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ON THE HOMOTOPY THEORY OF TOPOGENIC GROUPS AND GROUPOIDS

BY

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Introduction

In [1], following a line of development begun by Quillen [12] and pursued by Kan and Thurston [6], a new connection between group theory and homotopy theory was described. This is only one of a number of analogous relations between homotopy and various algebraic theories: simplicial sets, small categories [8] and monoids [10] are examples. Among these however, group theory is evidently unique in its intrinsic interest: there is reason to hope for a profitable flow of information between these two domains.

This paper is intended as a step in the exploitation of this connection. Its primary concern is to show how some fundamental notions and constructions of homotopy theory may be paralleled within group theory. It is of course our hope that these may prove useful for the development of group theory. A first indication of how this might happen is provided by a characterization entirely within group theory of the groups of Quillen's algebraic K-theory.

To be somewhat more precise, we shall introduce our homotopy theory in what we call here the category of topogenic groups, that is to say groups with a distinguished perfect normal subgroup. This homotopy theory corresponds to that of pointed connected CW complexes. It is obviously desirable to have a non-pointed, not-necessarily-connected theory as well. This is provided by the category of topogenic groupoids. Both of these categories are described in Section 1 below, and the connection with homotopy theory alluded to above is set forth in Section 2. In Section 3 we present in more detail our program for introducing homotopy theory in these categories; this is mediated by two notions of "fibration", viz. weak fibrations and Kan fibrations.

Weak fibrations are designed to exhibit the homotopy fibres, and more generally homotopy pullbacks, of morphisms of topogenic groups and groupoids. The principal result is an existence theorem (8.3) which, *inter alia*, allows a purely group-theoretic characterization (Section 9) of homotopy groups of topogenic groups and thus in particular of Quillen's algebraic K-theory. In the text this existence theorem is deduced from the theory of Kan fibrations. An alternate version, with some advantage of economy, appears in Appendix B.

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This description of the homotopy groups appears as a generalization of the Schur multiplicator, and thus also generalizes Milnor's description of the group $K_2(A)$, for a ring A, in terms of the Steinberg group (cf. Kervaire [7]). In Appendix A below we review some of the relevant facts about central extensions and comment, in the light of later developments, on some terminology and notation.

Kan fibrations are defined by a homotopy lifting property, in analogy with the corresponding notion in the category of simplicial sets. The analogy is less than perfect, but many of the maneuvers of homotopy theory survive the transition. In particular, for Kan groups (the analogue of Kan complexes) we can parallel the topological constructions of homotopy groups and function spaces, in virtue of a "homotopy extension" theorem (Section 12) which is among our principal results.

1. Topogenic groups and groupoids

We denote by $\mathcal{G}d$ the category of groupoids and homomorphisms (i.e. functors) between them and by $\mathcal{G}p$ the full subcategory of groups. A groupoid is uniquely a coproduct of connected subgroupoids; if each of these is a group we shall say that the groupoid is *totally disconnected*.

A congruence, in the sense of category theory, in a groupoid Γ is determined by the morphisms in Γ congruent to some identity, and thus by a totally disconnected subgoupoid $\Phi \subset \Gamma$, containing all the objects of Γ and subject to the condition that for any $s\colon x \to y$ in Γ, $s^{-1}\Phi(y, y)s \subset \Phi(x, x)$. We identify Φ with the congruence and write Γ/Φ for the corresponding quotient groupoid. In particular, a congruence in a group is just a normal subgroup.

A *topogenic groupoid* Γ is a groupoid Γ^0 provided with a congruence Γ^1 all of whose groups are perfect. These, provided with the obvious morphisms, viz. homomorphisms $\phi\colon \Gamma^0 \to \Lambda^0$ which restrict to $\phi^1\colon \Gamma^1 \to \Lambda^1$, form the category $\mathcal{G}d\mathcal{P}$. *Topogenic groups* are just topogenic groupoids which happen to be groups, and are the objects of the full subcategory $\mathcal{G}p\mathcal{P} \subset \mathcal{G}d\mathcal{P}$.

The *fundamental groupoid* functor $\pi\colon \mathcal{G}d\mathcal{P} \to \mathcal{G}d$ takes Γ to Γ^0/Γ^1. Its restriction $\pi_1\colon \mathcal{G}p\mathcal{P} \to \mathcal{G}\mathcal{P}$ is the *fundamental group* functor.

All four of these categories are complete and cocomplete. The structure of limits and colimits in $\mathcal{G}d$ and $\mathcal{G}p$ is well understood. In order to describe those in $\mathcal{G}d\mathcal{P}$ and $\mathcal{G}p\mathcal{P}$ we introduce the categories $\mathcal{P}d$ of totally disconnected perfect groupoids and $\mathcal{P}$ of perfect groups. Colimits in these are just colimits in $\mathcal{G}d$ and $\mathcal{G}p$; limits may be constructed in the larger category and reflected into the smaller one by taking the largest perfect subobject. Limits and colimits in $\mathcal{G}d\mathcal{P}$ and $\mathcal{G}p\mathcal{P}$ are then completely specified by the following proposition.

PROPOSITION 1.1. (i) *The functor* $\Gamma \mapsto \Gamma^1\colon \mathcal{G}d\mathcal{P} \to \mathcal{P}d$ *has the left adjoint* $\Lambda \mapsto (\Lambda, \Lambda)$.

(ii) *The functor* $\Gamma \mapsto \Gamma^0$: $\mathscr{G}d\mathscr{P} \to \mathscr{G}d$ *has the left adjoint* $\Lambda \mapsto (\Lambda, \Phi)$, *where* Φ *is the discrete groupoid on* ob Γ, *and the right adjoint* $\Lambda \mapsto (\Lambda, \tilde{\Lambda})$, *where* $\tilde{\Lambda}$ *is the largest perfect totally disconnected subgroupoid of* Λ.

(iii) *The functor* π: $\mathscr{G}d\mathscr{P} \to \mathscr{G}d$ *has the right adjoint* $\Lambda \mapsto (\Lambda, \Phi)$.

Entirely analogous statements hold for $\mathscr{G}p\mathscr{P}$. We shall in general use the functor $\Lambda \mapsto (\Lambda, \Lambda)$ to embed $\mathscr{P}d$ in $\mathscr{G}d\mathscr{P}$; in the same way we imbed $\mathscr{P}$ in $\mathscr{G}p\mathscr{P}$ and thus regard a perfect group P as a topogenic group.

We shall frequently have occasion to distinguish in $\mathscr{G}d$, $\mathscr{G}p$, $\mathscr{G}d\mathscr{P}$, $\mathscr{G}p\mathscr{P}$ the subcategories ${}^i\mathscr{G}d, \ldots$ of injective morphisms. From the theory of free products with amalgamation in group theory and its obvious extension to groupoids we get the following assertion.

PROPOSITION 1.2. *If a pushout in* $\mathscr{G}d\mathscr{P}$ ($\mathscr{G}p\mathscr{P}$) *has its two initial morphisms in* ${}^i\mathscr{G}d\mathscr{P}$ (${}^i\mathscr{G}p\mathscr{P}$) *then the other two are also in* ${}^i\mathscr{G}d\mathscr{P}$ (${}^i\mathscr{G}p\mathscr{P}$).

We shall refer to such diagrams as *i-pushouts*. In $\mathscr{G}p$ the corresponding notion is better known as a "free product with amalgamation".

Finally, intermediate between ${}^i\mathscr{G}d$ and $\mathscr{G}d$, or ${}^i\mathscr{G}d\mathscr{P}$ and $\mathscr{G}d\mathscr{P}$ are the categories ${}^f\mathscr{G}d$, ${}^f\mathscr{G}d\mathscr{P}$ containing the morphisms which as functors are faithful, i.e. injective on each of the groups $\Gamma(x, x)$.

2. The classifying space and the Quillen functor

We shall write CW for the category of CW complexes and cellular maps and $\mathrm{CW}^{\bullet}$ for the category of pointed connected CW complexes and basepoint preserving cellular maps. The corresponding homotopy categories are HoCW and $\mathrm{HoCW}^{\cdot}$. They are at once quotient categories with respect to the homotopy congruence and categories of fractions with respect to homotopy equivalences.

The fundamental groupoid functor π: CW $\to \mathscr{G}d$ has its usual sense except that for neatness we restrict ob πX to be the set of O-cells of X. The fundamental group π_1: $\mathrm{CW}^{\cdot} \to \mathscr{G}p$ is also the usual one. A local coefficient system on a CW complex X is a functor A: $\pi X \to Ab$. For a pointed connected CW complex X this is equivalent to a $\pi_1 X$ module; we shall not distinguish the notions. A map f: $X \to Y$ is a *homological equivalence* if for all local coefficient systems A in Y it induces isomorphisms $H(X; A \circ \pi f) \to H(Y; A)$, where $H(-; A)$ denotes homology with respect to the local coefficient system A.

The well-known theorem of J. H. C. Whitehead asserts that f: $X \to Y$ is a homotopy equivalence if and only if it is a homological equivalence and πf is an equivalence of groupoids.

The classifying-space functor B: $\mathscr{G}d \to$ CW takes a groupoid Γ to the geometrical realization of its nerve (cf. [14]). Thus, canonically, $\pi B\Gamma \approx \Gamma$; all higher homotopy groups are 0. Restricted to $\mathscr{G}p$, B takes its values in $\mathrm{CW}^{\bullet}$. We use the same letter for the functor $\mathscr{G}p \to \mathrm{CW}^{\bullet}$, which is nothing but the "classical" bar construction, as well as the composition of either with the canonical functor to the homotopy category, relying on the context to indicate which one is intended.

The homology of a groupoid Γ with coefficients $A: \Gamma \to ab$ is $H(B\Gamma, A)$.

The *Quillen functor* B^+: HoCW may be introduced by the following theorem, which summarizes results of the literature [12], [9].

THEOREM 2.1. *There exists a functor* $B^+: \mathscr{G}d\mathscr{P} \to$ HoCW, *supplied with a natural transformation* $b_\Gamma: B\Gamma^0 \to B^+\Gamma$, *characterized up to canonical isomorphism by either of the following equivalent conditions:*

(i) b_Γ: $B^+\Gamma$ *is universal for* $f: B\Gamma^0 \to X$ *such that* πf *is trivial on* Γ^1;

(ii) b_Γ *is a homological equivalence and induces an equivalence of groupoids* $\pi\Gamma \to \pi B^+\Gamma$.

Furthermore $B^+\Gamma$ *has in* CW *a representative* $B\Gamma^0 \cup X$ *such that* b_Γ *is represented by the inclusion and* X *is a three-dimensional complex with* $X^1 \subset B\Gamma^1$. *If* Γ^0 *is finitely generated (countable) then* X *may be taken to be finite (countable).*

In the pointed case $B^+: \mathscr{G}p\mathscr{P} \to \mathrm{HoCW}^\bullet$ and b_G are treated in entirely analogous fashion.

