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ALEXANDER POLYNOMIALS OF LINKS OF SMALL ORDER

BY

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Torres [12] has given necessary conditions for a polynomial to be the Alexander polynomial of a link in S^3. These conditions are analogous to Seifert's necessary and sufficient conditions for a polynomial of one variable to be the Alexander polynomial of a knot [11], but they have never been proved sufficient or insufficient. This paper attacks the sufficiency question in the case of two-component links with both components unknotted.

It has long been known that the genus [11] of a link places an upper bound on the total degree of its Alexander polynomial. We have shown elsewhere [7] that the degree of individual variables in the Alexander polynomial is also of geometric significance. (These variables are in 1-1 correspondence with the components of the link.) The *orders* or geometric intersection numbers of a link determine upper bounds for the degrees of the individual variables in its Alexander polynomial.

We begin with links of small order and work upward. In this paper, we characterize the Alexander polynomials of links of linking number ± 2 or 0, orders less than or equal to 4 and 2, linking number ± 1, orders less than or equal to 3 and 3, and linking number ± 3, order (3, 3). In the last-named case, a restriction in addition to the Torres conditions is required.

The question of whether the Torres conditions characterize the Alexander polynomials of links may be too broad, in the sense that links with quite different properties could have the same Alexander polynomial. In links with more than two components, the Alexander polynomial does not always determine the linking numbers of all pairs of components. Moreover, a link which has as a component a nontrivial knot with Alexander polynomial $\Delta(x) = 1$ may have the same Alexander polynomial $\Delta(x, y, \ldots)$ as a link with unknotted components. We exclude links with more than two components and links with knotted components from consideration.

In Section 1 below, we state the Torres conditions, the above-mentioned theorem concerning orders, and some needed lemmas. In Section 2, we show that, for small Alexander polynomials, the reduced Alexander polynomial $\Delta(t, t)$ determines the unreduced Alexander polynomial. In Section 3, we give the Seifert matrix computations necessary to characterize the Alexander polynomials in all the above-mentioned cases but the last. In Section 4, we establish our new restriction in the linking number ± 3, order (3, 3) case.

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1. Preliminary results

THEOREM 1 (The Torres conditions [12], [5]). *If a polynomial* Δ *in*

$$Z[t_1, t_1^{-1}, \ldots, t_\mu, t_\mu^{-1}]$$

is the Alexander polynomial of a link L *with* $\mu > 1$, *then:*

(1) $\Delta(t_1, \ldots, t_\mu) = (-1)^\mu t_1^{n_1} \cdots t_\mu^{n_\mu} \Delta(t_1^{-1}, \ldots, t_\mu^{-1})$ *for some integers* $n_1, \ldots, n_\mu$.

(2a) *If* $\mu(L) = 2$, *then*

$$\Delta(t_1, 1) = \frac{t_1^l - 1}{t_1 - 1} \Delta(t_1),$$

where l *is the linking number of* L *and* $\Delta(t_1)$ *is the Alexander polynomial of the first component of* L.

(2b) *If* $\mu(L) > 2$, *then* $\Delta(t_1, \ldots, t_{\mu-1}, 1) = (t_1^{l_1} t_2^{l_2} \cdots t_{\mu-1}^{l_{\mu-1}} - 1)\Delta(t_1, \ldots, t_{\mu-1})$, *where* l_i *is the linking number of the ith component of* L *with the* μ*th component and* $\Delta(t_1, \ldots, t_{\mu-1})$ *is the Alexander polynomial of the link consisting of the first* $\mu - 1$ *components of* L.

In the case of two-component links with both components unknotted, condition (2) reduces to

$$\Delta(t_1, 1) = \begin{cases} t_1^{l-1} + t_1^{l-2} + \cdots + 1 & \text{for } l \neq 0, \\ 0 & \text{for } l = 0. \end{cases}$$

We will usually write $\Delta(x, y)$ instead of $\Delta(t_1, t_2)$.

Throughout this paper, the Alexander polynomial of a two-component link will be written in a rectangular array, as follows:

$$\begin{array}{rlllll} \Delta(x, y) = & a_{00} & + a_{10}x & + a_{20}x^2 & + \cdots + a_{m0}x^m \\ & + a_{01} & + a_{11}xy & + a_{21}x^2y & + \cdots + a_{m1}x^my \\ & + & & & \\ & \vdots & & & \\ & + & & & \\ & a_{0n}y^n & + a_{1n}xy^n & + a_{2n}x^2y^n & + \cdots + a_{mn}x^my^n, & a_{ij} \in Z. \end{array}$$

(Negative powers of x and y can be eliminated by multiplication by units.) This notation allows ready display of the Torres conditions. For example, (1) states that $a_{ij} = a_{m-i,n-j}$. The dimensions of the array correspond to the x- and y-degrees of $\Delta(x, y)$. This notation is adopted in the table of knots and links and their Alexander polynomials in the back of Rolfsen's book *Knots and Links* [10, p. 388]. The term of smallest degree is in the *lower* left-hand corner, however.

DEFINITION. Let $L = K_1 \cup \cdots \cup K_\mu$ be a link in S^3 with K_1 unknotted. Then the *first order* of L is the nonnegative integer α_1 if there exists an embedded disk D_1 in S^3 with boundary K_1 which intersects $K_2 \cup \cdots \cup K_\mu$ in α_1 points, while no embedded disk with boundary K_1 intersects $K_2 \cup \cdots \cup K_\mu$ in fewer points. If K_i is unknotted, we define the *i*th *order* in the same way.

Order can be loosely described as an unsigned linking number. Unlike the linking number, the order may be asymmetric in two-component links. If the link L has all its components unknotted, so that μ different orders α_i are defined, then we say that L has order $(\alpha_1, \alpha_2, \ldots, \alpha_\mu)$.

DEFINITION. Let L be a link in S^3 with Alexander polynomial $\Delta(t_1, \ldots, t_\mu)$. Then $\deg_i \Delta \equiv$ (maximum t_i-power of any term of Δ) minus (minimum t_i-power of any term of Δ).

