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MAXIMAL CHAINS OF PRIME IDEALS IN INTEGRAL EXTENSION DOMAINS, III

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1. Introduction

All rings in this paper are assumed to be commutative with identity, and the terminology is, in general, the same as that in [2].

To briefly describe the results in this paper, let A' denote the integral closure of an integral domain A in its quotient field, and let $s(A)$ (resp., $c(A)$) denote the set of lengths of maximal chains of prime ideals in A (resp., in arbitrary integral extension domains of A). Also, when $P \in \operatorname{Spec} A$, let $s(P)$ and $c(P)$ denote $s(A_P)$ and $c(A_P)$. Finally, let $\mathscr{C}$ denote the class of quasi-local domains R such that $c(R) = s(R)$. ($\mathscr{C}$ is an important class, since it contains all local domains occurring in algebraic and analytic geometry and in number theory. (In fact, all these rings satisfy the more stringent condition $c(R) = \{\text{altitude } R\}$.) Also, by [5, (4.1)] it contains all Henselian local domains and all local domains of the form $R[X]_{(M,X)}$, where (R, M) is an arbitrary local domain and X is an indeterminate. On the other hand, [2, Example 2, pp. 203–205] in the case $m = 0$ shows that not all local domains are in $\mathscr{C}$—but it has been conjectured, the Upper Conjecture (3.4), that this is essentially the only type of local domain not in $\mathscr{C}$.)

Our first theorem, (2.2), shows that if A is any integral domain and $P \in \operatorname{Spec} A$ is such that $c(P') = c(P)$, for all $P' \in \operatorname{Spec} A'$ that lie over P, then $c(Q) = c(P)$ whenever $Q \in \operatorname{Spec} B$ lies over P and B is an integral extension domain of A. We then show in (2.4) that every semi-local domain R has finite integral extension domains B such that all maximal ideals N in all integral extension domains C of B satisfy $C_N \in \mathscr{C}$ and $c(N) = c(M')$, for some maximal ideal M' in R'. Finally, in Section 3 we consider a new conjecture related to the results in Section 2, and show that it lies (implicationwise) between two previously studied chain conjectures.

I am indebted to the referee for a number of suggestions on improving this paper, and also for a considerable strengthening of my original versions of (2.1) and (2.2).

2. Two theorems on $c(P)$

In this section we prove two theorems concerning the behavior of $c(P)$, where P is a prime ideal in an integral domain A. To prove the first of these,

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(2.2), we need the following lemma.

(2.1) LEMMA. *Let $A \subseteq B \subseteq C$ be integral domains with C the integral closure of A in an algebraic closure of its quotient field. Then the following statements hold:*

(2.1.1) *For $P \in \operatorname{Spec} A$,*

$$\begin{aligned} c(P) &= \cup\{c(Q);\, Q \in \operatorname{Spec} B \quad \textit{and} \quad Q \cap A = P\} \\ &= \cup\{s(N);\, N \in \operatorname{Spec} C \quad \textit{and} \quad N \cap A = P\}. \end{aligned}$$

(2.1.2) *If B is the integral closure of A in a normal extension field of the quotient field of A, and if Q_1, Q_2 are prime ideals in B such that $Q_1 \cap A' = Q_2 \cap A'$, then $s(Q_1) = s(Q_2)$ and $c(Q_1) = c(Q_2)$. Therefore $c(Q) = c(Q \cap A')$, for each $Q \in \operatorname{Spec} B$, and $s(N) = c(N \cap A')$, for each $N \in \operatorname{Spec} C$.*

Proof. (2.1.1) follows readily from the Going Up Theorem, and the first statement in (2.1.2) follows from the transitivity of the group of A'-automorphisms of B on the set of prime ideals in B lying over a given prime ideal in A'. The second statement in (2.1.2) follows easily from the first statement and (2.1.1). Q.E.D.

The following theorem is the first of our main results.

(2.2) THEOREM. *Let P be a prime ideal in an integral domain A such that $c(P') = c(P)$, for all $P' \in \operatorname{Spec} A'$ that lie over P. Then for each integral extension domain B of A and for each prime ideal Q in B such that $Q \cap A = P$, $c(Q) = c(P)$. Moreover, if there is only one such P' in A' and if $c(P) = s(P)$, then $c(Q) = s(Q)$, for all such Q* (cf. [5, (4.8.2)]).

Proof. By considering a normal extension field E of the quotient field of A that contains the quotient field of B, the first statement follows readily from the hypothesis and (2.1.1) and (2.1.2). The second statement also follows by considering E. The details are the same as in the proof of [5, (4.8.2)], Q.E.D.

We next give an example that is closely related to (2.2) and that gives a negative answer to the following question asked in [5, (4.5)]: if R is a local domain and S is a finite integral extension domain of R such that $c(R) = s(S)$, then is $S_N \in \mathscr{C}$, for all maximal ideals N in S? (In (2.4) it will be shown that if S is chosen a little more carefully, then the answer is yes.)

