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SOME REMARKS ON LAX-PRESHEAFS

BY

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Abstract. $[\mathscr{A}^{\mathrm{op}}, \mathscr{C}\mathrm{at}]$ is a reflective and coreflective sub-2-category of Fun $(\mathscr{A}^{\mathrm{op}}\mathscr{C}\mathrm{at})$. Lax ends and pointwise lax extensions can be expressed by indexed limits using the above coreflector.

1. We use the symbol $\oint$ for generalized lax ends [3].

For 2-categories $\mathscr{A}$ and $\mathscr{B}$, Fun $(\mathscr{A}, \mathscr{B})$ denotes the 2-category of lax functors, lax natural transformations and modifications from $\mathscr{A}$ to $\mathscr{B}$. We then have the standard formula Fun $(\mathscr{A}, \mathscr{B})(F, G) = \oint_A \mathscr{B}(FA, GA)$.

PROPOSITION. *The canonical embedding* $[\mathscr{A}^{\mathrm{op}}, \mathscr{C}\mathrm{at}] \to \mathrm{Fun}\ (\mathscr{A}^{\mathrm{op}}, \mathscr{C}\mathrm{at})$ *has both left and right adjoints.*

Proof. Given a lax-presheaf $F\colon \mathscr{A}^{\mathrm{op}} \to \mathscr{C}\mathrm{at}$, the 2-presheafs

$$\check{F}C = \oint_A \mathscr{A}(C, A) \times FA, \tag{1}$$

$$\hat{F}C = \oint_A [\mathscr{A}(A, C), FA] \tag{2}$$

are such that

$$[\mathscr{A}^{\mathrm{op}}, \mathscr{C}\mathrm{at}](H, \hat{F}) \simeq \mathrm{Fun}\ (\mathscr{A}^{\mathrm{op}}, \mathscr{C}\mathrm{at})(H, F),$$
$$[\mathscr{A}^{\mathrm{op}}, \mathscr{C}\mathrm{at}](\check{F}, H) \simeq \mathrm{Fun}\ (\mathscr{A}^{\mathrm{op}}, \mathscr{C}\mathrm{at})(F, H)$$

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respectively. Let us prove the first isomorphism:

$$
\begin{aligned}
[\mathscr{A}^{\mathrm{op}}, \mathscr{C}\mathrm{at}](H, \hat{F}) &= \int_C [HC, \hat{F}C] \\
&= \int_C \left[HC, \oint_A [\mathscr{A}(A, C), FA]\right] \\
&\simeq \int_C \oint_A \left[HC, [\mathscr{A}(A, C), FA]\right] \\
&\simeq \oint_A \int_C [HC \times \mathscr{A}(A, C), FA] \\
&\simeq \oint_A \left[\int^C HC \times \mathscr{A}(A, C), FA\right] \\
&\simeq \oint_A [HA, FA] \\
&= \mathrm{Fun}\,(\mathscr{A}^{\mathrm{op}}, \mathscr{C}\mathrm{at})(H, F).
\end{aligned}
$$

Comments (1) For $\mathscr{A}$ a category, Giraud [4, pp. 37–41] shows that the inclusion of $[\mathscr{A}^{\mathrm{op}}, \mathscr{C}\mathrm{at}]$ into the pseudo-functors and pseudo-natural transformations has both adjoints.

Formulas (1) and (2) cover this case if, instead of generalized lax ends, we take iso-lax ends (all 2-cells are invertibles).

(2) The result of this section is equivalent to a statement in [6 pp. 31–32]. However the proof here is quite different.

(3) In [8], Street establishes the existence of $\hat{F}$, for a lax functor F. Also, Street's 2-functor L_A[9, p. 171] is just $\hat{1}$, where 1 denotes the constant at 1 presheaf.

2. At the level of 2-categories there are three equivalent notions of limit: (i) lax limits, (ii) lax ends, (iii) indexed limits. In fact, by [9, Theorem 14], the existence of (i) implies that of (iii) and the converse comes from [9, Theorem 11].

On the other hand if a 2-category has lax limits then it has lax ends [7 pp. 52–53] and conversely [2, Remark a].

Now, there is a short path to pass from (iii) to (ii). Precisely, if $T: \mathscr{A}^{\mathrm{op}} \times \mathscr{A} \to \mathscr{B}$ is a 2-functor we have

$$\lim\,(\mathscr{A}(\check{-}, -), T) = \oint_A T(A, A),$$

either side existing if the other does. In fact

$$
\begin{aligned}
\mathscr{B}(B, \lim(\mathscr{A}(\check{-}, -), T)) &= \int_{C,D} [\mathscr{A}(\check{C}, D), \mathscr{B}(B, T(C, D))] \\
&\simeq \int_{C,D} \left[\oint^{A} \mathscr{A}(C, A) \times \mathscr{A}(A, D), \mathscr{B}(B, T(C, D)) \right] \\
&\simeq \oint_{A} \int_{C,D} [\mathscr{A}(C, A) \times \mathscr{A}(A, D), \mathscr{B}(B, T(C, D))] \\
&\cong \oint_{A} \int_{C} \int_{D} [\mathscr{A}(C, A), [\mathscr{A}(A, D), \mathscr{B}(B, T(C, D))]] \\
&\simeq \oint_{A} \int_{C} \left[\mathscr{A}(C, A), \int_{D} [\mathscr{A}(A, D), \mathscr{B}(B, T(C, D))] \right] \\
&\cong \oint_{A} \int_{C} [\mathscr{A}(C, A), \mathscr{B}(B, T(C, A))] \\
&\simeq \oint_{A} \mathscr{B}(B, T(A, A)).
\end{aligned}
$$

The last member may be identified with the category of lax wedges of vertex B over T.