Convention. If $\Delta(t_1, \ldots, t_\mu) \equiv 0$, then $\deg_i \Delta = -1$.

THEOREM 2. *Let $L = K_1 \cup \cdots \cup K_\mu$, $\mu > 1$, be a link in S^3 with Alexander polynomial $\Delta(t_1, \ldots, t_\mu)$. Assume K_1 is unknotted, and let α_1 be the first order of L. Then* $\deg_1 \Delta + 1 \leq \alpha_1$.

The proof of Theorem 2 is based on showing that an over presentation [3] for a special projection of L gives rise to an Alexander matrix in which just α_1 rows contain an entry "t_1", and that these rows contain t_1 to the first power. See [7].

James Bailey of University of British Columbia has extended this theorem to links with knotted components. His inequality involves the genus as well as the order of the component in question, and can be sharper than the inequality of Theorem 2 in the case of an unknotted component. Thus we do not prove Theorem 2 in detail.

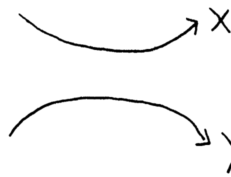
LEMMA 3. *Let $L = K_1 \cup \cdots \cup K_\mu$ be a link with Alexander polynomial $\Delta(t_1, \ldots, t_\mu)$. If L' is the link obtained from L by reversing the orientation of K_1, and $\Delta'(t_1, \ldots, t_\mu)$ is its Alexander polynomial, then*

$$\Delta'(t_1, \ldots, t_\mu) = \Delta(t_1^{-1}, t_2, \ldots, t_\mu),$$

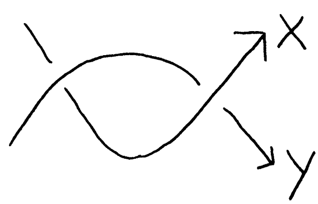
up to units.

This lemma, proved in [7], will be needed in Section 4. The following result of Conway's [1, p. 338] will be needed in Section 3.

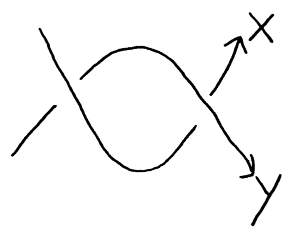
LEMMA 4. *Let L_{00} be a link which contains a configuration*



in its diagram. (x and y are segments from distinct knots.) Let L_{++} and L_{--} be the links obtained by replacing this configuration by



and



respectively. Then the Alexander polynomials Δ_{00}, Δ_{++}, and Δ_{--} of L_{00}, L_{++}, and L_{--} can be normalized so that $\Delta_{++} + \Delta_{--} = (1 + xy)\Delta_{00}$.

For a proof, see [7].

2. Reconstructing the Alexander polynomial from the reduced Alexander polynomial

The reduced Alexander polynomial $\Delta(t) \equiv \Delta(t, t)$ is much easier to compute than $\Delta(x, y)$, as we shall see in the next section. (Our definition of $\Delta(t)$ agrees with that of Torres [12] and, in the case of two-component links, Hosokowa [6]. It differs by a factor $(1 - t)$ from Crowell's definition [2].) In this section, we prove that $\Delta(x, y)$ is no more powerful than $\Delta(t)$ for links of order $(2n, 2)$ and $(3, 3)$.

First suppose that the link L has linking number ± 2, order $(2n, 2)$ and Alexander polynomial $\Delta(x, y)$. Then by Theorems 1 and 2,

$$\begin{aligned}\Delta(x, y) = {}& a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} + (1 - a_{n-1})x^n - \cdots - a_0 x^{2n-1} \\ & - a_0 y - a_1 xy - \cdots + (1 - a_{n-1})x^{n-1}y + a_{n-1}x^n y + \cdots + a_0 x^{2n-1}y.\end{aligned}$$

If we actually compute the reduced Alexander polynomial and obtain

$$\Delta(t) = b_0 + b_1 t + \cdots + b_{n-1}t^{n-1} + b_n t^n + \cdots + b_0 t^{2n},$$

then $a_0 = b_0$, $a_1 - a_0 = b_1$, $a_2 - a_1 = b_2, \ldots,$ and $a_{n-1} - a_{n-2} = b_{n-1}$. We can now solve for the a_i:

$$a_1 = a_0 + b_1 = b_0 + b_1,$$
$$a_2 = b_2 + a_1 = b_2 + b_1 + b_0, \ldots,$$
$$a_{n-1} = b_{n-1} + b_{n-2} + \cdots + b_0.$$

Thus the Alexander polynomial is completely determined by the reduced Alexander polynomial. A similar analysis holds for links of linking number 0, order $(2n, 2)$, provided the Alexander polynomial is not identically zero.

Now let us examine the case of linking number ± 1 or ± 3, order $(3, 3)$. In the first case,

$$\begin{aligned}\Delta(x, y) = \quad & a_{00} && + \quad a_{10}x && + (-a_{00} - a_{10})x^2 \\ + \quad & a_{01}y && + (1 - 2a_{01})xy + && a_{01}x^2y \\ + \; & (-a_{00} - a_{10})y^2 + && a_{10}xy^2 \quad + && a_{00}x^2y^2.\end{aligned}$$

In writing this expression, we have used the first Torres condition and the second Torres condition in the rows but not the columns. Applying the second Torres condition to the first column, we have $a_{00} + a_{01} + (-a_{00} - a_{10}) = 0$, or $a_{01} = a_{10}$. Thus letting $a_{00} = A$ and $a_{10} = a_{01} = B$, we have

$$\begin{aligned}\Delta(x, y) = \quad & A && + \quad Bx && - (A + B)x^2 \\ + \quad & By && + (1 - 2B)xy + && Bx^2y \\ - \; & (A + B)y^2 + && Bxy^2 \quad + && Ax^2y^2.\end{aligned}$$

Similarly, if the linking number is ± 3,

$$\begin{aligned}\Delta(x, y) = \quad & A && + \quad Bx && + (1 - A - B)x^2 \\ + \quad & By && + (1 - 2B)xy + && Bx^2y \\ + \; & (1 - A - B)y^2 + && Bxy^2 \quad + && Ax^2y^2.\end{aligned}$$

Suppose that in either case we actually compute the reduced Alexander polynomial and obtain

$$\Delta(t) = b_0 + b_1 t + b_2 t^2 + b_1 t^3 + b_0 t^4.$$

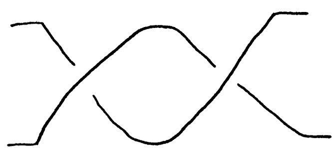
Then $b_0 = A$, $b_1 = 2B$, and we reconstruct the Alexander polynomial very easily.

