ILLINOIS JOURNAL OF MATHEMATICS
Volume 23, Number 3, September, 1979

SUBNORMAL OPERATORS AND HYPERINVARIANT SUBSPACES

BY

H. W. KIM AND CARL PEARCY

1. Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space, and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. An operator S in $\mathcal{L}(\mathcal{H})$ is said to be *subnormal* if there exists a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and a normal operator N in $\mathcal{L}(\mathcal{K})$ such that $N\mathcal{H} \subset \mathcal{H}$ and $N \mid \mathcal{H} = S$. (In this situation we say that N is a *normal extension* of S and that S is a *restriction* of N. Alternate characterizations of subnormal operators were given by Halmos [5] and Bram [2].) The operator N is called a *minimal* normal extension of S if the only reducing subspace for N containing $\mathcal{H}$ is $\mathcal{K}$ itself. It is well known that every subnormal operator has a minimal normal extension and that the minimal normal extension is unique up to unitary equivalence (cf. [5] or [7, p. 101]). Since subnormal operators are intimately related to their minimal normal extensions, and the spectral theorem guarantees the existence of a generous supply of invariant and hyperinvariant subspaces for (nonscalar) normal operators, the question whether every (nonscalar) subnormal operator in $\mathcal{L}(\mathcal{H})$ has a nontrivial invariant or hyperinvariant subspace has long been of interest, and remains open as of this writing. The purpose of this note is to make a modest contribution to this problem. We consider subnormal operators whose spectra have empty interior, and reduce the invariant subspace problem for this class of operators to a rather curious looking special case. This assumption on the spectrum is unpleasant, but a typical subnormal operator in this class has for spectrum a "Swiss cheese" with positive planar Lebesgue measure, and it is generally conceded that this class of subnormal operators is the most intractable with respect to the existence of invariant subspaces.

2. In what follows, the spectrum of an operator T will be denoted by $\sigma(T)$ and the essential spectrum (i.e., Calkin spectrum) of T by $\sigma_e(T)$. Let S be a nonscalar subnormal operator in $\mathcal{L}(\mathcal{H})$, and let N be a minimal normal extension of S acting on a Hilbert space $\mathcal{K} \supset \mathcal{H}$. If $\mathcal{K} = \mathcal{H}$, then S is normal and thus has nontrivial hyperinvariant subspaces, so we may assume that $\mathcal{K} \neq \mathcal{H}$ (which implies that S is not normal). We write $\mathcal{K} = \mathcal{H} \oplus (\mathcal{K} \ominus \mathcal{H})$ and note that it follows easily from the minimality of N that $\mathcal{K} \ominus \mathcal{H}$ has dimension $\aleph_0$ (i.e., $\mathcal{K} \ominus \mathcal{H}$ is neither finite dimensional nor nonseparable). We summarize these remarks as follows.

PROPOSITION 2.1. *Let S be a nonnormal subnormal operator in $\mathcal{L}(\mathcal{H})$. Then its minimal normal extension N may be taken to act on $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}$, and*

Received December 20, 1977.

associated with this decomposition of $\mathcal{H}$, *there is corresponding decomposition of* N *as*

$$(1) \qquad N = \begin{pmatrix} S & C \\ 0 & B \end{pmatrix},$$

where C *and* B *belong to* $\mathcal{L}(\mathcal{H})$.

Henceforth in this section the operators S *and* N *under discussion will always be those of Proposition* 2.1 *related as in* (1). We now begin to study the relations between $\sigma_e(S)$, $\sigma(S)$, $\sigma_e(N)$, $\sigma(N)$, $\sigma_e(B)$, and $\sigma(B)$. Recall first that it is known from [6] that $\sigma(N) \subset \sigma(S)$.

PROPOSITION 2.2. *If the subnormal operator* S *has no nontrivial hyperinvariant subspace, then* $\sigma_e(S) = \sigma(S) = \sigma_e(N) = \sigma(N)$.