If follows from the Whitehead theorem that if $\phi: \Gamma \to \Lambda$ in $\mathscr{G}p\mathscr{P}$ then

$$B^+\phi: B^+\Gamma \approx B^+\Lambda$$

if and only if for any $A: \Lambda \to ab$ it induces $H(\Gamma^0, A\phi^0) \approx H(\Lambda^0, A)$ and $\pi\phi: \pi\Gamma \to \pi\Lambda$ is an equivalence of groupoids. We shall call such morphisms *weak homotopy equivalences*. The pointed case is entirely analogous.

Denoting the categories of fractions with respect to weak homotopy equivalences by Ho $\mathscr{G}d\mathscr{P}$, Ho $\mathscr{G}p\mathscr{P}$ we get from B^+, functors

$$\bar{B}^+: \mathrm{Ho}\,\mathscr{G}d\mathscr{P} \to \mathrm{HoCW},\ \bar{B}^+: \mathrm{Ho}\,\mathscr{G}p\mathscr{P} \to \mathrm{HoCW}^\bullet.$$

THEOREM 2.2. *The functors* $\bar{B}^+$ *are equivalences of categories.*

This proved in the connected case in [1]; the other follows at once.

There is a slightly sharper characterization of morphisms in the homotopy categories (cf. [1]).

PROPOSITION 2.3. *Any morphism* $B^+\Gamma \to B^+\Lambda$ *is of the form* $(B^+\psi)^{-1}(B^+\phi)$ *where* ψ *is a weak homotopy equivalence.*

We record, finally, the following important property of $B^+: \mathscr{G}p\mathscr{P} \to \mathrm{HoCW}^\bullet$.

(2.4) If G, K are topogenic groups with $K^0 \subset G^0$ and $G^1 = K^1$ then $B^+K \to B^+G$ is the covering space corresponding to $\pi_1 K \subset \pi_1 G$. In particular $K^0 = K^1 = G^1$ gives the universal covering.

3. Homotopy theory in $\mathscr{G}d\mathscr{P}$ and $\mathscr{G}p\mathscr{P}$

In view of Theorem 2.2 it becomes resonable to ask whether, in some sense, it is possible to construct in $\mathscr{G}d\mathscr{P}$ and $\mathscr{G}p\mathscr{P}$ the kind of homotopy theories familiar in categories of spaces or simplicial sets. Among the senses which come to mind the foremost is that formalized by Quillen under the name of "model category" [13]. While we shall not attempt to formulate a theorem asserting that these categories cannot be provided with such a structure, a number of considerations, not least among them the existence of arbitrarily large simple groups, make it seem most unlikely that this might be done. Alternate proposals, e.g. "*h-c* categories" [4], seem even less promising.

What, then, is to count as "doing homotopy theory" in $\mathcal{G}d\mathcal{P}$ and $\mathcal{G}p\mathcal{P}$? Without in the least attempting to be exhaustive we may pose, as a start, two test questions.

(i) in Ho $\mathcal{G}d\mathcal{P} \simeq$ HoCW and Ho $\mathcal{G}p\mathcal{P} \simeq$ HoCW$^{\bullet}$ there exist certain homotopy limits and colimits, viz. homotopy pullbacks and pushouts: among these are the homotopy fibres and cofibres. Can these be identified within $\mathcal{G}d\mathcal{P}$ and $\mathcal{G}p\mathcal{P}$?

(ii) In any model category there is a homotopy congruence such that for large classes of suitable objects the morphisms in the homotopy category are just the homotopy classes of morphisms in the model category. Does a similar situation obtain in our categories?

We shall, in the course of this paper, answer both of these questions in the affirmative.

Let us begin by identifying the homotopy pushouts and pullbacks in HoCW and HoCW$^{\bullet}$. Since the pointed and unpointed cases are completely parallel we shall confine our attention to the latter. Furthermore, in the interest of brevity, we shall replace HoCW by the equivalent category Ho $\mathcal{T}$, where $\mathcal{T}$ is the category of spaces of the homotopy type of a CW complex.

By a *c-pushout* in $\mathcal{T}$ we mean a pushout in which at least one of the two initial maps is a cofibration. The smallest class of commutative squares in Ho $\mathcal{T}$, closed with respect to isomorphism of diagrams and containing the images of the c-pushouts in $\mathcal{T}$, is the class of *homotopy pushouts* on Ho $\mathcal{T}$. We shall describe this situation by saying that the homotopy pushouts are *determined* by the c-pushouts in $\mathcal{T}$. The *homotopy pullbacks* in Ho $\mathcal{T}$ are in analogous fashion determined by the f-pullbacks in $\mathcal{T}$, i.e. by the pullbacks in $\mathcal{T}$ in which at least one of the two terminal maps is a fibration.

We shall show next that the homotopy pushouts in Ho $\mathcal{G}d\mathcal{P} \simeq$ Ho $\mathcal{T}$, Ho $\mathcal{G}p\mathcal{P} \simeq$ Ho $\mathcal{T}^{\bullet}$ are in the same sense determined by the i-pushouts in $\mathcal{G}d\mathcal{P}$ and $\mathcal{G}p\mathcal{P}$, thus dealing with half of question (i). The other half needs more work. We shall in fact introduce below two notions of fibration—weak fibrations and Kan fibrations—and show that f-pullbacks in either sense similarly determine homotopy pullbacks.

Finally, associated with the Kan fibrations we shall discover a class of topogenic groups—the Kan groups—whose homotopy theory is particularly simple, giving us an appropriate answer to question (ii).

Returning to homotopy pushouts we now prove the following assertion.

THEOREM 3.1. *The i-pushouts in $\mathscr{G}d\mathscr{P}$ ($\mathscr{G}p\mathscr{P}$) determine in the associated homotopy category the family of homotopy pushouts.*

We may confine ourselves to the case of $\mathscr{G}d\mathscr{P}$, the other being analogous. The theorem is immediately implied by the following three lemmas.

LEMMA 3.2. *B^+ takes i-pushouts in $\mathscr{G}d\mathscr{P}$ into homotopy pushouts in* HoCW $\simeq$ Ho $\mathscr{T}$.

Suppose that

$$\Gamma_0 \xrightarrow{\gamma_1} \Gamma_1 \longrightarrow \Gamma \longleftarrow \Gamma_2 \xleftarrow{\gamma_2} \Gamma_0$$

is a pushout with γ_1, γ_2 injective and let

$$B = (B\Gamma_0^0 \to B\Gamma_1^0 \to Q \leftarrow B\Gamma_2^0 \leftarrow B\Gamma_0^0)$$

be a pushout in CW.

By a standard argument, Q has the homotopy type of $B\Gamma$. Further, represent $B^+\gamma_1$, $B^+\gamma_2$ by inclusions of subcomplexes and construct the pushout

$$B^+ = (B^+\Gamma_0 \to B^+\Gamma_1 \to Q^+ \leftarrow B^+\Gamma_2 \leftarrow B^+\Gamma_0).$$

There is then a morphism $B \to B^+$ composed of maps b_i representing b_{Γ_i}, $i = 0$, 1, 2 and q: $Q \to Q^+$.

It follows from the generalized van Kampen theorem that Q^+ has fundamental groupoid equivalent to $\pi\Gamma$ and from the exactness of the Mayer-Vietoris theorem with local coefficients that q is a homological equivalence, so that q: $Q \to Q^+$ represents b_Γ: $B\Gamma^0 \to B^+\Gamma$.

LEMMA 3.3. *Any groupoid can be imbedded in an acyclic one.*

Since any groupoid imbeds in a connected one we may as well begin with a connected groupoid Γ. Choose $x \in$ ob Γ and imbed $\Gamma(x, x)$ in an acyclic group D. Then the pushout of $\Gamma \leftarrow \Gamma(x, x) \to D$ is acyclic.

LEMMA 3.4. *If f: $B^+\Gamma \to X$ in* HoCW *then there is an injection γ: $\Gamma \to \Lambda$ in $\mathscr{G}d\mathscr{P}$ such that $f \approx B^+\gamma$ in* ($B^+\Gamma \downarrow$ HoCW).

In view of Theorem 1.1 we may as well suppose that $f = B^+\phi$ for some ϕ: $\Gamma \to \Phi$. Let δ: $\Gamma \to \Delta$ imbed Γ in an acyclic Δ and set

$$\gamma = \langle \phi\ \delta \rangle : \Gamma \to \Phi \times \Delta.$$

4. On the equivalence $\bar{B}^+$: some nonfunctorial constructions

We proceed next to a closer analysis of the equivalence of categories asserted in Theorem 2.2. For brevity we confine ourselves to the pointed case. In [1] there was constructed a functor L^0: ${}^i\mathscr{K}^\bullet \to \mathscr{G}p$, where ${}^i\mathscr{K}^\bullet$ is the category of pointed connected simplicial complexes, and a natural transformation t_K: $BL^0K \to |K|$, where $|K|$ is the geometric realization, such that for each K, t_K is a homological equivalence and $\pi_1 t_K$ is surjective. It follows that the kernel

L^1K of $\pi_1 t_K$ is perfect. Thus we have defined a functor $L\colon {}^i\mathscr{K}^\bullet \to \mathscr{G}p\mathscr{P}$ and t_K factors canonically as

$$BL^0K \xrightarrow{b_{LK}} B^+LK \xrightarrow{\bar{t}_K} |K|$$

with $\bar{t}_K$ an isomorphism.

The functor L^0 has the further property that if K is a countable simplicial complex then L^0K is a countable group.

While we cannot compose B^+ and L in the other order we shall attempt to achieve a similar end by means of the following nonfunctorial construction. Suppose that G is a topogenic group. Then $b_G\colon BG^0 \to B^+G$ may be lifted in many ways) to ${}^i\mathscr{K}^\bullet$. That is to say we may find $\hat{b}_G\colon \hat{B}G \to \hat{B}^+G$ in ${}^i\mathscr{K}^\bullet$ such that $|\hat{b}_G|$ is isomorphic in HoCW$^\bullet$ to b_G: we shall allow ourselves to write, by a harmless abuse of notation, $|\hat{b}_G| = b_G$. It follows from 2.1 that if G is countable then $\hat{B}^+G$ may be chosen countable as well.

Now $t(\hat{B}G)\colon BL^0\hat{B}G \to |\hat{B}G| = BG$ and, setting $\varepsilon_G = \pi_1 t(\hat{B}G)\colon L^0\hat{B}G \to G$ we have $t(\hat{B}G) = B\varepsilon_G$. Let $M^0G = L^0\hat{B}G$ and $M^1G = \varepsilon_G^{-1}(G^1)$. Since $t(\hat{B}G)$ is a homological equivalence M^1G is perfect and we have constructed a topogenic group MG and a morphism $\varepsilon_G\colon MG \to G$ in $\mathscr{G}p\mathscr{P}$.

It follows immediately from the definition of B^+G that $L^0\hat{b}_G$ takes M^1G into $L^1\hat{B}^+G$ and thus defines a morphism $\beta_G\colon MG \to L\hat{B}^+G$.