For Alexander polynomials of higher degree in x and y, this method no longer works, at least not without additional information. The number of independent parameters in $\Delta(x, y)$ becomes larger than the number of independent parameters in $\Delta(t)$.

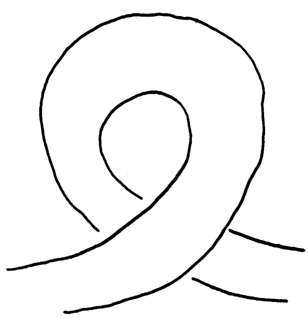
3. Computations using the Seifert matrix

In this section and the next, we apply the method of the classic paper [11] to compute reduced Alexander polynomials of links using Seifert surfaces and matrices. From geometric properties of the links, we will then be able to reconstruct their Alexander polynomials, in the manner of the previous section.

Every link in S^3 spans an embedded, orientable surface of some genus h. For a two-component link, this "Seifert surface" can be represented as a disk with $2h + 1$ attached bands, as shown in Figure 1. These bands can be twisted, knotted, and linked. By replacing twists



in the bands by loops



one arrives at a projection in which only one side of the orientable surface is visible.

The paths $a_1, a_2, \ldots, a_{2h+1}$ which trace the midlines of the bands form a basis for the first homology group of the surface. The paths $a_i, i = 1, \ldots, 2h$ are oriented so that a_{2i-1} crosses a_{2i} from left to right at their point of intersection in the disk. The orientation of a_{2h+1} coincides with that of the boundary component K_2 which includes the edges of all the other bands.

The Seifert matrix is defined in terms of the "overcrossing numbers" v_{ij} of the paths a_i. Let v_{ij} be the number of times a_i crosses over a_j from left to right minus the number of times a_i crosses over a_j from right to left. If $a_i \cap a_j = \emptyset$, then $v_{ij} = v_{ji}$ (this is a linking number) and if $i = 2k - 1$, $j = 2k$, then $v_{2k-1,2k} = v_{2k,2k-1} + 1$; see [4, p. 152]. The Seifert matrix of L, as defined in [6], is then

$$\begin{pmatrix}
v_{11}(1-t) & v_{12}(1-t)+t & \cdots & v_{1,2h-1}(1-t) & v_{1,2h}(1-t) & v_{1,2h+1}(1-t) \\
v_{12}(1-t)-1 & v_{22}(1-t) & & v_{2,2h-1}(1-t) & v_{2,2h}(1-t) & v_{2,2h+1}(1-t) \\
\vdots & \vdots & & \vdots & \vdots & \vdots \\
v_{1,2h-1}(1-t) & v_{2,2h-1}(1-t) & \cdots & v_{2h-1,2h-1}(1-t) & v_{2h-1,2h}(1-t)+t & v_{2h-1,2h+1}(1-t) \\
v_{1,2h}(1-t) & v_{2,2h}(1-t) & \cdots & v_{2h-1,2h}(1-t)-1 & v_{2h,2h}(1-t) & v_{2h,2h+1}(1-t) \\
v_{1,2h+1} & v_{2,2h+1} & \cdots & v_{2h-1,2h+1} & v_{2h,2h+1} & v_{2h+1,2h+1}
\end{pmatrix}$$

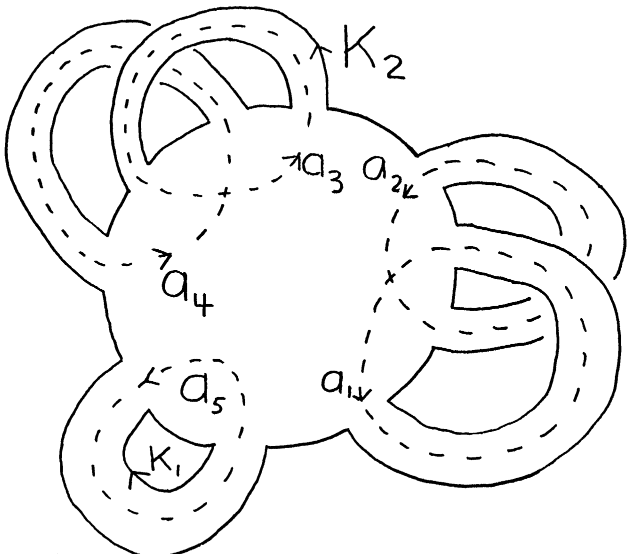


FIG. 1

The determinant of this matrix, a link invariant, is the reduced Alexander polynomial [12, pp. 64–73].

We proceed to calculate the reduced Alexander polynomials of a class of links of linking number 2, order (2, 2). Figure 2 shows the class of links together with their Seifert surfaces and a suitable basis for the first homology groups of the surfaces. The path a_3 must be parallel to the component K_1 in order to play the role of a_{2h+1} in Figure 1. The only intersection of paths is that of a_1 and a_2. There are no intersections, merely overcrossings, at the crossing points of the link projection. For example, at the crossing labelled c in Fig. 2, a_3 overcrosses a_1, which overcrosses a_2.