Proof. It is an easy consequence of the Fredholm theory that if T is any operator such that $\sigma(T) \backslash \sigma_e(T) \neq \emptyset$, then either T or T^* has point spectrum, and thus T has a nontrivial hyperinvariant subspace. Thus, by hypothesis, we conclude that $\sigma_e(S) = \sigma(S)$. Furthermore, if $\sigma(N) \backslash \sigma_e(N) \neq \emptyset$, then N has point spectrum—say λ is an eigenvalue for N. If (f_1, f_2) in $\mathcal{H} \oplus \mathcal{H}$ is any eigenvector for N corresponding to λ, then, since N is normal, (f_1, f_2) is also an eigenvector for N^* corresponding to the eigenvalue $\bar{\lambda}$, and according to (1) we have $S^* f_1 = \bar{\lambda} f_1$. Since S^* cannot have point spectrum, $f_1 = 0$, which implies that the eigenspace $\mathcal{E}_\lambda$ for N corresponding to λ is a subspace of $0 \oplus \mathcal{H}$. Since $\mathcal{E}_\lambda$ and $\mathcal{H} \ominus \mathcal{E}_\lambda$ are reducing for N, this contradicts the minimality of N, and hence $\sigma_e(N) = \sigma(N)$. Finally, if $\sigma(S) \neq \sigma(N)$ and $\lambda \in \sigma(S) \backslash \sigma(N)$, then $N - \lambda$ is invertible, and since $S - \lambda = (N - \lambda) \mid (\mathcal{H} \oplus 0)$, $S - \lambda$ must be bounded below and have closed range. Since $\lambda \in \sigma(S)$, range $(S - \lambda) \neq \mathcal{H}$, and hence kernel $(S - \lambda)^* \neq (0)$. Thus, once again, S^* has point spectrum, which is contrary to the hypothesis that S has no nontrivial hyperinvariant subspace, and the result follows. ■

PROPOSITION 2.3. *If the subnormal operator* S *has no nontrivial hyperinvariant subspace, then the operator* B *in* (1) *satisfies* $\sigma(B) = \sigma(N) = \sigma(S)$.

Proof. We know from Proposition 2.2 that $\sigma(N) = \sigma(S)$. If $\lambda \notin \sigma(N)$, then the 2×2 matrix $N - \lambda$ is invertible, and the (1,1) entry $S - \lambda$ is also invertible. In this situation it is always true (and easy to see) that the (2, 2) entry $B - \lambda$ must also be invertible, so $\sigma(B) \subset \sigma(N)$. Suppose next that $\lambda \in \sigma(N) \backslash \sigma(B)$. Then $B - \lambda$ is invertible and bounded below—say by γ. Let δ be the open disc in $\mathbf{C}$ with center λ and radius $\gamma/2$, and let $E(\delta) \neq 0$ be the spectral projection for N corresponding to δ (i.e., let $E(\delta)$ be the value of the spectral measure $E(\cdot)$ of N at δ). Let (f_1, f_2) be any unit vector in $\mathcal{H} \oplus \mathcal{H}$ belonging to the range of $E(\delta)$. Then, by the spectral theorem and (1), for every positive integer n we have

$$\gamma^n \|f_2\| \leq \|(B - \lambda)^n f_2\| \leq \|(N - \lambda)^n (f_1, f_2)\| \leq (\gamma/2)^n,$$

which implies that $f_2 = 0$. This says that the range of $E(\delta)$ is contained in $\mathcal{H} \oplus 0$, and thus that N has a nonzero reducing subspace contained in $\mathcal{H} \oplus 0$. It follows easily from (1) that the range of $E(\delta)$ is also a reducing subspace for S and that S restricted to this subspace is normal. But this implies that S has a nontrivial hyperinvariant subspace (cf. [4, Theorem 1.4]), contrary to hypothesis. Thus $\sigma(B) = \sigma(N) = \sigma(S)$. ■

Recall now that a vector x in $\mathcal{H}$ is said to be a *rational cyclic vector* for an operator T in $\mathcal{L}(\mathcal{H})$ if the linear manifold consisting of all vectors of the form $r(T)x$ where r is a rational function with poles off $\sigma(T)$ is dense in $\mathcal{H}$. Recall also that an operator T in $\mathcal{L}(\mathcal{H})$ is *essentially normal* if T has a compact self-commutator, or, equivalently, if the image $\pi(T)$ of T in the Calkin algebra is normal.