(2.3) *Example.* There exist a local domain R and a finite integral extension domain S of R such that $c(R) = s(S)$ and $S_N \notin \mathscr{C}$, for some maximal ideal N in S.

Proof. Let R_0 be as in [2, Example 2, pp. 203–205] in the case $m = 0$, so R_0 is a local domain such that R_0' is a finite R_0-algebra and is a regular domain with two maximal ideals (whose heights are one and $r+1>1$). Therefore $c(R_0') = \{1, r+1\}$, and $c(R_0) = c(R_0')$, by (2.1.1). Let A be a finite integral extension domain of R_0' such that there exist two maximal ideals in A that lie over the height one maximal ideal in R_0', and let $R = R_0 + J$ with J the Jacobson radical of A. Then R is a local domain (since A is a finite R_0-algebra and $R_0 \subset R \subset A$), $c(R) = c(R_0) = \{1, r+1\}$, and there are at least two height one maximal ideals in $R' = A'$. Let b in one of the height one maximal ideals in R' such that $1-b$ is in all other maximal ideals in R' and let $S = R[b]$. Then clearly $c(R) = s(S)$ and there exists a maximal ideal N in S such that $r+1 = \text{altitude } S_N > 1$ and S_N' has a height one maximal ideal, so $1 \in c(N)$ and $1 \notin s(N)$, hence $S_N \notin \mathscr{C}$, Q.E.D.

The other questions asked in [5, Section 4] will be briefly considered following (3.7).

We now come to the second of our main results. In (2.4) we show, in particular, that every local domain has certain finite integral extension domains that behave nicely with respect to the function c on their prime spectrum.

(2.4) THEOREM. *If A is an integral extension domain of a semi-local domain R, then there exists a finite integral extension domain B of A such that, for each integral extension domain C of B and for each maximal ideal N in C, $C_N \in \mathscr{C}$ and $c(N) = c(M')$, for some maximal ideal M' in R'.*

Proof. Assume first that the result is known for the case $A = R$ and let B_0 be such a finite integral extension domain of R. Then for the general case when A is an arbitrary integral extension domain of R let $B = A[B_0]$. Then B is a finite integral extension domain of A, and each integral extension domain C of B is an integral extension domain of B_0. Therefore the conclusion readily follows from the properties of B_0. Thus it suffices to prove the theorem for the case $A = R$ in a semi-local domain.

For this case, there exists a finite integral extension domain $S \subseteq R'$ of R such that S and R' have the same number of maximal ideals. Then it clearly suffices to prove the theorem for S, so we assume to begin with that R and R' have the same number of maximal ideals. Now [1, (1.10)] says that for each maximal ideal M_i in R and for each $n \in c(M_i)$ there exists a finite integral extension domain D_i of R_{M_i} such that $n \in s(D_i)$. Then, since D_i is a quotient ring (with respect to $R - M_i$) of a finite integral extension domain of R, and since each $c(M_i)$ is finite (and since R is semi-local), it follows from the Going Up Theorem that there exists a finite integral extension domain D of R such that for each M_i and for each $n \in c(M_i)$ there exists a maximal ideal P in D such that $P \cap R = M_i$ and $n \in s(P)$. Then it is seen (by adjoining all conjugates of the elements in D) that there exists a finite normal extension field E of the quotient field of R such that this continues to hold for the integral closure R'' of R in E. Fix a maximal ideal Q in R'', so $c(Q) = c(Q \cap R')$, by (2.1.2). Also, $c(Q \cap R') = c(Q \cap R)$, by (2.1.1) (and since R and R' have the same number of maximal ideals). Therefore if

$n \in c(Q)$, then $n \in c(Q \cap R)$, so $n \in s(P)$, for some maximal ideal P in R'' such that $P \cap R = Q \cap R$. Then $P \cap R' = Q \cap R'$, so $n \in s(Q)$, by (2.1.2), hence it follows that $s(Q) = c(Q) = c(Q \cap R')$.

Since only finitely many prime ideals in R'' lie over a given prime ideal in R, there exists a finite integral extension domain $B \subseteq R''$ of R such that B and R'' have the same number of maximal ideals. Then, for all maximal ideals Q in R'', $s(Q) = s(Q \cap B)$, by [4, (3.1)], and $c(Q) = c(Q \cap B)$, by (2.1.1). Therefore, since $s(Q) = c(Q) = c(Q \cap R')$, it follows that, for each maximal ideal M in B, $c(M) = s(M) = c(M')$, for some maximal ideal M' in R'. Finally, if C is an integral extension domain of B and N is a maximal ideal in C, then by (2.2) we have $c(N) = s(N) = c(M')$, for some maximal ideal M' in R', Q.E.D.