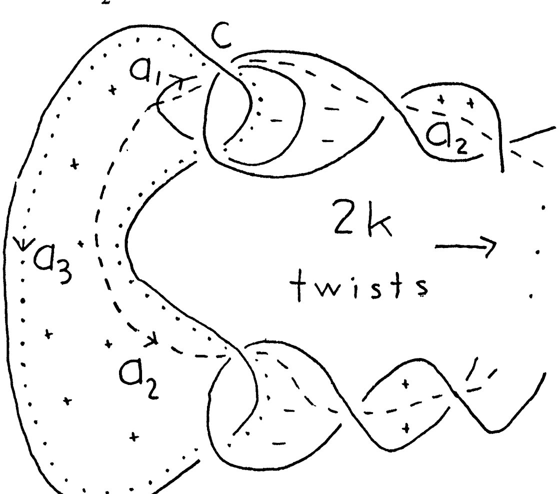


FIG. 2

The overcrossing numbers of the projection in Fig. 2 are $v_{11} = 1$, $v_{22} = 1 + k$, where k is any integer, (negative k corresponds to twists in the opposite direction), $v_{33} = 2$ (the linking number), $v_{12} = 0$, $v_{13} = -1$, and $v_{23} = 0$. Thus the Seifert matrix is

$$\begin{pmatrix} (1-t) & t & (t-1) \\ -1 & (1+k)(1-t) & 0 \\ -1 & 0 & 2 \end{pmatrix}$$

Calculation of its determinant gives $\Delta(t) = (1 + k) - 2kt + (1 + k)t^2$. In the language of Section 2, $b_0 = (1 + k)$, and hence can equal any integer. Since both orders are 2, the complete Alexander polynomial is

$$\begin{aligned} \Delta(x, y) = (1 + k) &- kx \\ - \quad ky &+ (1 + k)xy. \end{aligned}$$

There is only one independent parameter, and it can take any integer value. Thus the Alexander polynomials of links of linking number ± 2, order (2, 2) are characterized by the Torres conditions.

We proceed to a class of links which includes the previous class but in which the first order may be 4 or 2. A typical member of the class is shown in Fig. 3. This time we allow three of the overcrossing numbers, namely v_{22}, v_{44}, and v_{24}, to take arbitrary integer values. This is accomplished by adding twists in the bands which include a_2 and a_4 and giving a_2 and a_4 an arbitrary linking number, without changing K_1 or the Seifert surface near K_1. (It is geometrically clear that v_{22} and v_{44} are not really independent: only the sum $v_{22} + v_{44}$ is a link-type invariant.) Compute the other overcrossing numbers; the entire Seifert matrix is

$$\begin{pmatrix} (1-t) & 1 & 0 & 0 & (t-1) \\ -t & v_{22}(1-t) & 0 & v_{24}(1-t) & 0 \\ 0 & 0 & (1-t) & t & (t-1) \\ 0 & v_{24}(1-t) & -t & v_{44}(1-t) & 0 \\ -1 & 0 & -1 & 0 & 2 \end{pmatrix}$$

Taking the determinant yields the reduced Alexander polynomial

$$\begin{aligned} \Delta(t) = {} & -v_{24} + (v_{22} + v_{44} + 2v_{24})t + (2 - 2(v_{22} + v_{44}) - 2v_{24})t^2 \\ & + (v_{22} + v_{44} + 2v_{24})t^3 - v_{24}t^4. \end{aligned}$$

Let $m = v_{22} + v_{44}$. Then $b_0 = -v_{24}$ is an arbitrary integer and $b_1 = m + 2v_{24}$ can be made arbitrary by varying m. Again, the Alexander polynomials of links with linking number ± 2, first order less than or equal to 4, and second order 2 are characterized by the Torres conditions. For in this case

$$\begin{aligned} \Delta(x, y) = {} & a_0 + a_1 x + (1 - a_1)x^2 - a_0 x^3 \\ & - a_0 y + (1 - a_1)xy + a_1 x^3 y + a_0 x^3 y \end{aligned}$$

where $a_0 = -v_{24}$ and $a_1 = m + v_{24}$.

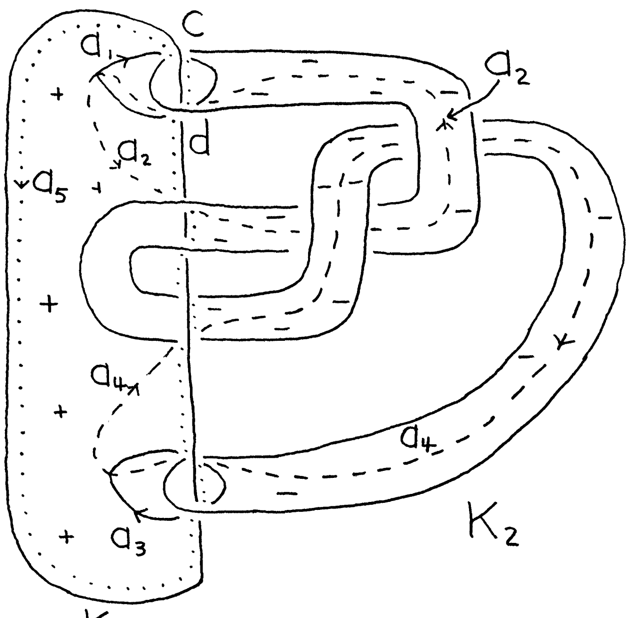


FIG. 3

With little additional work, we can extend these results to characterize Alexander polynomials of links of linking number 0, first order less than or equal to 4, second order less than or equal to 2. The case of order (0, 0) is trivial: the only possible Alexander polynomial is $\Delta(x, y) \equiv 0$, and the splittable link with two unknotted components *has* this Alexander polynomial.