Let us now consider the following diagram in HoCW$^\bullet$:

(4.1)
$$\begin{tikzcd}
BL^0\hat{B}^+G \arrow[d, "b(L\hat{B}^+G)"'] & BL^0\hat{B}G \arrow[l, "BL^0\hat{b}_G"'] \arrow[r, "B\varepsilon_G = t(\hat{B}G)"] \arrow[d, "b(MG)"] & BG \arrow[d, "b_G"] \\
B^+L\hat{B}^+G \arrow[rr, bend right, "\bar{t}(\hat{B}^+G)"'] & B^+MG \arrow[l, "B^+\beta_G"'] \arrow[r, "B^+\varepsilon_G"] & B^+G
\end{tikzcd}$$

We make the following claim.

LEMMA 4.2. *The diagram* 4.1 *commutes and* $B^+\beta_G$, $B^+\varepsilon_G$ *are isomorphisms.*

The two squares commute because of the naturality of b. Further, each instance of b is a homological equivalence, thus also $BL^0\hat{b}_G$. Similarly $t(\hat{B}G)$ is a homological equivalence. By construction, $\pi_1 B^+\beta_G$ and $\pi_1 B^+\varepsilon_G$ are isomorphisms. It follows from the Whitehead theorem that $B^+\beta_G$ and $B^+\varepsilon_G$ are isomorphisms. Finally, the commutativity

(4.3)
$$\bar{t}(\hat{B}^+G)B^+\beta_G = B^+\varepsilon_G$$

follows from the universal property of $b(MG)$, since $\bar{t}(\hat{B}^+G)b(L\hat{B}^+G) = t(\hat{B}^+G)$ and, by the naturality of t, $t(\hat{B}^+G)(BL\hat{b}_G) = b_G t(\hat{B}G)$, so that

$$\bar{t}(\hat{B}^+G)(B^+\beta_G)b(MG) = (B^+\varepsilon_G)b(MG).$$

The constructions we have just introduced enjoy the following naturality property. Suppose ϕ: $W \to G$ in $\mathscr{G}p\mathscr{P}$ and let the following commutative diagram in ${}^{i}\mathscr{K}^{\bullet}$ lift the corresponding diagram in $\mathrm{HoCW}^{\bullet}$:

$$\begin{array}{ccc} \hat{B}W & \xrightarrow{\hat{b}_W} & \hat{B}W \\ {\scriptstyle \hat{B}\phi}\downarrow & & \downarrow{\scriptstyle \hat{B}^{+}\phi} \\ \hat{B}G & \xrightarrow{\hat{b}_G} & \hat{B}^{+}G. \end{array}$$

Then $L^0\hat{B}\phi$ takes M^1W into M^1G and thus defines $M\phi$: $MW \to MG$. The diagram

$$\begin{array}{ccccc} L\hat{B}^{+}W & \xleftarrow{\beta_W} & MW & \xrightarrow{\varepsilon_W} & W \\ {\scriptstyle L\hat{B}\phi}\downarrow & & \downarrow{\scriptstyle M\phi} & & \downarrow{\scriptstyle \phi} \\ L\hat{B}^{+}G & \xleftarrow{\beta_G} & MG & \xrightarrow{\varepsilon_G} & G \end{array} \tag{4.4}$$

commutes.

5. Relative homotopy and conjugacy

We shall have to deal with relative as well as absolute homotopy. If ϕ and ψ are morphisms $G \to W$ in $\mathscr{G}p\mathscr{P}$ agreeing on $U \subset G$ we should like to be able to say what it means for $B^+\phi$ and $B^+\psi$ to be "homotopic rel B^+U". Since B^+ takes its values in $\mathrm{HoCW}^{\bullet}$ rather than $\mathrm{CW}^{\bullet}$ this does not *a priori* make sense, but we can explicate it in the following way.

If A is in $\mathrm{CW}^{\bullet}$ we denote by $\mathrm{CW}^{\bullet}$ (rel A) the category of CW complexes containing A as a subcomplex with, as morphisms, maps extending the identity on A. Homotopy rel A, i.e. stationary on A, is a congruence in this category. The category $\mathrm{HoCW}^{\bullet}$ (rel A) is also the category of fractions with respect to maps such that $X \to Y$ is a homotopy equivalence.

$\mathrm{CW}^{\bullet}$ (rel A) is itself a covariant functor of A: if g: $A \to B$ then $A \to X$ goes to its pushout along g. If g is a homotopy equivalence this gives rise to an equivalence of categories $\mathrm{HoCW}^{\bullet}$ (rel A) $\to$ $\mathrm{HoCW}^{\bullet}$ (rel B).

Similarly we define, for a topogenic group K, the category $\mathscr{G}p\mathscr{P}$(rel K) whose objects are topogenic groups containing K as a subgroup. Now let us fix, in $\mathrm{CW}^{\bullet}$, a model for $B^+K \supset BK^0$. We may define a functor

$$B^{+}_{\mathrm{rel}}\colon \mathscr{G}p\mathscr{P}(\mathrm{rel}\ K) \to \mathrm{HoCW}^{\bullet}(\mathrm{rel}\ B^{+}K)$$

by observing that, for $G \supset K$ in $\mathscr{G}p\mathscr{P}$, a morphism

$$B^{+}K \cup BG^{0} \to B^{+}G \tag{5.1}$$

which restricts to the obvious ones on B^+K and BG^0 is universal in $\mathrm{HoCW}^{\bullet}$(rel B^+K) for morphisms f such that $(\pi_1 f)G^1 = 1$. The argument, by obstruction theory, is identical to the one for the absolute case, since the map of 5.1 is easily seen to be a homological equivalence.

This whole construction is, up to equivalence of categories, independent of the choice of a model for B^+K. Thus we shall allow ourselves to write

$$B^+_{\mathrm{rel}}\colon \mathcal{G}p\mathcal{P}(\mathrm{rel}\ K) \to \mathrm{HoCW}^\bullet(\mathrm{rel}\ B^+K)$$

even when no special choice is envisaged. We may now express the notion of relative homotopy of morphisms ϕ, ψ as above by interpreting it simply as $B^+_{\mathrm{rel}}\phi = B^+_{\mathrm{rel}}\psi$.

It would be possible to proceed analogously in $\mathcal{G}d\mathcal{P}$ and CW. But relative homotopy is the nontrivial case is always pointed so that there would seem to be nothing to be gained by this generalization.

That there is a connection between conjugacy of homomorphisms and homotopy seems intuitively obvious. We shall begin now to explore this connection. Here however it is useful to distinguish the unpointed case, i.e. $B^+\colon \mathcal{G}d\mathcal{P} \to \mathrm{HoCW}$.

PROPOSITION 5.2. *If ϕ, $\psi\colon \Gamma \to \Lambda$ in $\mathcal{G}d\mathcal{P}$ and there exists a natural transformation $\phi \to \psi$ then $B^+\phi = B^+\psi$.*

Such a natural transformation would give a morphism $\Gamma \times I \to X$ in $\mathcal{G}d\mathcal{P}$ where I is the indiscrete groupoid on two objects. It is clear however that $B^+(\Gamma \times I) \to B^+\Gamma$, via the projection, is an isomorphism.

The existence of such a natural transformation, which is of course always an isomorphism, is a congruence in $\mathcal{G}d\mathcal{P}$, as well as in the subcategories ${}^f\mathcal{G}d\mathcal{P}$, ${}^i\mathcal{G}d\mathcal{P}$. We shall write $\mathrm{Ho}_c\mathcal{G}d\mathcal{P}$, $\mathrm{Ho}_c{}^f\mathcal{G}d\mathcal{P}$, $\mathrm{Ho}_c{}^i\mathcal{G}d\mathcal{P}$ for the corresponding quotient categories.

A caution is necessary here. If Γ and Λ are groups then a natural transformation $\phi \to \psi$, where ϕ, $\psi\colon \Gamma \to \Lambda$, is just a conjugation C_x for some $x \in \Lambda^0$ with $C_x\phi = \psi$. From its existence it does not follow that $B^+\phi = B^+\psi$ where B^+ is the "pointed" functor $\mathcal{G}p\mathcal{P} \to \mathrm{HoCW}^\cdot$. This last relation we shall do better to understand by considering it as a special case of relative homotopy, to which we next proceed.

The functor

$$U \longmapsto (K \xrightarrow{\langle 0\ K\rangle} U \times K)$$

from $\mathcal{G}p\mathcal{P}$ to $\mathcal{G}p\mathcal{P}(\mathrm{rel}\ K)$ has the right adjoint $G \mapsto ZG$ where Z^0G is the centralizer of K^0 in G^0 and Z^1G is the largest perfect subgroup of $Z^0G \cap G^1$. We may refer to Z as the *centralizer* functor. The bijection

$$(5.3) \qquad \zeta_{U,K}\colon \mathcal{G}p\mathcal{P}(U, ZG) \approx \mathcal{G}p\mathcal{P}(\mathrm{rel}\ K)(U \times K, G)$$

is defined by multiplication in G^0.

If K is centerless—i.e. the center of K^0 is trivial—then also

$$(5.4) \qquad \zeta_{U,G}\colon {}^i\mathcal{G}p\mathcal{P}(U, ZG) \approx {}^i\mathcal{G}p\mathcal{P}(\mathrm{rel}\ K)(U \times K, G).$$

In $\mathcal{G}p\mathcal{P}(\mathrm{rel}\ K)$ we introduce a congruence $\sim_c$ making $\phi, \psi: G \to W$ congruent ($\phi \sim_c \psi(\mathrm{rel}\ K)$) if there is an $x \in Z^1W$ such that $\psi = C_x\phi$. In the pointed absolute case, i.e., in $\mathcal{G}p\mathcal{P}$, this means that $\psi = C_x\phi$ for some $x \in W^1$. This congruence restricts to ${}^i\mathcal{G}p\mathcal{P}(\mathrm{rel}\ K)$ and we may thus construct the quotient categories $\mathrm{Ho}_c\mathcal{G}p\mathcal{P}(\mathrm{rel}\ K)$, $\mathrm{Ho}_c{}^i\mathcal{G}p\mathcal{P}(\mathrm{rel}\ K)$. We write $\mathrm{Ho}_c\mathcal{G}p\mathcal{P}$ for $\mathrm{Ho}_c\mathcal{G}p\mathcal{P}(\mathrm{rel}\ 1)$; but note that even if U, G are groups, $\mathrm{Ho}_c\mathcal{G}d\mathcal{P}(U, G)$ and $\mathrm{Ho}_c\mathcal{G}p\mathcal{P}(U, G)$ are not in general the same.

The bijection 5.4 induces one on the quotients as well:

$$(5.5) \qquad \zeta_{U,G}: \mathrm{Ho}_c{}^i\mathcal{G}p\mathcal{P}(U, ZG) \approx \mathrm{Ho}_c{}^i\mathcal{G}p\mathcal{P}(\mathrm{rel}\ K)(U \times K, G).$$

The connection between conjugacy and relative homotopy, finally, is expressed by the following proposition.

