. The roots of $Spin(2k+1)$ are the same as the roots of $SO(2k+1)$. They are $\pm x_r$, $x_i - x_j$ and $\pm(x_i + x_j)$, $i \neq j$, $1 \leq i, j, r \leq k$. Let the Fundamental Weyl Chamber be described by $0 < x_1 < \cdots < x_k$; then the postitive roots are x_r, $x_i - x_j$ for $i > j$ and $x_i + x_j$ for $i \neq j$. Therefore β is $(x_1 + 3x_2 + \cdots)/2$ and as a representation $\beta = x_1^{1/2} \cdots x_k^{(2k-1)/2}$, an element of $R(\tilde{T}_k)$.

Let the set $\{1, \Delta^+\}$ be called $\{a_i\}_{i=1}^2$, and let $\{b_j\}_{j=1}^2$ be the set $\{\Delta^+, 1\}$. $F(a_i, b_j) = A(\beta' a_i b_j)/A(\beta)$. The first thing to note is that $A(\beta') = 0$, which implies that the term above the diagonal is zero. This follows because

$$\beta' = x_2 \cdots x_k^{k-1}$$

is fixed under $\rho \in W(Spin(2k+1))$ where ρ is the element permuting x_1 and x_1^{-1}.

To complete the proof we just have to note that $A(\beta'\Delta^+) = A(\beta)$. This follows because

$$\Delta^+ = \sum_{\pi\varepsilon_i = +1} x_1^{\varepsilon_1/2} \cdots x_k^{\varepsilon_k/2},$$

with $x_1^{1/2} \cdots x_k^{1/2}\beta' = \beta$, and for any other summand, γ of Δ^+, $\beta'\gamma$ is symmetric in some (i, j).

By the inductive lemma we also have the following:

THEOREM 14. *$F : R(\tilde{T}_k) \times R(\tilde{T}_k) \to R(Spin(2k+1))$ is strongly non-singular.*

Remark. For the case $G = Spin^c(2k+1)$ we have

$$Spin^c(2k) \leq Spin^c(2k+1)$$

induced from the monomorphic inclusion of $SO(2k)$ in $SO(2k+1)$. This induces

$$R(Spin^c(2k+1)) \leq R(Spin^c(2k)).$$

As in the case of $Spin^c(2k)$ it is easy to see that

$$W(Spin^c(2k+1)) \cong W(SO(2k+1)),$$

and that

$$R(Spin^c(2k+1)) \cong Z[x_{k+1}^{\pm 1}, \rho_1, \cdots, \rho_{k-1}, x_{k+1}^{1/2}\Delta_k] \leq R(\tilde{T}_{k+1}).$$

Noticing also that the roots of $Spin^c(2k+1)$ are the "same" as the roots of $SO(2k+1)$ will yield the following theorem.

THEOREM 15. *$F : R(Spin^c(2k)) \times R(Spin^c(2k)) \to R(Spin^c(2k+1)$ is strongly non-singular.*

Proof. Analogous to the case $Spin(2k) \leq Spin(2k+1)$.

Theorem 16 together with the inductive lemma yields the following concluding theorem.

THEOREM 16. *$F : R(\tilde{T}_{k+1}) \times R(\tilde{T}_{k+1}) \to R(Spin^c(2k+1)$ is strongly non-singular.*

9. Applications

As remarked in the beginning of the paper, the main theorem gives another proof of the famous conjecture of Atiyah-Hirzebruch. The assertion is that

$$\alpha : R(H) \to K(G/H)$$

is onto for suitable compact connected Lie groups $H \leq G$. (See [2] for notation and a proof of the conjecture for a large class of Lie groups; see [8] for a general proof where $\pi_1(G)$ has no torsion.) The proof of the conjecture, for the G and H discussed in this paper, will imply that $\{\alpha(a_i)\}$ is a basis for the free abelian group $K(G/H)$, with $\{a_i\}$ being one of the dual bases. (See the main theorem.)

I learned this argument from A. Vasquez in the name of D. Anderson, though apparently Atiyah and Hirzebruch were aware of the approach even at the time of [2]. As pointed out in [2] it suffices to consider the case where H is a maximal torus, T.

COROLLARY. *Let G be any of the classical groups considered in this paper. Then*

$$\alpha : R(T) \to K(G/T)$$

is onto.

Proof. Let $\{a_i\}_{i=1}^N$ and $\{b_j\}_{j=1}^N$ be a pair of dual bases satisfying the conditions of the main theorem (i.e., $F(a_i, b_j) = \delta_{ij}$). By 8.2 of [3],

$$\langle \tau \text{ ch } (\alpha(a_i b_j), [G/T] \rangle = \delta_{ij},$$

where ch denotes the Chern character and τ is the generalized Todd genus of the $Spin^c$-structure on G/T. By [4], $\langle \tau \text{ ch } (xy), [G/T] \rangle$ is an integer for any $x, y \in K(G/T)$. Now it is known [2] that $K(G/T)$ is a free abelian group of rank N. Let

$$c_1, \cdots, c_N \in K(G/T)$$

be a basis. Let $\alpha(a_i) = \sum n_{i_k} c_k$ and $\alpha(b_j) = \sum m_{lj} c_l$. We deduce that the $N \times N$ identity matrix is the product of 3 matricies with integer entries, viz.

$$\| n_{ij} \|, \quad \| \langle \tau \text{ ch } (c_k, c_l), [G/T] \rangle \| \quad \text{and} \quad \| m_{lj} \|.$$

Thus they are all invertible matricies. In particular $\{\alpha(a_i)\}_{i=1}^N$ is a basis for $K(G/T)$.

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Washington University
St. Louis, Missouri