We move straight to the analogue of Fig. 3, since these links include the links of Fig. 2 as a special case. We apply Lemma 4 (Conway's lemma) to the crossings labelled c and d in Fig. 3. Figure 4(a) gives a larger picture of these crossings. The links of Fig. 3 play the role of L_{++} in Lemma 4. The links L_{--} (Fig. 4(b)) have linking number 0: we will compute their Alexander polynomials. In all cases, L_{00} is the standard link of linking number 1. By Lemma 4, $\Delta_{++} + \Delta_{--} = (1 + xy)\Delta_{00}$. In order to maintain the symmetry of the first Torres condition, we must take $\Delta_{00}(x, y) = x$. Then

$$\begin{aligned}\Delta_{--} = (x + x^2y) - \Delta_{++} = &- a_0 + (1 - a_1)x + (a_1 - 1)x^2 + a_0x^3 \\ &+ a_0y + (a_1 - 1)xy + (1 - a_1)x^2y - a_0x^3y,\end{aligned}$$

where a_0 and a_1 can be arbitrary integers by the analysis of the linking number 2 case. Thus the Alexander polynomials of links of linking number 0, orders less than or equal to 4 and 2 are also characterized by the Torres conditions.

A link of linking number ± 1, order (1, 1) can only have $\Delta(x, y) \equiv 1$, (up to units) and the standard link of linking number 1 *has* this Alexander polynomial. We move on to the case of linking number ± 1, first and second order less than or equal to 3. Figure 5 illustrates the class of links we will use to character-

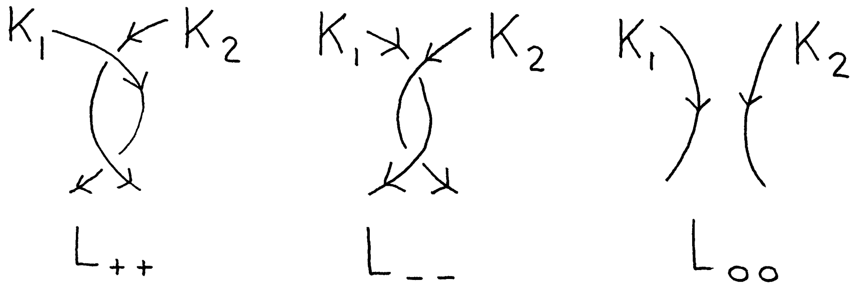


FIG. 4

ize the Alexander polynomial in this case. Again, we leave three crossing numbers undetermined, namely v_{24}, v_{22}, and v_{44}. These are related to the number of twists in the "arms" of the link by $v_{24} = k_3, v_{22} = k_1 + k_3, v_{44} = k_2 + k_3$. Since k_1, k_2, and k_3 can be arbitrary integers, so can v_{24}, v_{22}, and v_{44}. Compute the other overcrossing numbers; the Seifert matrix is

$$\begin{pmatrix} (1-t) & t & 0 & 0 & (t-1) \\ -1 & v_{22}(1-t) & 0 & v_{24}(1-t) & 0 \\ 0 & 0 & (1-t) & 1 & (t-1) \\ 0 & v_{24}(1-t) & -t & v_{44}(1-t) & 0 \\ -1 & 0 & -1 & 0 & +1 \end{pmatrix}$$

Computing its determinant gives

$$\begin{aligned} \Delta(t) = {} & (v_{24}^2 - v_{22}v_{44} - v_{24}) + (4v_{22}v_{44} - 4v_{24}^2 + 2v_{24})t \\ & + (1 - 2v_{24} + 6v_{24}^2 - 6v_{22}v_{44})t^2 + (4v_{22}v_{44} - 4v_{24}^2 + 2v_{24})t^3 \\ & + (v_{24}^2 - v_{22}v_{44} - v_{24})t^4. \end{aligned}$$

Thus the parameters A and B of Section 2 are given by

(1) $A = v_{24}^2 - v_{22}v_{44} - v_{24}$,
(2) $B = 2v_{22}v_{44} - 2v_{24}^2 + v_{24}$.

Let $m = v_{22}v_{44}$. Then m and v_{24} can take arbitrary integer values. We shall show that A and B then can take arbitrary integer values, and thus that the Torres conditions are sufficient in this case.

Given A and B, we must have $2A + B = -v_{24}$, by (1) and (2). Then

$$A = (2A + B)^2 - m + (2A + B),$$

so $m = (2A + B)^2 + (A + B)$. But we must also have $B = 2m - 2(2A + B)^2 - (2A + B)$, or $2m = 2(2A + B)^2 + (2A + 2B)$, or again $m = (2A + B)^2 + A + B$. Since these two expressions for m are identical, we can generate arbitrary A and B. For example, suppose we wish to find a link with $A = -5$, $B = 6$. Then $2A + B = -4$, so $v_{24} = 4$. $m = (2A + B)^2 + A + B = 16 + 1 = 17$, so we let

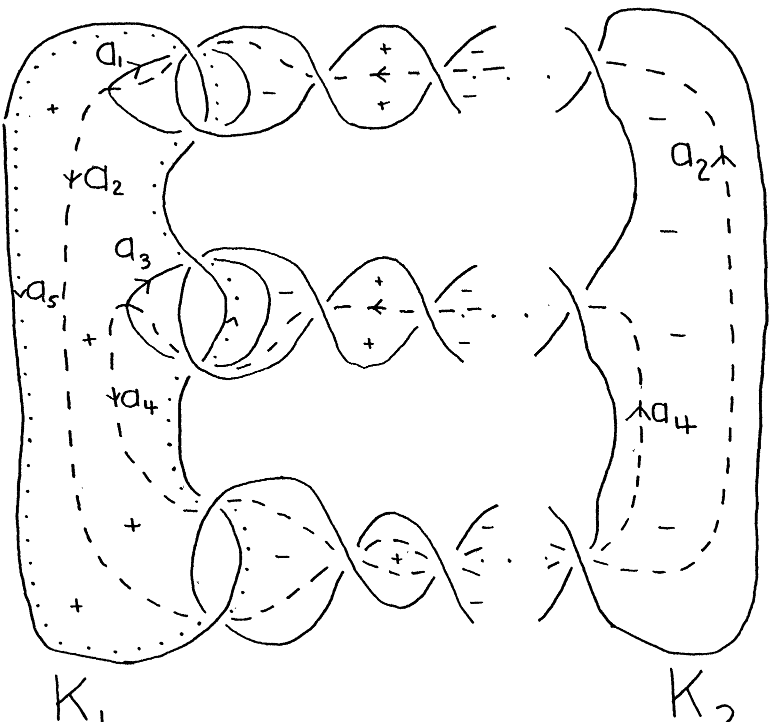


FIG. 5

$v_{22} = 1$, $v_{44} = 17$. For the number of twists $2k_1, 2k_2, 2k_3$ in the arms (Fig. 5), we have $k_3 = v_{24} = 4$, $k_1 = v_{22} - k_3 = -3$, $k_2 = v_{44} - k_3 = 13$.