PROPOSITION 2.4. *If the subnormal operator S has a rational cyclic vector and has no nontrivial hyperinvariant subspace, then S can be written as $S = N_1 + K_1$ where N_1 is normal and K_1 is compact. Furthermore, in this case the operator C in* (1) *belongs to the Hilbert–Schmidt class, and the operator B in* (1) *is essentially normal.*

Proof. According to [1], every hyponormal operator in $\mathcal{L}(\mathcal{H})$ with a rational cyclic vector has a trace-class self-commutator. Since subnormal operators are hyponormal and $S^*S - SS^* = CC^*$ can be deduced from (1) and the normality of N, it follows from the hypothesis that CC^* belongs to the trace-class, and hence that C belongs to the Hilbert-Schmidt class. In particular, S is an essentially normal operator, and since $\sigma_e(S) = \sigma(S)$ by Proposition 2.2, it follows from [3, Corollary 11.2] that S has the form $S = N_1 + K_1$ where N_1 is normal and K_1 is compact. Finally, to see that B is essentially normal, one uses the normality of N and (1) to obtain $BB^* - B^*B = C^*C$, and that C^*C is compact has already been observed. ■

PROPOSITION 2.5. *If the subnormal operator S has no nontrivial hyperinvariant subspace and has the further property that $\sigma(S)$ has empty interior, then*

$$\sigma(N) = \sigma_e(N) = \sigma(S) = \sigma_e(S) = \sigma(B) = \sigma_e(B).$$

Proof. The first four equalities follow from Propositions 2.2 and 2.3, so its suffices to prove that $\sigma(B) \backslash \sigma_e(B) = \emptyset$. Suppose, on the contrary, that $\sigma(B) \backslash \sigma_e(B) \neq \emptyset$. Then one knows (cf. [8, §1]) that the difference $\sigma(B) \backslash \sigma_e(B)$ consists of the union of various holes in $\sigma_e(B)$ together with some isolated eigenvalues of B. Since a hole in $\sigma_e(B)$ is a nonempty open set and $\sigma(B)$ $(=\sigma(S))$ by hypothesis contains no nonempty open set, it follows that $\sigma(B) \backslash \sigma_e(B)$ must consist only of isolated eigenvalues of B. If λ is such a point, then $\bar{\lambda}$ is an eigenvalue of B^* (since $B - \lambda$ is a Fredholm operator of index zero). Since $0 \oplus \mathcal{H}$ is an invariant subspace for $N^* - \bar{\lambda}$ and $(N^* - \bar{\lambda}) \mid (0 \oplus \mathcal{H}) = B^* - \bar{\lambda}$, it follows that $\bar{\lambda}$ belongs to the point spectrum of N^*. Arguing just as in the proof of Proposition 2.2, we see that this leads to a contradiction of the minimality of N, and it follows that $\sigma_e(B)=\sigma(B)$. ■

3. We are now prepared to establish our main structure theorem.

THEOREM 3.1. *If S is a nonnormal subnormal operator in $\mathcal{L}(\mathcal{H})$ such that*

(a) *$\sigma(S)$ has empty interior,*
(b) *S has a rational cyclic vector, and*
(c) *S has no nontrivial hyperinvariant subspaces,*

then S can be written as $S=N_1+K_1$ where N_1 is normal and K_1 is compact, and S has a minimal normal extension $\tilde{N}$ acting on $\mathcal{H}\oplus\mathcal{H}$ of the form

$$\tilde{N}=\begin{pmatrix} N_1+K_1 & C_1 \\ 0 & N_1+K_2 \end{pmatrix} \tag{2}$$

where K_2 is compact and C_1 is a Hilbert-Schmidt operator, and where

$$\sigma(N)=\sigma_e(N)=\sigma(S)=\sigma_e(S)=\sigma(N_1+K_2)=\sigma_e(N_1+K_2).$$