The proof of (2.4) shows that (2.4) could be extended to arbitrary integral domains R such that $c(R)$ is finite and R' has only finitely many maximal ideals, if the referenced results in [1] and [4] could be appropriately generalized.

3. Some questions and chain conjectures

In this section, some questions closely related to the results in Section 2 are asked, and then it is shown that the questions are related to some previously stated conjectures concerning saturated chains of prime ideals.

In [4, (3.14)] it was shown that if S is a quasilocal integral extension domain of a local domain R, then $S \in \mathscr{C}$ if and only if $R \in \mathscr{C}$. Therefore it follows that (3.1) holds when R' is quasi-local, and it seems quite possible that the following conjecture is true. If this can be shown, then a problem open since 1956, the Normal Chain Conjecture (3.5), will also be settled—see (3.6).

(3.1) Conjecture. Let R be a local domain such that $c(M') = c(R)$, for all maximal ideals M' in R'. Then $R \in \mathscr{C}$ if and only if each $R'_{M'} \in \mathscr{C}$.

It is somewhat surprising (and will be shown below) that if "$\Leftarrow$" holds, then so does "$\Rightarrow$". However, "$\Leftarrow$" will probably be difficult to either prove or disprove, since it implies the Normal Chain Conjecture (3.5) and it is implied by the Upper Conjecture (3.4) (and these latter two conjectures have been open for some time). To show this, we will first restate (3.1) as two questions and then briefly recall the two named conjectures.

(3.2) Question. If R is a local domain such that, for each maximal ideal M' in R', $R'_{M'} \in \mathscr{C}$ and $c(M') = c(R)$, then is $R \in \mathscr{C}$?

(3.3) Question. If R is a local domain such that $R \in \mathscr{C}$ and $c(M') = c(R)$, for all maximal ideals M' in R', then is each $R'_{M'} \in \mathscr{C}$?

To avoid having to recall the definitions of a number of chain conditions, we state the next two conjectures and the Catenary Chain Conjecture ((*) in

(3.7.2)) in a different form than usual. The reader will find a history of these conjectures and proofs that the given versions of them are equivalent to the usual versions in Chapters 9, 12, and 11, respectively, of [7].

(3.4) UPPER CONJECTURE. If R is a local domain, then either $R \in \mathscr{C}$ or altitude $R > 1$ and there exists a height one maximal ideal in R'.

(3.5) NORMAL CHAIN CONJECTURE. If R is a local domain such that $s(R) = \{\text{altitude } R\}$ and R' is quasi-local, then $c(R) = s(R)$.

We now show the relationship between these questions and conjectures.

(3.6) *Remark.* (3.4) $\Rightarrow$ (3.2) $\Rightarrow$ (3.3) and (3.5).

Proof. Assume (3.4) holds and let R be as in (3.2). Then the conditions on R' show that all maximal ideals in R' have the same height, so $R \in \mathscr{C}$, by (3.4), hence (3.4) $\Rightarrow$ (3.2).

Assume (3.2) holds and let R be as in (3.3). Let $S \subseteq R'$ be a finite integral extension domain of R such that R' and S have the same number of maximal ideals, let M' be a maximal ideal in R', and let $M = M' \cap S$. Then $R'_{M'} \in \mathscr{C}$ if and only if $S_M \in \mathscr{C}$, by [4, (3.14)], so it suffices to prove that $S_M \in \mathscr{C}$, for all maximal ideals M in S. By (2.4), let B be a finite integral extension domain of R such that, for each maximal ideal N in each integral extension domain C of B, $C_N \in \mathscr{C}$ and $c(N) = c(R)$ (since all $c(M')$ are equal to $c(R)$). Then $A = B[S]$ is a finite integral extension domain of B and $B \subseteq A \subseteq B'$. Therefore, by the properties of B, $A'_P \in \mathscr{C}$ and $c(P) = c(R)$, for all maximal ideals P in A'. Thus, for each maximal ideal M in S, $D = A_{S-M}$ is a finite integral extension domain of S_M and, for all maximal ideals Q in D', $D'_Q \in \mathscr{C}$ and $c(Q) = c(R)$. Let J be the Jacobson radical of D and let $L = S_M + J$. Then L is a finite S_M-algebra (since $S_M \subseteq L \subseteq D$) and J is the only maximal ideal in L, so L is a local domain. Also, L and D have the same quotient field, so $L' = D'$. Therefore, since all $c(Q)$ are equal, they are all equal to $c(L)$, by (2.1.1). Thus $L \in \mathscr{C}$, by (3.2), hence $S_M \in \mathscr{C}$, by [4, (3.14)], and so (3.2) $\Rightarrow$ (3.3).