Levine [8] has proved that any polynomial which satisfies the Torres conditions for the Alexander polynomial of a two-component link of linking number ± 1 is the Alexander polynomial of such a link. His method does not seem to restrict the knot types of the components, however. Our results are a start toward showing that the Alexander polynomials of links with *unknotted* components of linking number ± 1 are also characterized by the Torres conditions.

4. Insufficiency of the Torres conditions in the linking number ± 3, order $(3, 3)$ case

Up to now, we have not tried to display all links of a given linking number and order, but only a selected class which generates all Alexander polynomials allowed by the Torres conditions. In establishing a new restriction, however, we must be careful to consider all possibilities. In the linking number ± 3, order $(3, 3)$ case, this is not too difficult. The assumption that the absolute value of the linking number equals the order severely limits the ways in which embedded disks spanned in the components of the link can intersect. This in turn limits the possible Seifert matrices of such a link.

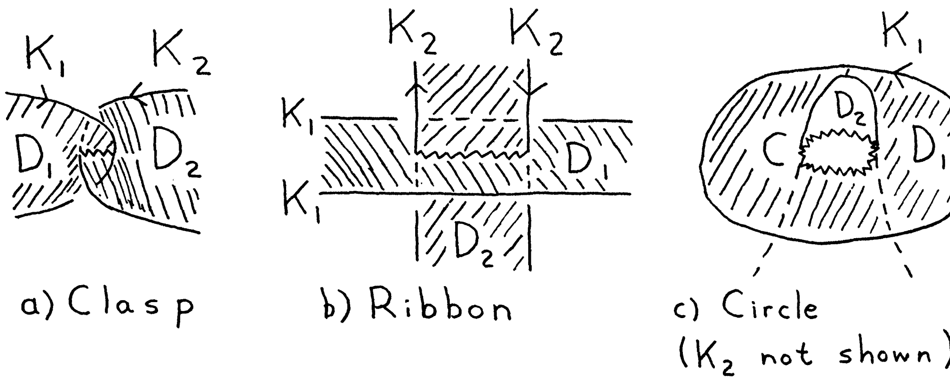


FIG. 6

The three ways in which two embedded disks in general position in S^3 can intersect are illustrated in Figures 6(a), (b), and (c). We call these three types of intersections *clasps*, *ribbons* and *circles* respectively.

The links of order (4, 2) in Section 3 contain a ribbon intersection. Indeed, ribbon intersections are necessary to make the order asymmetrical. It is not clear that circle intersections are ever necessary. In the case we are considering, both circles and ribbons can be eliminated.

LEMMA 5. *Let $L = K_1 \cup K_2$ be a link of linking number $\pm n$, order (n, n). Then K_1 and K_2 span disks which intersect only in n clasps.*

Proof. Let D_1 and D_2 be spanning disks for K_1 and K_2 respectively which intersect the opposite component the minimum possible number of times. Suppose D_1 and D_2 are in general position and have a ribbon intersection, with K_2 intersecting D_1 in two points, as in Fig. 6(b). Then K_2 must have opposite orientations at the two points. Thus their combined contribution to the linking number is 0, but their contribution to the order is 2. This contradicts the assumption that the order equals the absolute value of the linking number.

A circle intersection C divides D_1 and D_2 into interior disks D_1° and D_2° and exterior annuli. If D_1° or D_2° contains no further intersection with $D_1 \cup D_2$, as in Fig. 6(c), then the circle intersection can be eliminated by a surgery. Nested circle intersections can be handled by induction. (See [7].) If K_1 or K_2 intersect $D_1^\circ \cup D_2^\circ$, as in Figure 7, then K_1 must enter and leave through D_2°, since it cannot intersect D_1°, and similarly for K_2 and D_1°. This creates one or more ribbon intersections. ■

Thus any link of linking number ± 3, order (3, 3) contains a pair of spanning disks which intersect only in three clasps. We can find a regular projection of the part of the link near K_1 and the spanning disks of the type shown in Figure 8(a) and (b).

Our next task will be to construct a Seifert surface for the links depicted in Fig. 8(a). We will then use Lemma 3 to determine the Alexander polynomial for the standard form shown in Fig. 8(b).

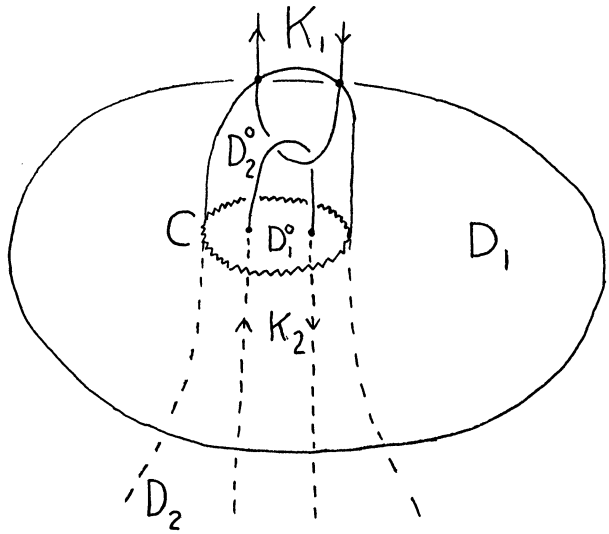


FIG. 7

Applying Seifert's algorithm [11] for finding a Seifert surface to the portion of a link depicted in Fig. 8(a) produces one complete Seifert circuit and three partial Seifert circuits, as depicted in Figure 9. A disk that incorporates these three partial Seifert circuits can be constructed from the disk D_2 of Fig. 8(a) by eliminating the parts of D_2 that project into the interior of D_1 (Fig. 9). In this way, we obtain a Seifert surface F for any link L in the class depicted in Fig. 8(a).