PROPOSITION 5.6. *If $\phi \sim_c \psi(\mathrm{rel}\ K)$ then $B^+_{\mathrm{rel}}\phi = B^+_{\mathrm{rel}}\psi$.*

For its proof we shall need the following lemma.

LEMMA 5.7. *If F is a free group, P is a perfect group and $\psi: F \to P$ is a homomorphism then ψ may be factorized as $F \to D \to P$ with D acyclic. If F is countable then D may also be taken to be countable.*

Choose free generators for F. For each generator x we can find finitely many elements $u_{x,i}, u'_{x,i}$ of P such that $\psi_x = \Pi_i[u_{x,i}, u'_{x,i}]$. Let F_1 be the free group generated by the set $\{(x, i), (x, i)'\}$; define $\psi_1: F_1 \to P$ by $(x, i) \mapsto u_{x,i}$, $(x, i)' \mapsto u'_{x,i}$ and $\phi: F \to F_1$ by $x \to \Pi_i[(x, i), (x, i)']$. Then $\psi_1\phi = \psi$ and the image of ϕ is contained in $[F_1, F_1]$. Iterating this construction by similarly factorizing ψ_1 we arrive at a sequence $F \to F_1 \to F_2 \cdots$ whose colimit is the required D.

Returning now to 5.6, observe that we need only prove that for G in $\mathcal{G}p\mathcal{P}(\mathrm{rel}\ K)$ and $x \in Z^1G$, $B^+C_x = 1_{B^+G}$. By 5.7 we can find an acyclic D, a homomorphism $\psi: D \to Z^1G$ and an element $\alpha \in D$ such that $\psi\alpha = x$. In the pushout

$$\begin{array}{ccc} K & \xrightarrow{\langle K\ 0\rangle} & K \times D \\ \Big\downarrow{\scriptstyle\gamma} & & \Big\downarrow{\scriptstyle\omega} \\ G & \xrightarrow{\ \theta\ } & W \end{array}$$

in $\mathcal{G}p\mathcal{P}$, $B^+\langle K\ 0\rangle$ is an isomorphism, hence also $B^+_{\mathrm{rel}}\theta$. Define $\rho_0, \rho_1: W \to G$ by $\rho_0\theta = \rho_1\theta = 1_G$, $\rho_0\omega = \gamma\,\mathrm{pr}_K$, $\rho_1\omega = \gamma \cdot \psi$ where $\cdot$ means multiplication in G^0, which gives a homomorphism because ψ maps D into the centralizer of K^0. Then $B^+_{\mathrm{rel}}\rho_0 = B^+_{\mathrm{rel}}\rho_1$. But $\rho_0 C_{\omega(1,\alpha)}\theta = 1_G$ and $\rho_1 C_{\omega(1,\alpha)}\theta = C_x$.

There is a connection between the pointed and the unpointed cases which we may express in the following way. First we notice that for K, G in $\mathcal{G}p\mathcal{P}$, $\pi_1 G = G^0/G^1$ operates on $\mathrm{Ho}_c\mathcal{G}p\mathcal{P}K, G)$ (on $\mathrm{Ho}_c{}^i\mathcal{G}p\mathcal{P}(K, G)$) with orbit space $\mathrm{Ho}_c\mathcal{G}d\mathcal{P}(K, G)$(resp. $\mathrm{Ho}_c{}^f\mathcal{G}d\mathcal{P}(K, G)$). This is analogous to the familiar operation, for X, Y in $\mathrm{CW}^{\bullet}$, of $\pi_1 Y$ on $\mathrm{HoCW}^{\bullet}(X, Y)$, whose orbit space is $\mathrm{HoCW}(X, Y)$.

PROPOSITION 5.8. *If K, G are $\mathscr{G}p\mathscr{P}$ then B^+ induces a $\pi_1 B^+ G$-equivariant map* $\mathrm{Ho}_c{}^i\mathscr{G}p\mathscr{P}(K, G) \to \mathrm{HoCW}^{\bullet}(B^+K, B^+G)$ *and*

$$\begin{array}{ccc} \mathrm{Ho}_c{}^i\mathscr{G}p\mathscr{P}(K, G) & \longrightarrow & \mathrm{HoCW}^{\bullet}(B^+K, B^+G) \\ \Big\downarrow & & \Big\downarrow \\ \mathrm{Ho}_c{}^f\mathscr{G}d\mathscr{P}(K, G) & \longrightarrow & \mathrm{HoCW}(B^+K, B^+G) \end{array}$$

commutes.

6. B^+: countable factorizations

We have observed above that B and B^+, as well as the functor L out of which we constructed a "homotopy inverse" of B^+ preserve countability. The factorization lemmas below will also be needed. For the first, compare Kan-Thurston [6].

LEMMA 6.1. *If G_1 is a countable subgroup of the group G then there is a countable $G_1 \subset G_0 \subset G$ such that $HG_0 \to HG$ is injective.*

Using for example the standard resolution we see that HG_1 is countable, hence also the kernel of $HG_1 \to HG$. But each cycle belonging to a class in this kernel in fact bounds in some subgroup of G containing G_1 and generated by finitely many additional elements. Thus there is a countable $G_1 \subset G_2 \subset G$ such that the kernel of $HG_1 \to HG_2$ is the same as that of $HG_1 \to HG$. Iterating this procedure we get a sequence $G_1 \subset G_2 \subset \cdots$ whose union G_0 has the required property.

LEMMA 6.2. *Suppose that X is a countable* CW*-complex, G is a topogenic group and f: $X \to B^+G$. Then there is a countable $W \subset G$ such that f factors as $X \to B^+W \to B^+G$.*

We may take for B^+G a CW-complex $BG^0 \cup Y$ where Y is a 3-dimensional complex with $Y^1 \subset BG^1$. Choosing a representative for f in $\mathrm{CW}^{\bullet}$ we see that its image lies in $BU \cup Z$ where U is a countable subgroup of G^0 and Z is a countable subcomplex of Y with $Z^1 \subset B(U \cap G^1)$. With the aid of 6.1 we can find a countable $W \subset G$ with $W^0 \supset U$ and $W^1 \supset U \cap G^1$ such that $HW^1 \to HG^1$ is injective.

In particular $\pi_1 B^+ W^1 = 0$ and $\pi_2 B^+ W = H_2 W^1 \to \pi_2 B^+ G = H_2 G^1$ is injective. An easy obstruction-theoretic argument gives the factorization.

COROLLARY 6.3. *Suppose that X is a countable* CW*-complex, U is countable, θ: $U \subset G$ is an inclusion of topogenic groups, f_0, f_1: $X \to B^+U$ and $(B^+\theta)f_0 = (B^+\theta)f_1$. Then there is a countable*

$$U \overset{\omega}{\subset} W \subset G$$

such that $(B^+\omega)f_0 = (B^+\omega)f_1$.

In our discussion of Kan groups below we shall want the following application of these lemmas.

PROPOSITION 6.4. *Suppose that K is a countable topogenic group, that U and G are in $\mathcal{G}p\mathcal{P}$(rel K), and that U is also countable. If*

$$f\colon B^+_{\mathrm{rel}}U \to B^+_{\mathrm{rel}}G$$

in $\mathrm{HoCW}^{\bullet}$(rel B^+K) *then there is a countable $K \subset W \subset G$ such that f factors as $B^+_{\mathrm{rel}}U \to B^+_{\mathrm{rel}}W \to B^+_{\mathrm{rel}}G$.*

7. Weak fibrations in $\mathcal{P}$

We introduce now a notion of weak fibrations in $\mathcal{G}p\mathcal{P}$ and $\mathcal{G}d\mathcal{P}$; for convenience we begin with homomorphisms in $\mathcal{P}$, the category of perfect groups, corresponding under B^+ to the category of 1-connected CW-complexes.

A normal subgroup N of a group G is *homologically central* if the operation of G/N on the integral homology HN of N is trivial. Thus for example a homologically central abelian subgroup is central. If N is homologically central in G it follows immediately that $N/[N, N] = H_1 N$ is homologically central in $G/[N, N]$, and thus central.

An epimorphism $\phi\colon U \to V$ of perfect groups is a *weak fibration* if both its kernel N^0 and $N^1 = [N^0, N^0]$ are homologically central in U.

PROPOSITION 7.1. *If $\phi\colon U \to V$ is a weak fibration in $\mathcal{P}$ then N^1 is perfect, so that $N = (N^0, N^1)$ is a topogenic group.*

From the homological centrality it follows that the commutative diagram

$$\begin{tikzcd}
H_2(N^0/N^1) \arrow[r] \arrow[d, "="'] & H_2(U/N^1) \arrow[r] \arrow[d] & H_2 V \arrow[d] & \\
H_2(N^0/N^1) \arrow[r] & H_1 N^1 \arrow[r] \arrow[d] & H_1 N^0 \arrow[r, "\approx"] \arrow[d] & H_1(N^0/N^1) \\
 & 0 & 0 &
\end{tikzcd}$$

has exact rows and columns. But also (cf. for example [7] or Appendix A)

$$H_2(U/N^1) \to H_2 V$$

is injective. Thus $H_1 N^1 = 0$.

LEMMA 7.2. *If $\phi\colon U \to V$ is a weak fibration in $\mathcal{P}$ then B^+N is a nilpotent space.*

From the homological centrality of N^1 it follows that $\pi_1 B^+N = N^0/N^1$ operates trivially on the homology of B^+N^1, its universal covering space. Nilpotency follows [11].

PROPOSITION 7.3. *If ϕ: $U \to V$ is a surjective homomorphism of perfect groups with kernel N^0 and $N^1 = [N^0, N^0]$, and $\tilde{N}$ is a perfect normal subgroup of N^0, then the following are equivalent:*
(i) *$\tilde{N} = N^1$ and ϕ is a weak fibration;*
(ii) *$B^+(N^0, \tilde{N}) \to B^+U$ is the homotopy fibre of $B^+\phi$.*
Furthermore when these conditions hold then any subgroup M such that $N^1 \subset M \subset N^0$ is homologically central.

It is convenient to work in $\mathscr{T}^{\bullet}$, the category of pointed spaces of the homotopy type of a pointed CW-complex, instead of CW$^{\bullet}$. If ϕ is a weak fibration we may construct in $\mathscr{T}^{\cdot}$ a commutative diagram

(7.4)
$$\begin{tikzcd} BN^0 \arrow[r] \arrow[d, "f"'] & BU^0 \arrow[r] \arrow[d, "b_U"] & BV^0 \arrow[d, "b_V"] \\ F \arrow[r] & B^+U \arrow[r] & B^+V \end{tikzcd}$$

in which the rows are fibrations with their fibres (we have of course labeled maps and spaces as their images in Ho $\mathscr{T}^{\bullet}$). The Serre spectral sequences of the rows have constant coefficients and the comparison theorem implies that Hf: $HBN^0 \approx HF$.