3. A lax natural transformation

$$
\text{(I)} \qquad
\begin{tikzcd}
\mathscr{A} \arrow[rr, "K"] \arrow[dr, "G"'] & \arrow[d, phantom, "\varepsilon"] & \mathscr{C} \arrow[dl, "R"] \\
& \mathscr{B} &
\end{tikzcd}
$$

is said to exhibit R as a *right lax extension* of G along K when pasting ε at R determines an isomorphism of categories $[\mathscr{C}, \mathscr{B}](S, R) \simeq \mathrm{Fun}(\mathscr{A}, \mathscr{B})(SK, G)$.

We say that (I) is *pointwise* if it is respected by the 2-representables

$$\mathscr{B}(B, -): \mathscr{B} \to \mathscr{C}\mathrm{at}$$

for each $B \in \mathscr{B}$.

We have the following limit-formulas for R.

(i) For each $C \in \mathscr{C}$, let d_C be the canonical projection from the comma 2-category $[\ulcorner C \urcorner, K]$ to $\mathscr{A}$.

If $l \varprojlim G \cdot d_C$ exists, then R exists and we have $RC = l \varprojlim G \cdot d_C$ [5], [6]

(ii) Suppose now that for each $C \in \mathscr{C}$ the $\oint_A [\mathscr{C}(C, KA), GA]$ exists; then R exists and $RC = \oint_A [\mathscr{C}(C, KA), KA), GA]$ [2, Theorem 7].

(iii) As far as the indexed limit version we have the following result.

PROPOSITION. *The pointwise right lax extension of G along K exists if and only if, for each $C \in \mathscr{C}$, the indexed limit* $\lim(\check{\mathscr{C}}(C, K), G)$ *exists. In this case*

$$(*) \qquad RC = \lim(\check{\mathscr{C}}(C, K), G).$$

Proof. We first prove that R defined by $(*)$ is a right lax extension:

$$\begin{aligned}
[\mathscr{C}, \mathscr{B}](S, R) &= \int_C \mathscr{B}(SC, RC) \\
&= \int_C \mathscr{B}(SC, \lim(\check{\mathscr{C}}(C, K), G)) \\
&= \int_{C,A} [\check{\mathscr{C}}(C, KA), \mathscr{B}(SC, GA)] \\
&= \int_{C,A} \oint_{\bar{A}} [\mathscr{A}(\bar{A}, A) \times \mathscr{C}(C, K\bar{A}), \mathscr{B}(SC, GA)] \\
&= \oint_{\bar{A}} \int_A \left[\mathscr{A}(\bar{A}, A), \int_C (\mathscr{C}(C, K\bar{A}), \mathscr{B}(SC, GA))\right] \\
&= \oint_{\bar{A}} \int_A [\mathscr{A}(\bar{A}, A), \mathscr{B}(SK\bar{A}, GA)] \\
&= \oint_{\bar{A}} \mathscr{B}(SK\bar{A}, G\bar{A}) \\
&= \operatorname{Fun}(\mathscr{A}, \mathscr{B})(SK, G).
\end{aligned}$$

The preservation property is evident.

Conversely, if (I) is pointwise then for each $B \in \mathscr{B}$ and each $Q: \mathscr{C} \to \mathscr{C}\mathrm{at}$ we have

$$[\mathscr{C}, \mathscr{C}\mathrm{at}](Q, \mathscr{B}(B, R)) \simeq \operatorname{Fun}(\mathscr{A}, \mathscr{C}\mathrm{at})(QK, \mathscr{B}(B, G)).$$

For $Q = \mathscr{C}(C, -)$, the above isomorphism gives

$$[\mathscr{C}, \mathscr{C}\mathrm{at}](\mathscr{C}(C, -), \mathscr{B}(B, R)) \simeq \operatorname{Fun}(\mathscr{A}, \mathscr{C}\mathrm{at})(\mathscr{C}(C, K), \mathscr{B}(B, G))$$

or

$$\mathscr{B}(B, RC) \simeq [\mathscr{A}, \mathscr{C}\mathrm{at}](\check{\mathscr{C}}(C, K), \mathscr{B}(B, G)),$$

that is $RC = \lim(\check{\mathscr{C}}(C, K), G)$ as desired.

Remarks (1) It is clear that $\hat{F}$ and $\check{F}$ are lax extensions along an identity 2-functor.

(2) Recall the formula for ordinary ($=$ Cat) right extensions: $RC = \lim(\mathscr{C}(C, K), G)$ ([1, Theorem 8.3] with Street's notation). So, we see that the symbol $\vee$ gives the measure of laxness for extensions.

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