Figure 9 also shows a basis for the first homology group of F. Three generators, a_1, a_3 and a_5, are completely shown, and the remaining two, a_2 and a_4, are partially shown. (The arrangement is entirely analogous to that of the linking

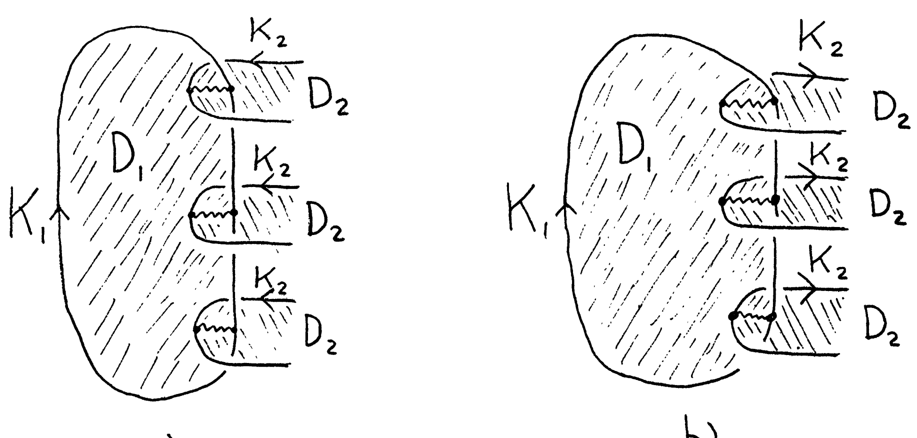


FIG. 8

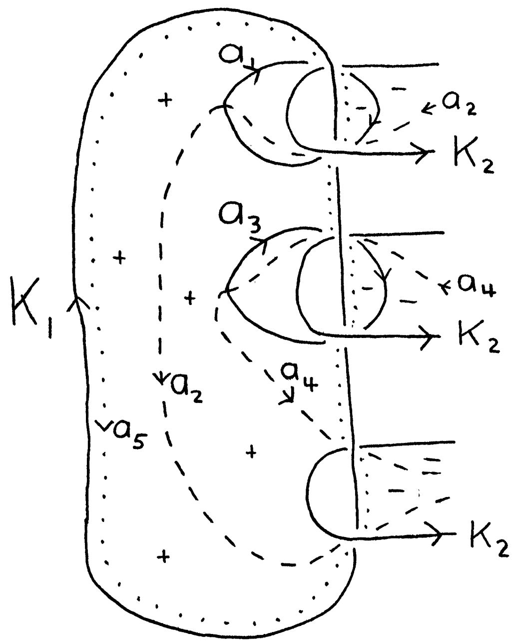


FIG. 9

number ± 1, order (3, 3) links of Section 3.) Thus all the overcrossing numbers v_{ij} can be determined from Fig. 9 except v_{22}, v_{24}, and v_{44}. These three overcrossing numbers can be arbitrary integers, as the class of links depicted in Figure 10 shows. That is, we choose

(1) k_3 so that $k_3 = v_{24}$,
(2) k_1 so that $k_1 + k_3 + 1 = v_{22}$,
(3) k_2 so that $k_2 + k_3 + 1 = v_{24}$.

The Seifert matrix, with v_{22}, v_{24}, and v_{44} undetermined, is

$$\begin{pmatrix} (1-t) & 1 & 0 & 0 & (t-1) \\ -t & v_{22}(1-t) & 0 & v_{24}(1-t) & 0 \\ 0 & 0 & (1-t) & t & (t-1) \\ 0 & v_{24}(1-t) & -1 & v_{44}(1-t) & 0 \\ -1 & 0 & -1 & 0 & 3 \end{pmatrix}$$

Taking the determinant, we obtain the reduced Alexander polynomial

$$\begin{aligned} \Delta(t) = {} & (v_{22}v_{44} - v_{24}^2 - v_{24}) + (4v_{24}^2 - 4v_{22}v_{44} + 2v_{22} + 2v_{24} + 2v_{44})t \\ & + (3 - 4v_{22} - 2v_{24} - 4v_{44} - 6v_{24}^2 + 6v_{22}v_{44})t^2 \\ & + (4v_{24}^2 - 4v_{22}v_{44} + 2v_{22} + 2v_{24} + 2v_{44})t^3 + (v_{22}v_{44} - v_{24}^2 - v_{24})t^4 \end{aligned}$$