Since B^+U is 1-connected $\pi_1 F$ is abelian and thus $\pi_1 F$: $N^0 \to \pi_1 F$ vanishes on $\tilde{N} = N^1$. Thus f factors in Ho $\mathscr{T}^{\cdot}$ as

$$BN^0 \to B^+N \xrightarrow{\bar{f}} F$$

and $H\bar{f}$ is once more an isomorphism. But B^+N is nilpotent by 7.2 and F, as fibre of a fibration of 1-connected spaces, is also nilpotent [5] it follows from Dror's generalization of the Whitehead theorem [3] that $\bar{f}$ is a homotopy equivalence and from the universal property of $BN^0 \to B^+N$ that the composition $B^+N \to F \to B^+U$ comes from the inclusion $N \to U$.

Conversely if $B^+N \to B^+U$ is the homotopy fibre of $B^+\phi$ then, since

$$H_2 V \approx \pi_2 B^+ V \to \pi_1 B^+(N^0, \tilde{N}) = N^0/\tilde{N},$$

$\tilde{N} \supset [N^0, N^0] = N^1$. But $\tilde{N}$ is perfect, hence $\tilde{N} = N^1$. It remains only to show that all $N^1 \subset M \subset N^0$ are homologically central.

Referring to 3.4, this time with $F = B^+N$, we see that the operation of $V = \pi_1 BV$ on $HN^0 = HBN^0 \simeq HB^+N$ factors through $\pi_1 b_V = 0$. Thus N^0 is homologically central in U, N^0/N^1 is central in U/N^1 and any $N^1 \subset M \subset N^0$ is normal.

For any such M let us construct in $\mathscr{T}^{\bullet}$ the diagram

$$\begin{tikzcd}
W \arrow[r, "\hat{n}"] \arrow[d, "="'] & B^+N \arrow[r, "n"] \arrow[d, "t"] \arrow[dr, phantom, "*"] & B(N^0/M) = B^+(N^0/M, 1) \arrow[d, "\hat{v}"] \\
W \arrow[r, "\hat{u}"] & B^+U \arrow[r, "u"] \arrow[d, "vu"'] & B^+(U/M) \arrow[d, "v"] \\
& B^+V \arrow[r] & B^+V
\end{tikzcd}$$

by taking u and v to be fibrations in the obvious homotopy classes, $\hat{u}$ and $\hat{v}$ their fibres, and the starred square a pullback, so that n is a fibration with fibre $\hat{n}$. We have identified $\hat{v}$ and t by out previous argument, the construction assuring us that t is the fibre of vu.

But $B(N^0/M)$ classifies covering spaces. Thus, up to homotopy, $\hat{n}$ is the covering of B^+N corresponding to the subgroup $M/N^1 \subset N/N^1 = \pi_1 B^+N$. Thus $W = B^+(M, N^1)$ and, as above, U/M operates trivially on $HB^+(M, N^1) = HM$.

COROLLARY 7.5. *If*

$$\begin{tikzcd}
U_1 \arrow[r] \arrow[d, "\phi_1"'] & U \arrow[d, "\phi"] \\
V_1 \arrow[r] & V
\end{tikzcd}$$

is a pullback in $\mathscr{P}$ and ϕ is weak fibration then so also is ϕ_1.

Recall that U_1 is the largest perfect subgroup of the pullback W in $\mathscr{G}p$. If N^0 is, as above, the kernel of ϕ in $\mathscr{G}p$ and $N^1 = [N^0, N^0]$ then $N^0 \to W \to V_1$ and $N^0/N^1 \to W/N^1 \to V_1$ are extensions, the latter being central. It follows that $[W/N^1, W/N^2]$ is perfect. But

$$N^1 \to [W, W] \to [W/N^1, W/N^1]$$

is also an extension so that $[W, W]$ is perfect. Thus $U_1 = [W, W]$ and ϕ_1 is surjective. Its kernel is $N_1^0 = N^0 \cap [W, W] \supset N^1$, with $[N_1^0, N_1^0] = N^1$. The conditions concerning homological centrality follow from 7.3.

Proposition 7.3 allows us to cast the the light of hindsight on some results of Kervaire [7]: cf. Appendix A.

8. Weak fibrations in $\mathscr{G}d\mathscr{P}$ and $\mathscr{G}p\mathscr{P}$

Let us recall the notion of a fibration of groupoids [2]. A morphism $\phi\colon \Gamma \to \Lambda$ in $\mathscr{G}d$ is a fibration if it has the "path-lifting" property: for any $x \in \text{ob}\,\Gamma$ and $s \in \text{mor}\,\Lambda$ with domain ϕx there is a $t \in \text{mor}\,\Gamma$ with domain x such that

$\phi t = s$. Note that when Γ, Λ are groups this means simply that ϕ is surjective.

By a *weak fibration* in $\mathscr{G}d\mathscr{P}$ we mean a morphism $\phi\colon \Gamma \to \Lambda$ such that $\phi^0\colon \Gamma^0 \to \Lambda^0$ is a fibration in $\mathscr{G}d$ and, for each $x \in \operatorname{ob}\Gamma$, $\phi^1\colon \Gamma^1(x, x) \to \Lambda^1(\phi^0 x, \phi^0 x)$ is a weak fibration in $\mathscr{P}$. The *weak fibrations* in $\mathscr{G}p\mathscr{P}$ are just the weak fibrations in $\mathscr{G}d\mathscr{P}$ which lie in the subcategory, i.e., the morphisms $\phi\colon G \to K$ with ϕ^0 surjective and ϕ^1 a weak fibration in $\mathscr{P}$.

From 7.5, attending to the structure of pullbacks in $\mathscr{G}d\mathscr{P}$ (cf. 1.1), we deduce easily the following statement.

PROPOSITION 8.1. *A pullback in $\mathscr{G}d\mathscr{P}$ ($\mathscr{G}p\mathscr{P}$) of a weak fibration is again a weak fibration.*

In either category we define the *f-pullbacks* to be those pullback squares with one of the two terminal morphisms a weak fibration. Among our objectives is the following theorem (compare with 3.1) which completes our answer to question (i) of Section 3.

THEOREM 8.2. *The f-pullbacks in $\mathscr{G}d\mathscr{P}$ ($\mathscr{G}p\mathscr{P}$) determine in the associated homotopy category the family of homotopy pullbacks.*

This is easily seen to be a consequence of the following two statements:

THEOREM 8.3. *Any morphism ϕ in $\mathscr{G}d\mathscr{P}$ (in $\mathscr{G}p\mathscr{P}$ with $\pi_1\phi$ surjective) can be factored as a weak homotopy equivalence followed by a weak fibration.*

The proof of this is deferred to Section 11 below.

PROPOSITION 8.4. *B^+ takes f-pullbacks in $\mathscr{G}d\mathscr{P}$ ($\mathscr{G}p\mathscr{P}$) into homotopy pullbacks in* HoCW (HoCW$^\bullet$).

Suppose that

$$\begin{array}{ccc} \Gamma_1 & \xrightarrow{\psi} & \Gamma \\ \phi_1 & & \phi \\ \Lambda_1 & \xrightarrow{\theta} & \Lambda \end{array} \tag{8.5}$$

is a pullback in $\mathscr{G}d\mathscr{P}$ and that ϕ is a weak fibration. We can construct in $\mathscr{T}$ a commutative square

$$\begin{tikzcd} B^+\Gamma_1 \arrow[r] \arrow[d, "B^+\phi_1"'] & B^+\Gamma \arrow[d, "B^+\phi"] \\ B^+\Lambda_1 \arrow[r] & B^+\Lambda \end{tikzcd} \tag{8.6}$$

lifting the image of 8.5 under $B^+\colon \mathscr{G}d\mathscr{P} \to \text{HoCW} \simeq \text{Ho}\,\mathscr{T}$, and such that both $B^+\phi$ and $B^+\phi_1$ are fibrations. We must show that the canonical map of $B^+\Gamma_1$ to the pullback X in 8.6 is a homotopy equivalence.

We proceed via a sequence of reductions. First, since both B^+ and pullbacks preserve coproducts we may without loss of generality suppose that Λ_1, Λ and Γ are connected. If ob $\Lambda_1 \ni x_1 \mapsto x \in$ ob Λ then

$$(\Lambda_1^0(x_1, x_1), \Lambda_1^1(x_1, x_1)) \to \Lambda_1 \quad \text{and} \quad (\Lambda^0(x, x), \Lambda^1(x, x)) \to \Lambda,$$

as well as the pullbacks of these inclusions along ϕ and ϕ_1, are full, and so are weak homotopy equivalences. Once more without loss of generality, we may assume that Λ_1 and Λ are topogenic groups.

In order to show that the fibre map $B^+\Gamma_1 \to X$ over the connected space $B^+\Lambda$, is a homotopy equivalence it is enough to show that it induces a homotopy equivalence on the fibre. Thus we may in fact assume $\Lambda_1 = 1$.

The components of Γ_1, in this case, are easily seen to be isomorphic to one another and to be in bijective correspondence with the cosets of the image of $\Gamma^0(y, y)/\Gamma^1(y, y) \to \Lambda^0/\Lambda^1$, that is to say of

$$\pi_1(B^+\Gamma, y) \to \pi_1 B^+\Lambda.$$

But the same description applies to the components of X, which is now the fibre of $B^+\Gamma \to B^+\Lambda$. Thus, we may assume that Γ is also a topogenic group and ϕ is surjective, i.e. is a fibration in $\mathscr{G}p\mathscr{P}$, and Γ_1 is thus also a group.

Finally, recall that the universal coverings of $B^+\Gamma$ and $B^+\Lambda$ are $B^+\Gamma^1$ and $B^+\Lambda^1$ with $B^+\phi$ lifting to $B^+\phi^1$. The fibre of this map is the covering space $\tilde{X}$ of X belonging to the kernel of $\pi_1 X \to \pi_1 B^+\Gamma = \Gamma^0/\Gamma^1$. But by 7.3 this is just $B^+(\ker \phi^1, \Gamma^1)$, the covering group in both instances being the kernel of $\Gamma^0/\Gamma^1 \to \Lambda^0/\Lambda^1$. Thus $B^+(\ker \phi^1, \Gamma^1) \to \tilde{X}$ is a homotopy equivalence, whence $B^+\Gamma \tilde{\to} X$.

9. Homotopy groups of topogenic groups

If G is a topogenic group it seems natural to refer to the homotopy groups of B^+G as homotopy groups of G (but cf. Appendix A). Thus for example $\pi_1 G = G^0/G^1$, in accord with our convention above. We shall see that these homotopy groups may be described completely within the context of group theory.

We cannot of course expect an easy computation: these $\pi_1 G$ are just as general as homotopy groups of spaces. At most we might hope for information in special cases, such as the topogenic groups $(Gl(A), \mathscr{E}(A))$ where $Gl(A)$ is the general linear group, i.e. $Gl(A) = \bigcup_n Gl(n, A)$ of a ring A and $\mathscr{E}(A)$ is the subgroup generated by elementary matrices. Here of course we have $\pi_n(Gl(A), \mathscr{E}(A)) = K_n(A)$, $n = 1, 2, \ldots$. In any case, we exhibit here a new description of these groups.