Proof. According to Propositions 2.1–2.5, S can be written as $S=N_1+K_1$ where N_1 is normal and K_1 is compact, and S has a minimal normal extension N acting on $\mathcal{H}\oplus\mathcal{H}$ of the form

$$N=\begin{pmatrix} N_1+K_1 & C \\ 0 & B \end{pmatrix},$$

where B is essentially normal, C belongs to the Hilbert–Schmidt class, and $\sigma(N)=\sigma_e(N)=\sigma(S)=\sigma_e(S)=\sigma(B)=\sigma_e(B)$. Since $S=N_1+K_1$ and B are both essentially normal and they have the same spectral picture (cf. [8, §1]), it is a consequence of [3] that there exist a unitary operator U and a compact operator K_3 in $\mathcal{L}(H)$ such that $U(N_1+K_1)U^*+K_3=B$. We now define

$$\tilde{N}=\begin{pmatrix} 1 & 0 \\ 0 & U^* \end{pmatrix}\begin{pmatrix} S & C \\ 0 & B \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix}=\begin{pmatrix} N_1+K_1 & C' \\ 0 & N_1+K_2 \end{pmatrix},$$

where $C'=CU$ and $K_2=K_1+U^*K_3U$. It is clear that K_2 is compact and that C' belongs to the Hilbert–Schmidt class. Furthermore $\tilde{N}$ is unitarily equivalent to N and is by inspection a normal extension of S. Since N is a minimal normal extension of S, the same must be true of $\tilde{N}$, and the proof is complete. ■

COROLLARY 3.2. *If there exists a subnormal operator in $\mathcal{L}(\mathcal{H})$ whose spectrum has empty interior and which has no nontrivial invariant subspaces, then such an operator must be of the form of $S=N_1+K_1$ in Theorem 3.1, and moreover one may assume that the minimal normal extension of S has the form (2). Furthermore there exists a compact operator K such that $(S+K)^*$ is also subnormal.*

Proof. In view of Theorem 3.1, it suffices to prove the last statement. Since $(\tilde{N})^*$ is normal with $\tilde{N}$ and

$$(\tilde{N})^* \mid (0 \oplus \mathcal{H}) = N_1^* + K_2^*,$$

it follows that $N_1^* + K_2^*$ is subnormal, and $N_1^* + K_2^* = (S+K)^*$ where K is the compact operator $K_2 - K_1$. ■

The structure of the operator $\tilde{N}$ in (2) leads to some interesting questions. Since $\tilde{N}$ is a compact perturbation of the normal operator $N_1 \oplus N_1$, is there any nice relation between the spectral measures of these two normal operators? Can the assumption that $\sigma(S)$ has no interior be removed from Theorem 3.1 without altering the conclusion? What is a concrete model for the operator $\tilde{N}$? Finally and most importantly, can Theorem 3.1 be used to solve the invariant subspace problem for subnormal operators S such that $\sigma(S)$ has empty interior?

Added in proof. Recently, Scott Brown; in the brilliant paper *Some invariant subspaces for subnormal operators*, Integral Equations and Operator Theory, vol. 1 (1978), pp. 310–333, showed that every subnormal operator in $\mathcal{L}(\mathcal{H})$ has a nontrivial *invariant* subspace. Thus far no one has been able to use his results and techniques to solve the hyperinvariant subspace problem for subnormal operators.

REFERENCES

1. C. BERGER and B. J. SHAW, *Self-commutators of multicyclic hyponormal operators are always trace class*, Bull. Amer. Math. Soc., vol. 79 (1973), pp. 1193–1199.
2. J. BRAM, *Subnormal operators*, Duke Math. J., vol. 22 (1955), pp. 75–94.
3. L. BROWN, R. G. DOUGLAS and P. FILLMORE, *Unitary equivalence modulo the compact operators and extensions of C^*-algebras*, Proc. Conf. Operator Theory, Springer Lecture Notes in Mathematics #345, 1973, pp. 58–128.
4. R. G. DOUGLAS and C. PEARCY, *Hyperinvariant subspaces and transitive algebras*, Michigan Math. J., vol. 19 (1972), pp. 1–12.
5. P. R. HALMOS, *Normal dilations and extensions of operators*, Summa Brasil. Math., vol. 2 (1950), pp. 125–134.
6. ———, *Spectra and spectral manifolds*, Ann. Soc. Pol. Math., vol. 25 (1952), pp. 43–49.
7. ———, *A Hilbert space problem book*, Van Nostrand, Princeton, 1967.
8. C. PEARCY, *Some recent developments in operator theory*, CBMS Regional Conference Series Lecture Notes, Amer. Math. Soc., 1978.

BUCKNELL UNIVERSITY
LEWISBURG, PENNSYLVANIA
UNIVERSITY OF MICHIGAN
ANN ARBOR, MICHIGAN