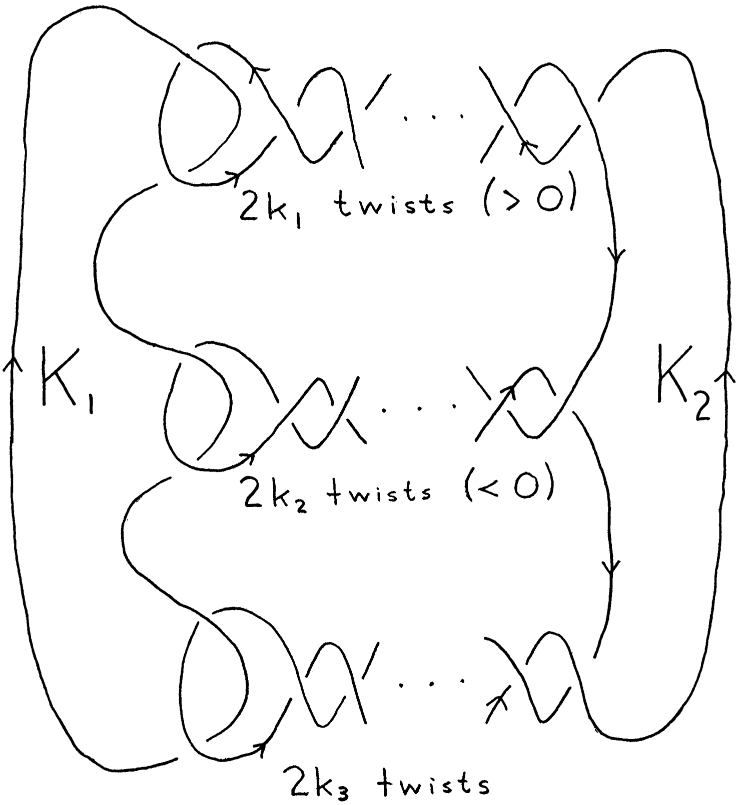


FIG. 10

The unreduced Alexander polynomial may be written

$$\begin{aligned}\Delta(x, y) = \quad & A + Bx + (1 - A - B)x^2 \\ & + By + (1 - 2B)xy + Bx^2y \\ & + (1 - A - B)y^2 + Bxy^2 + Ax^2y^2\end{aligned}$$

where now $A = v_{22}v_{44} - v_{24}^2 - v_{24}$ and $B = 2v_{24}^2 - 2v_{22}v_{44} + v_{22} + v_{24} + v_{44}$. Thus we are faced with the number-theoretic question of whether v_{22}, v_{24} and v_{44} can be varied to produce arbitrary A and B. The problem becomes more symmetrical if we replace v_{22}, v_{24} and v_{44} by k_1, k_2 and k_3, using equations (1)–(3). Then we have

$$A = k_1k_2 + k_1k_3 + k_2k_3 + k_1 + k_2 + k_3 + 1,$$

$$B = -2k_1k_2 - 2k_1k_3 - 2k_2k_3 - k_1 - k_2 - k_3$$

Notice that the expression $(1 - A - B)$, which also appears in $\Delta(x, y)$, is simply $k_1k_2 + k_1k_3 + k_2k_3$. Thus we let $a = (1 - A - B)$, $b = -B$, and ask whether the equations

(4) $k_1k_2 + k_1k_3 + k_2k_3 = a,$
(5) $2k_1k_2 + 2k_1k_3 + 2k_2k_3 + k_1 + k_2 + k_3 = b$

have simultaneous solutions in integers for arbitrary integers a and b.

We can reduce equations (4) and (5) to one equation in two unknowns. For if $c = k_1 + k_2 + k_3$, then equations (4) and (5) combine to give $2a + c = b$, or

(6) $c = b - 2a = k_1 + k_2 + k_3.$

Using equation (6) to eliminate k_3 from equation (4), we have

$$k_1k_2 + k_1(c - k_1 - k_2) + k_2(c - k_1 - k_2) = a,$$

which simplifies to

(7) $-k_1^2 - k_1k_2 - k_2^2 + ck_1 + ck_2 = a.$

In order to make our Alexander polynomial symmetric under the change of orientation depicted in Fig. 8(b), we assume $A = 1 - A - B$. Then

$$\Delta(x, y) = \Delta(x, y^{-1}) = \Delta(x^{-1}, y).$$

This condition translates into $b = 2a - 1$. Then $c = b - 2a = -1$. Now assume $a = 1 - A - B = A > 0$. Then equation (7) becomes

(8) $k_1^2 + k_1k_2 + k_2^2 + k_1 + k_2 = -a.$

Equation (8) does not have a real solution, much less an integer solution, because the function of two variables $f(x, y) = x^2 + xy + y^2 + x + y$ assumes its minimum value of $-1/3$ at $(-1/3, -1/3)$. We summarize these results in a theorem.

THEOREM 6. *A polynomial*

$$\begin{array}{ccc} A & + \quad Bx & + (1 - A - B)x^2 \\ + \quad By & + (1 - 2B)xy + & Bx^2y \\ + (1 - A - B)y^2 + & Bxy^2 \quad + & Ax^2y^2 \end{array}$$

is the Alexander polynomial of a link of linking number ± 3, *order* $(3, 3)$ *if and only if there are integers* k_1, k_2 *and* k_3 *such that*

$$B = -2k_1k_2 - 2k_1k_3 - 2k_2k_3 - k_1 - k_2 - k_3$$

and

$$A = k_1k_2 + k_1k_3 + k_2k_3 + k_1 + k_2 + k_3 + 1 \text{ or } k_1k_2 + k_2k_3 + k_3k_1.$$

This is a genuine restriction. For example

$$\begin{array}{ccc} A & + \ (1 - 2A)x \ + & Ax^2 \\ + (1 - 2A)y + & (4A - 1)xy & + (1 - 2A)x^2y \\ + \quad Ay^2 & + (1 - 2A)xy^2 + & Ax^2y^2, \end{array}$$

$A > 0$, is *not* the Alexander polynomial of any link of linking number ± 3, order $(3, 3)$ although it satisfies all the Torres conditions.

We emphasize again that a link of the class depicted in Fig. 8(b) cannot have this polynomial as its Alexander polynomial, for reversing the orientation of K_2 would produce a link of the class depicted in Fig. 8(a) with the same Alexander polynomial.

The following are two classes of links that could generate the missing polynomials of Theorem 6:

(a) Links of linking number ± 3 with unknotted components and orders greater than 3, whose larger orders are not detected by the Alexander polynomial. (Theorem 2 is only an inequality.)

(b) Links of linking number ± 3 with knotted components whose components have trivial Alexander polynomials. (Such knots are given in the final section of [11], for example.)

To determine the Alexander polynomials of such links, one must presumably analyze the universal Abelian covering space of the link's complement. For more complicated links than those of this paper, this double-infinite-cyclic cover is more powerful than the infinite-cyclic cover which is described by the Seifert matrix.

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