If G is a topogenic group then B^+G^1 is the universal covering of B^+G, so that $\pi_q B^+G = \pi_q B^+G^1$ for $q > 1$. Thus we may as well confine our attention to perfect groups. If P is a perfect group we define a *topogenic resolution* of P to be a chain complex

$$X = \Big(\cdots \longrightarrow X_n \xrightarrow{d_n} X_{n-1} \longrightarrow \cdots \longrightarrow X_2 \xrightarrow{d_2} X_1 \longrightarrow 1 \longrightarrow 1 \longrightarrow \cdots\Big)$$

of nonabelian groups such that $X_1 = P$ and

(i) X_n is acyclic for $n > 1$,

(ii) X is exact in $\mathscr{P}$,

(iii) each d_q is a weak fibration onto its image $B_{q-1}X$.

It follows at once that $B_q X$ is the derived group of $Z_q X$, the kernel (in $\mathscr{G}p$) of d_q and thus that $H_q X$ is abelian for all q and trivial for $q < 2$.

THEOREM 9.1. *Every perfect group P has a topogenic resolution X and, for any such resolution, $\pi_q P \approx H_q X$ for all q.*

Since B^+X_n is contractible for $n > 1$ it follows from 7.3 that $B^+(Z_nX, B_nX)$ has the homotopy type of $\Omega^{n-1}B^+P$, which implies the latter conclusion. The existence of topogenic resolutions follows, of course, from 8.3 or Theorem B1 of Appendix B.

10. Kan fibrations

A prefatory remark would seem to be in order. The homotopy category of the category of simplical sets, which is equivalent to HoCW or Ho $\mathscr{T}$, appears naturally as a category of fractions but not as a quotient category. However for the subcategory of simplicial sets satisfying Kan's extension condition the dual identification is correct.

These "Kan complexes" may be identified as the relative injectives with respect to a set of morphisms, viz. the inclusions of cones on the boundary in simplices; in fact they are equally well the relative injectives with respect to all inclusions which are weak homotopy equivalences.

This state of affairs has been formalized by Quillen under the name of "model categories". In fact this theory, following the lead of Kan's original treatment, identifies more generally the "relative injectives" over a fixed object as fibrations over that object, the Kan complexes being those fibred over the terminal object.

The categories of topogenic groups and groupoids do not lend themselves to a fully analogous treatment. It is nonetheless profitable to pursue the analogy as far as it will go.

We shall describe as *test morphisms* in $\mathscr{G}d\mathscr{P}$ the injective weak homotopy equivalences $\Phi \to \theta$ with both Φ and θ countable. Up to isomorphism these constitute a set and we shall allow ourselves to speak as though in fact there were only a set of them. The test morphisms in $\mathscr{G}p\mathscr{P}$ are defined in exactly the same way.

We shall say that $\phi\colon \Gamma \to \Lambda$ in $\mathcal{G}d\mathcal{P}$ is a *Kan fibration* if the following "homotopy lifting condition" holds: for any commutative diagram

$$\begin{tikzcd} \Phi \arrow[r, "\gamma"] \arrow[d, "\psi"'] & \Gamma \arrow[d, "\phi"] \\ \Theta \arrow[r, "\lambda"] & \Lambda \end{tikzcd} \tag{10.1}$$

in which ψ is a test morphism and γ is faithful (i.e. injective on each of the sets $\Phi^0(x, y)$) there is a faithful $\mu\colon \theta \to \Gamma$ with $\phi\mu = \lambda$, $\mu\psi = \gamma$.

If $\Gamma \to 1$ is a Kan fibration we say that Γ is a *Kan groupoid* or a groupoid of *Kan type*. These are thus characterized by an extension condition: ${}^f\mathcal{G}d\mathcal{P}(\psi, \Gamma)$ is surjective for ψ a test morphism.

PROPOSITION 10.2. *The following conditions on a morphism $\phi\colon \Gamma \to \Lambda$ are equivalent:*

(i) *ϕ is a Kan fibration;*

(ii) *$\phi^0\colon \Gamma^0 \to \Lambda^0$ is a fibration of groupoids and the homotopy lifting condition of* 10.1 *holds for test morphisms ψ in $\mathcal{G}p\mathcal{P}$;*

(iii) *$\pi\phi\colon \pi\Gamma \to \pi\Lambda$ is a fibration of groupoids and the homotopy lifting condition holds for test morphisms ψ in $\mathcal{G}p\mathcal{P}$.*

To see that (i) implies (ii) it is sufficient to consider test morphisms $1 \to \theta$ where θ is the indiscrete groupoid with two objects. For the reverse implication observe first that it is sufficient to consider Φ (and thus θ) connected. If further ψ is bijective on objects then a lifting of any $\theta(x, x)$ extends uniquely to one on all of θ; if ψ is full then the fact that ϕ^0 is a fibration of groupoids provides a lifting in $\mathcal{G}d$, which however must preserve the congruences, i.e. take θ^1 to Γ^1. But any ψ factors in this fashion.

Evidently (ii) implies (iii). To see that (iii) implies (ii) it will be enough to show that for any $x \in \text{ob}\,\Gamma^0$, $\Gamma^1(x, x) \to \Lambda^1(\phi x, \phi x)$ is surjective. But for any $s \in \Lambda^1(\phi x, \phi x)$ there is, by 5.7, a homomorphism $D \to \Lambda^1(\phi x, \phi x)$ with D acyclic countable whose image contains s. Then the lifting condition applied to the test morphism $1 \to D$ provides an element in $\Gamma^1(x, x)$ mapping onto s.

PROPOSITION 10.3. *The pullback in $\mathcal{G}d\mathcal{P}$ ($\mathcal{G}p\mathcal{P}$) of a Kan fibration along a faithful morphism is again a Kan fibration.*

This is, essentially, an immediate consequence of the definition. But notice that the condition that the pullback be along a faithful morphism is essential (cf. 13.5 below).

The notion of Kan fibration is, of course, stronger than that of weak fibration.

THEOREM 10.4. *A Kan fibration in $\mathcal{G}d\mathcal{P}$ ($\mathcal{G}p\mathcal{P}$) is a weak fibration.*

In view of 10.2,3 we may confine our attention to a Kan fibration $\rho\colon U \to V$ where U and V are perfect groups. We must show that ρ is surjective and that both the kernel N of ρ and its derived group $[N, N]$ are homologically central in U. The former statement is implied by 10.2(ii). The latter is clearly a consequence of the following assertion: if K is a countable subgroup of N and $x \in U$ then for some $y \in N^1$, $C_x | K = C_y | K$.

Now by 5.7 there is a homomorphism $\delta\colon D \to V$ with D acyclic together with an element $d \in D$ such that $\delta d = \rho x$. Applying the lifting condition of 9.1 to the test morphism $(K, 1) \to (K \times D, D)$ we get $\theta\colon K \times D \to U$, extending the inclusion of K in N and satisfying $\rho\theta = \delta pr_D$. Thus for any $x \in U$ there is an $x' = x\theta(1, d)^{-1}$ in N such that $C_{x'} | K = C_x | K$.

Since U is perfect we may write $x = \Pi[u_i, v_i]$. Let K_1 be the smallest group containing K which is normalized by all u_i, v_j. Then K_1 is still countable and there are u_i', v_j' in N such that

$$C_{u_i'} | K_1 = C_{u_i} | K_1, \qquad C_{v_j'} | K_1 = C_{v_j} | K_1.$$

If we set $y = \Pi[u_i', v_i'] \in [N, N]$ then $C_y | K_1 = C_x | K_1$.

11. Kan fibrations, continued

Among the characteristic properties of a model category is the fact that any morphism can be factorized into a weak homotopy equivalence followed by a fibration. We shall show that $\mathcal{G}d\mathcal{P}$ and $\mathcal{G}p\mathcal{P}$ possess the analogous properties.

If Γ is a groupoid and $\Gamma^{\#}$ denotes the morphism category of Γ then the domain and codomain functors $\delta_0, \delta_1\colon \Gamma^{\#} \to \Gamma$ are both equivalences of groupoids and $\langle \delta_0\, \delta_1 \rangle\colon \Gamma^{\#} \to \Gamma \times \Gamma$ is a fibration of groupoids. For any $\phi\colon \Lambda \to \Gamma$ we may construct the pullback

$$\begin{tikzcd} \Lambda^{\#} \arrow[r, "\phi^{\#}"] \arrow[d, "\lambda"'] & \Gamma^{\#} \arrow[d, "\delta_0"] \\ \Lambda \arrow[r, "\phi"] & \Gamma. \end{tikzcd}$$

The unit functor $v\colon \Gamma \to \Gamma^{\#}$ is right inverse to δ_0 and δ_1 and thus, together with 1_Λ, defines a right inverse μ of λ, which is of course an equivalence of categories. But $\delta_1, \phi^{\#}$ is a fibration of groupoids and $\delta_1, \phi^{\#}\mu = \delta_1 v\phi = \phi$.

If Λ and Γ are supplied with perfect congruences these extend canonically to congruences on $\Lambda^{\#}$ and $\Gamma^{\#}$; if ϕ preserves the congruences so also do δ_i, $\phi^{\#}$ and μ. We have accordingly proved the following lemma.

LEMMA 11.1. *Any morphism in $\mathcal{A}d\mathcal{P}$ factors as $\phi\psi$ with ψ an injective weak homotopy equivalence and ϕ^0 a fibration of groupoids.*

We may now proceed to the proof of the following theorem.

THEOREM 11.2. *Any morphism in $\mathscr{G}d\mathscr{P}$ may be factorized as $\phi\psi$ where ψ is an injective weak homotopy equivalence and ϕ is a Kan fibration.*

By 11.1 we may start with a morphism $\theta\colon \Gamma \to \Lambda$ with θ^0 a fibration of groupoids. Let J be the set of commutative diagrams in $\mathscr{G}d\mathscr{P}$

$$\begin{tikzcd} K \arrow[r, "\varkappa"] \arrow[d, "\tau"'] & \Gamma \arrow[d, "\theta"] \\ L \arrow[r, "\lambda"] & \Lambda \end{tikzcd}$$

with τ a test morphism of topogenic groups and $\varkappa$ injective and define Γ_1 by the pushout in $\mathscr{G}d\mathscr{P}$

$$\begin{tikzcd} \coprod_J K \arrow[r, "(\tau)"] \arrow[d, "(\varkappa)"'] & \coprod_J L \arrow[d, "\lambda_1"] \\ \Gamma \arrow[r, "\gamma"] & \Gamma_1 \end{tikzcd}$$

It is easy to see that γ is an injective w.h.e., that Γ_1 is provided with a unique $\theta_1\colon \Gamma_1 \to \Lambda$ such that $\theta_1\gamma = \theta$ and $\theta_1\lambda_1 = (\lambda)$, and that θ_1^0 is a fibration of groupoids.