Finally, assume (3.2) holds, let R be a local domain such that $s(R) = \{\text{altitude } R\}$ and R' is quasi-local, and let B be a finite integral extension domain of R as in (2.4). Then $B'_P \in \mathscr{C}$ and $c(P) = c(R)$, for all maximal ideals P in B'. Let K be the Jacobson radical of B and let $L_1 = R + K$, so L_1 is a local domain such that $L'_1 = B'$ (as in the preceding paragraph). Therefore $L_1 \in \mathscr{C}$, by (3.2) (and (2.1.1)), so $R \in \mathscr{C}$, by [4, (3.14)]. Hence $s(R) = c(R)$, so (3.2) $\Rightarrow$ (3.5), Q.E.D.

(3.7) *Remark.* As noted in the first paragraph of the proof of (3.6), the condition "$c(M') = c(R)$, for all maximal ideals M' in R'" implies "all maximal ideals in R' have the same height". If this latter conditions is substituted for the former in (3.1), then:

(3.7.1) The Upper Conjecture implies "$\Leftarrow$" holds.

(3.7.2) "$\Leftarrow$" implies "$\Rightarrow$" and (*) hold, where (*) is "if R is a local domain such that $s(R) = \{\text{altitude } R\}$ and all maximal ideals in R' have the same height, then $c(R) = s(R)$".

(The statement (*) in (3.7.2) is an equivalence of the Catenary Chain conjecture, and this conjecture implies the Normal Chain Conjecture—see [7, (11.1.1) $\Leftrightarrow$ (11.1.4) and (3.3.8) $\Rightarrow$ (3.3.9)].)

The proof that (3.7.1) and (3.7.2) hold is similar to the proof of (3.6), so it will not be given here.

In [5, Section 4] four questions concerned with $s(R)$, $c(R)$, and $\mathscr{C}$ were asked. One of these has been answered in (2.3), and this paper will be closed by briefly considering the other three.

[5, (4.9)] asked if maximal chains of prime ideals in local integral extension domains of a local domain R must contract in R to maximal chains of prime ideals. The answer was shown to be no in [6, (2.10)].

[5, (4.6)] asked a more general question than (3.3), and the answer is still unknown.

The remaining question in [5] is also somewhat related to (3.3), since it is concerned with $L \in \mathscr{C}$, where $R \in \mathscr{C}$ and L is a locality over R that is contained in F, the quotient field of R. Specifically, in [5, (4.7)] the following question is asked: if (R, J) is a local domain such that $R \in \mathscr{C}$, if L is a locality over R of the form A_P, where

$$A = R[X_1, \ldots, X_k, b_1, \ldots, b_h]$$

($X_1, \ldots, X_k$ are indeterminates, $b_1, \ldots, b_h$ are in the quotient field of $R[X_1, \ldots, X_k]$, and $P \in \operatorname{Spec} A$ lies over J), and if R is a subspace of L, then is L in $\mathscr{C}$? In (3.8) we show that the answer is no, even when $k = 0$ (so $L \subseteq F$).

(3.8) *Example.* There exist a local domain (R, J) and b in the quotient field of R such that $R \in \mathscr{C}$, R is a subspace of $L = R[b]_P$ where $P \in \operatorname{Spec} R[b]$ and $P \cap R = J$, and $L \notin \mathscr{C}$.

Proof. Let (R, J) be as in [2, Example 2, pp. 203–205] in the case $m = r = 1$. Then R is a local domain and $(R'; M, N)$ is a finite R-algebra and is a regular domain such that height $M = 2$ and height $N = 3$, so $C(R') = \{2, 3\}$. Therefore $c(R) = \{2, 3\}$, by (2.1.1), and it is easily seen that $s(R) = \{2, 3\}$, so $R \in \mathscr{C}$. Let

$$A = R[(x^2 - x)/y] \quad \text{and} \quad B = R'[(x^2 - x)/y]$$

(with x and y as in [2]), so

$$\text{height } (x^2 - x, y)R = \text{height } (x^2 - x, y)R' = 2,$$

hence JA, MB, and NB are prime ideals of height two, one, and two, respectively, by [3, Lemmas 4.3 and 4.2]. Let $L = A_{JA}$. Then altitude $L = 2$ and there exists a height one maximal ideal in L', since B is integral over A

and $MB \cap A = JA$, so $L \notin \mathscr{C}$. Finally, R is a subspace of L, as will now be shown. Namely, the completion R^* of R has only two prime divisors of zero (since R is a subspace of R' and the completion R'^* of R' is a direct sum of two regular local rings), and $A^\# = R^*[(x^2 - x)/y]$ has the same total quotient ring as R^*. Also, both (minimal) prime divisors of zero in $A^\#$ are contained in $JA^\#$, by [3, Remark 4.4(i)], so R is a subspace of L, by [3, Lemma 4.5(i)], Q.E.D.

Examples similar to (3.8) can be given with $k \geq 0$, $m \geq 1$, and $r \geq 1$.

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