We now define, for any countable ordinal α,

$$\Gamma_\alpha = \left(\operatorname*{colim}_{\beta<\alpha} \Gamma_\beta\right)_1$$

and set $\Gamma_* = \operatorname{colim} \Gamma_\alpha$. If $\gamma_*\colon \Gamma \to \Gamma_*$ and $\theta_*\colon \Gamma_* \to \Lambda$ are defined to be the appropriate colimits then they provide the required factorization of θ.

COROLLARY 11.3. *If $\phi\colon G \to K$ is a morphism in $\mathscr{G}p\mathscr{P}$ such that $\pi_1\phi\colon \pi_1 G \to \pi_1 K$ is surjective then ϕ may be factorized in $\mathscr{G}p\mathscr{P}$ as an injective weak homotopy equivalence followed by a Kan fibration.*

COROLLARY 11.4. *Any topogenic groupoid (topogenic group) may be imbedded in a Kan groupoid (group) by a weak homotopy equivalence.*

The constructions above are of course all functorial in the appropriate senses.

We remark that 11.2,3 in conjunction with 10.4 proves Proposition 8.3 and thus completes the proof of Theorem 8.2.

12. The homotopy extension theorem

We turn next to the second question raised in Section 3, and show that when K is a countable topogenic group and G a Kan group then morphisms $B^+K \to B^+G$ in $\mathrm{HoCW}^\bullet$ can be represented as homotopy classes of morphisms

$K \to G$, and not merely as classes of fractions. Analogy—albeit partial—with the classical topological state of affairs suggest that we describe Theorem 12.2 below, which expresses this fact, as a homotopy extension theorem.

We shall need the following "general position" lemma.

LEMMA 12.1. *Suppose that*

$$K \xrightarrow{\alpha} U \xrightarrow{\bar{\alpha}} W \xleftarrow{\bar{\beta}} V \xleftarrow{\beta} K$$

is an i-pushout of countable topogenic groups, that G is a Kan group, that ϕ: $W \to G$ and that $\phi\bar{\alpha}$, $\phi\bar{\beta}$ are injective. Then there is an injective ψ: $W \to G$ such that $\psi\bar{\alpha} = \phi\bar{\alpha}$ and $\psi\bar{\beta} \sim_c \phi\bar{\beta}$ (rel K).

Let

$$W \xrightarrow{\phi'} E \xrightarrow{\varepsilon} G$$

factor ϕ through its image E and suppose that D is an acyclic group with $1 \neq d \in D$. In the i-pushout

$$\begin{array}{ccc} K & \xrightarrow{\langle K\ 0\rangle} & K \times D \\ {\scriptstyle \phi'\bar{\alpha}\alpha}\downarrow & & \downarrow{\scriptstyle \gamma} \\ E & \xrightarrow{\theta} & M, \end{array}$$

θ is a weak homotopy equivalence so that ε extends to an injective μ: $M \to G$. Define ψ': $W \to M$ by $\psi'\bar{\alpha} = \theta\phi'\bar{\alpha}$, $\psi'\bar{\beta} = C_{\gamma(1,d)}\theta\phi'\bar{\beta}$. Then ψ' is injective and we may set $\psi = \mu\psi'$.

THEOREM 12.2. *Suppose that υ: $K \to U$ and γ: $K \to G$ in ${}^i\mathcal{G}p\mathcal{P}$, that U is countable, and that G is a Kan group. Then B^+ induces a bijection*

$$\bar{B}^+_{\mathrm{rel}}\colon \mathrm{Ho}_c{}^i\mathcal{G}p\mathcal{P}(\mathrm{rel}\ K)(U, G) \to \mathrm{HoCW}^\bullet(\mathrm{rel}\ B^+K)(B^+_{\mathrm{rel}}U, B^+_{\mathrm{rel}}G).$$

We begin by showing that ${}^i\mathcal{G}p\mathcal{P}(\mathrm{rel}\ K)(U, G)$ maps onto

$$\mathrm{HoCW}^\bullet(\mathrm{rel}\ B^+K)(B^+_{\mathrm{rel}}U, B^+_{\mathrm{rel}}G).$$

Suppose, accordingly, that f: $B^+U \to B^+G$ in $\mathrm{HoCW}^\cdot$ and that $fB^+\upsilon = B^+\gamma$. We must find an injection ψ: $U \to G$ such that $\psi\upsilon = \gamma$ and $B^+\phi = f$.

By 6.4 there is a countable subgroup ι: $W \to G$, containing the image of γ, such that f factorizes as

$$B^+U \xrightarrow{\delta} B^+W \xrightarrow{B^+\iota} B^+G$$

with $gB^+\upsilon = B^+\omega$, where $\gamma = \iota\omega$. Using mapping cylinders as needed we may construct in ${}^i\mathscr{K}^\bullet$ a commutative diagram

$$\text{(12.3)}\qquad \begin{tikzcd} \hat{B}U^0 \arrow[d, "\hat{b}U"] & \hat{B}K^0 \arrow[l, "\hat{B}\upsilon^0"'] \arrow[r, "\hat{B}\omega^0"] \arrow[d, "\hat{b}K"] & \hat{B}W^0 \arrow[d, "\hat{b}W"] \\ \hat{B}^+U \arrow[rr, bend right, "\hat{g}"'] & \hat{B}^+K \arrow[l, "\hat{B}^+\nu"'] \arrow[r, "\hat{B}^+\omega"] & \hat{B}^+W \end{tikzcd}$$

which lifts the corresponding diagram in HoCW$^\bullet$ defined by the data above.

We apply to 12.3 the constructions of Section 4. Since MW is countable we may imbed it in a countable acyclic group, say by $\delta\colon MW \to D$. Then $\delta(M\omega)$ imbeds MK in D and we may construct the i-pushout

$$MK \xrightarrow{\langle \varepsilon_K\ \delta(M\omega)\rangle} K \times D \xrightarrow{\alpha} p \xleftarrow{\mu} MU \xleftarrow{M\upsilon} MK$$

and define $\rho\colon P \to U$ by $\rho\alpha = \upsilon pr_K$, $\rho\mu = \varepsilon_U$. Note that μ is a weak homotopy equivalence.

By imbedding the pushout in a countable acyclic group E we may, further, construct a commutative square

$$K \xrightarrow{\upsilon} U \xrightarrow{\sigma} E \xleftarrow{\tau} P \xleftarrow{\alpha\langle K\ 0\rangle} K$$

in ${}^i\mathscr{G}p\mathscr{P}$.

We now assemble these constructions into a commutative diagram of countable groups in ${}^i\mathscr{G}p\mathscr{P}$:

$$\begin{tikzcd} U \times E \arrow[d, "\langle \rho\ \tau\rangle"'] & K \arrow[l, "\langle \upsilon\ \sigma\upsilon\rangle"'] \arrow[r, "\omega"] \arrow[d, "\langle K\ 0\rangle"] & W \arrow[d, "\langle W\ 0\rangle"] \\ P \arrow[d, "\mu"'] & K \times D \arrow[l, "\alpha"'] \arrow[r, "\omega \times D"] \arrow[d, "\langle \varepsilon_K\ \delta(M\omega)\rangle"] & W \times D \arrow[d, "\langle \varepsilon_W\ \delta\rangle"] \\ MU \arrow[d, "\beta\upsilon"'] & MK \arrow[l, "M\upsilon"'] \arrow[r, "M\omega"] \arrow[d, "\beta K"] & MW \arrow[d, "\beta W"] \\ L\hat{B}^+U \arrow[rr, bend right, "L\hat{g}"'] & L\hat{B}^+K \arrow[l, "L\hat{B}^+\upsilon"'] \arrow[r, "L\hat{B}^+\omega"] & L\hat{B}^+W \end{tikzcd}$$

Since G is a Kan group we may construct injections of the terms of 11.4 as follows. First, $i: W \to G$ extends to $i': W \times D \to G$, since $\langle W\ 0\rangle$ is a weak homotopy equivalence. Similarly $i'\langle \varepsilon_W \delta\rangle$ extends to $\lambda: L\hat{B}^+ W \to G$. Next, using the general position Lemma 12.1, we find $\theta: P \to G$ such that $\theta\alpha = i'(\omega \times D)$ and $\theta\mu \sim_c \lambda(L\hat{g})\beta_U$ (rel MK). Finally θ extends to $\phi: U \times E \to G$. We claim that $\psi = \phi\langle U\ \sigma\rangle$ is the required morphism.

That $\psi\upsilon = i\omega = \gamma$ follows at once from the commutativity of 12.4. If we apply B^+ to 12.4 all the vertical arrows become isomorphisms and, by 4.2, the following equalities hold:

$$\begin{aligned} &(B^+\langle W\ 0\rangle)^{-1}B^+\langle \varepsilon_W\ \delta\rangle(B^+\beta_W)^{-1} = \bar{t}(\hat{B}^+ W) \\ (12.5)\qquad &(B^+\langle K\ 0\rangle)^{-1}B^+\langle \varepsilon_K\ \delta(M\omega)\rangle(B^+\beta_K)^{-1} = \bar{t}(\hat{B}^+ K) \\ &(B^+\langle U\ 0\rangle)^{-1}B^+\langle \rho\ \tau\rangle B^+\mu(B^+\beta_U)^{-1} = \bar{t}(\hat{B}^+ U). \end{aligned}$$

Furthermore $(B^+\theta)(B^+\mu) = B^+(\lambda(L\hat{g})\beta_U)$ by 5.6 so that, in view of the acyclicity of E,

$$\begin{aligned} B^+\psi &= (B^+\phi)B^+\langle U\ \sigma\rangle = (B^+\theta)(B^+\rho)^{-1} \\ &= (B^+\theta)(B^+\mu)(B^+\varepsilon_\upsilon)^{-1} \\ &= (B^+\lambda)(B^+L\hat{g})\bar{t}(\hat{B}^+ U) \quad \text{(by 4.3)} \\ &= (B^+\lambda)\bar{t}(\hat{B}^+ W)g \quad \text{(by the naturality of } \bar{t}) \\ &= (B^+ i)g = f \quad \text{(by 12.5)}. \end{aligned}$$

It remains only to show that if $\phi, \phi': U \to G$ are injective morphisms extending γ and $B^+_{\text{rel}}{}^{\phi+} = B^+_{\text{rel}\,\phi'}$ then $\phi \sim_c \phi'$ (rel K). Let us construct i-pushouts

$$K \xrightarrow{\ \upsilon\ } U \xrightarrow{\ v\ } V \xleftarrow{\ v'\ } U \xleftarrow{\ \upsilon\ } K,$$

$$K \xrightarrow{\langle K\ 0\rangle} K \times D \xrightarrow{\ \delta\ } W \xleftarrow{\ \omega\ } U \xleftarrow{\ v\ } K$$

with D acyclic and $1 \neq d \in D$. The morphism ω is a weak homotopy equivalence and $\delta(1, d)$ centralizes the image of K in W. If $\theta: V \to W$ is defined by $\theta v = \omega$, $\theta v' = C_{\delta(1,d)}\omega$ then it lies in ${}^i\mathscr{G}p\mathscr{P}$(rel K).

By the general position Lemma 12.1 there is a $\psi: V \to G$ in ${}^i\mathscr{G}p\mathscr{P}$(rel K) such that $\psi v = \phi$, $\psi v' \sim_c \phi'$(rel K). Since $B^+_{\text{rel}}\phi = B^+_{\text{rel}}\phi'$ there is an $f: B^+ W \to B^+ G$ in HoCW$\cdot$(rel B^+K) with $f(B^+\theta) = B^+\psi$. Our surjectivity result, applied in ${}^i\mathscr{G}p\mathscr{P}$(rel V), implies that $f = B^+_{\text{rel}}\mu$ for some $\mu: W \to G$ with $\mu\theta = \phi$. But then $\phi' \sim_c \psi v' = C_{\mu\delta(1,d)}\phi$(rel K).

The unpointed analogue of 12.2 follows easily.

THEOREM 12.6. *If Λ is a locally countable topogenic groupoid and Γ is a groupoid of Kan type then B^+ induces a bijection*

$$\mathrm{Ho}_c{}^f\mathscr{G}d\mathscr{P}(\Lambda, \Gamma) \to \mathrm{HoCW}(B^+\Lambda, B^+\Gamma).$$

Since both functors take coproducts in the contravariant variable into products we may, without loss of generality, suppose Λ connected. Thus restricted, both functors preserve coproducts in the covariant variable, and we may accordingly suppose Γ connected as well. But in this case we may evidently replace Λ, Γ by the groups $\Lambda(x, x) = (\Lambda^0(x, x), \Lambda^1(x, x))$, $\Gamma(y, y)$, for any x, y. Thus we may as well assume that both Λ and Γ are groups. The conclusion now follows at once from 12.2 and 5.8.

13. Some applications of the homotopy extension theorem

In Section 9 above we gave a description within $\mathscr{G}p\mathscr{P}$ of the homotopy groups of a topogenic group. We now give a quite different one. From [1] we may deduce the existence of topogenic groups $S_1 = (Z, 1)$, $S_2, \ldots,$ geometrically finite and thus countable, with B^+S_n of the homotopy type of S^n.

Since any topogenic group G imbeds by a weak homotopy equivalence in a Kan group $\hat{G}$ we may describe its homotopy groups in the following way.

PROPOSITION 13.1. $\pi_n B^+ G \approx \mathrm{Ho}_c {}^i\mathscr{G}p\mathscr{P}(S_n, \hat{G})$, i.e. *$\pi_n G$ consists of $\hat{G}^1$ conjugacy classes of imbeddings of S_n in $\hat{G}$.*

We have here omitted the description of the group operation in

$$\mathrm{Ho}_c {}^i\mathscr{G}p\mathscr{P}(S_n, \hat{G}),$$

which may however easily be supplied.

Of course this gives us yet another "algebraic" description of algebraic K-theory.

If Γ is a Kan groupoid then $B^+\Gamma$ may be supplied with a basepoint by choosing any $x \in \mathrm{ob}\,\Gamma$; we see immediately that

$$\pi_n(B^+\Gamma, x) \approx \mathrm{Ho}_c {}^i\mathscr{G}p\mathscr{P}(S_n, \Gamma(x, x)). \tag{13.2}$$

Cohomology with constant coefficients, at least for countable groups, may be treated in the following way. For any abelian group A and $n = 2, \ldots$ there is a Kan group $\mathbf{K}(A, n)$ such that $B^+\mathbf{K}(A, n)$ has the homotopy type of the space $\mathbf{K}(A, n)$.

PROPOSITION 13.3. *If W is a countable group then*

$$H^n(W; A) \approx \mathrm{Ho}_c {}^i\mathscr{G}p\mathscr{P}((W, 1), \mathbf{K}(A, n)).$$

In other words, cohomology classes are just conjugacy classes of imbeddings of W in $\mathbf{K}(A, n)^0 = \mathbf{K}(A, n)^1$. Once again, we have omitted the description of the group structure. We omit also a description of cohomology with coefficients in a module as conjugacy classes of cross-sections of a suitable Kan fibration.

We observe next that inner automorphisms are highly transitive in Kan groups.

PROPOSITION 13.4. *If G is a perfect Kan group then any isomorphism of countable free subgroups of G is the restriction of an inner automorphism. In particular, any two elements of infinite order are conjugate..*

This is in general false for elements of finite order, e.g. in the group $\mathbf{K}(Z/m, 3)$. But the transivity is sufficient to ensure the following fact.

PROPOSITION 13.5. *A perfect Kan group is simple. Hence any perfect group imbeds by a weak homotopy equivalence in a simple group.*

If N is a proper normal subgroup of the perfect Kan group G then either all elements of infinite order in G are in N or they are all in $G - N$. We may exclude the former case at once, since if $x \in G - N$ is of order n we may imbed $Z/n \times D$, with D a torsion free acyclic group, in G with a generator of Z/n going to x. If $1 \neq d \in D$ goes to y then xy is an element of infinite order in $G - N$.

But the latter case is similarly excluded. Suppose $x \in N$ is of order n. Then there is an injection $Z/n * Z \to G$ taking a generator y of Z/n to x. But the normal subgroup of $Z/n * Z$ generated by y contains elements of infinite order.

We conclude with a partial converse to a theorem of [1], which asserts that algebraically closed groups are acyclic.

PROPOSITION 13.6. *A perfect acyclic Kan group is algebraically closed.*

For if G is such a group then B^+G is contractible. Thus if $K \subset W$ are countable groups any imbedding of K (i.e. of $(K, 1)$) in G extends to an imbedding of W. But this clearly implies algebraic closure.

14. Function spaces

Although the categories CW and $\mathscr{T}$ do not have function spaces the category HoCW is cartesian closed; the adjoint $\mathscr{T}(X, -)$ to the product functor $- \times X$ may be described as the functor taking a space Y to the geometric realization of the singular complex of the function space Y^X in the category of compactly generated spaces. We shall see there that under favorable circumstances this function space can be directly computed within the category of topogenic groupoids.

Let us begin by recalling that the category $\mathscr{G}d$ of groupoids is cartesian closed, the adjoint to the product being given by the groupoids Γ^Λ of functors and natural transformations from Λ to Γ. If Λ and Γ are topogenic groupoids we define a topogenic groupoid $\mathscr{M}(\Lambda, \Gamma) = (\mathscr{M}^0(\Lambda, \Gamma), \mathscr{M}^1(\Lambda, \Gamma))$ by letting $\mathscr{M}^0(\Lambda, \Gamma)$ be the full subgroupoid of $\Gamma^{0\Lambda^\bullet}$ containing $\mathscr{G}d\mathscr{P}(\Delta, \Gamma)$ as its objects and, for $\phi: \Lambda \to \Gamma$, taking as $\mathscr{M}^1(\Lambda, \Gamma)(\phi, \phi)$ the largest perfect subgroup of

$$\{\sigma: \phi \to \phi \mid \forall_{x \in \mathrm{ob}\,\Lambda}\ \sigma_x \in \Gamma^1(\phi x, \phi x)\}.$$

PROPOSITION 14.1. *If Λ is a topogenic groupoid then $- \times \Lambda$ is left adjoint to $\mathscr{M}(\Lambda, —): \mathscr{G}d\mathscr{P} \to \mathscr{G}d\mathscr{P}$. Thus $\mathscr{G}d\mathscr{P}$ is cartesian closed.*

This is a straightforward, if tedious, verification.

The subcategory ${}^f\mathcal{G}d\mathcal{P}$ containing the faithful morphisms is closed under products in $\mathcal{G}d\mathcal{P}$ containing the faithful morphisms is closed under products in $\mathcal{G}d\mathcal{P}$ but these are not in general products in the subcategory. They do however make ${}^f\mathcal{G}d\mathcal{P}$ a monoidal category. While ${}^f\mathcal{G}d\mathcal{P}$ is not monoidal closed a fragment of such structure does nevertheless exist. Let us define, for Λ, Γ in $\mathcal{G}d\mathcal{P}$, ${}^f\mathcal{M}(\Lambda, \Gamma) = ({}^f\mathcal{M}^0(\Lambda, \Gamma), {}^f\mathcal{M}^1(\Lambda, \Gamma))$ to consist of the full subcategories of $\mathcal{M}^i(\Lambda, \Gamma)$ containing the faithful morphisms $\Lambda \to \Gamma$. The following statement is easily verified.

PROPOSITION 14.2. *If Λ is connected and centerless then*

$${}^f\mathcal{M}(\Lambda, -)\colon {}^f\mathcal{G}d\mathcal{P} \to {}^f\mathcal{G}d\mathcal{P}$$

is right adjoint to $- \times \Lambda$.

Both hypotheses are necessary, the first because $\times$ is not a product in ${}^f\mathcal{G}d\mathcal{P}$, the second as in 5.4. Pursuing the latter point we observe further that if K and G are topogenic groups and $\phi\colon K \to G$ is injective then G can be considered as an object of $\mathcal{G}p\mathcal{P}(\text{rel } K)$ and

$$(14.3) \qquad {}^f\mathcal{M}(K, G)(\phi, \phi) = ZG$$

where Z is the centralizer functor of Section 5.

Now for Λ, Γ in $\mathcal{G}d\mathcal{P}$ we have

$$\begin{aligned} 1_{\mathcal{M}(\Lambda,\Gamma)} \in \mathcal{G}d\mathcal{P}(\mathcal{M}(\Lambda, \Gamma), \mathcal{M}(\Lambda, \Gamma)) &\approx \mathcal{G}d\mathcal{P}(\mathcal{M}(\Lambda, \Gamma) \times \Lambda, \Gamma) \\ &\to \mathrm{HoCW}(B^+\mathcal{M}(\Lambda, \Gamma) \times B^+\Lambda, B^+\Gamma) \\ &\approx \mathrm{HoCW}(B^+\mathcal{M}(\Lambda, \Gamma), \mathcal{T}(B^+\Lambda, B^+\Gamma)) \end{aligned}$$

giving us a natural transformation $B^+\mathcal{M}(\Lambda, \Gamma) \to \mathcal{T}(B^+\Lambda, B^+\Gamma)$. This restricts to a map $w_{\Lambda,\Gamma}\colon B^+\, {}^f\mathcal{M}(\Lambda, \Gamma) \to \mathcal{T}(B^+\Lambda, B^+\Gamma)$ which we shall think of as a natural transformation in the covariant variable, restricted to ${}^f\mathcal{G}d\mathcal{P}$.

THEOREM 14.4. *Suppose that Λ is locally countable and centerless and that Γ is a Kan groupoid.*

(i) *If Λ is connected then ${}^f\mathcal{M}(\Lambda, \Gamma)$ is a Kan groupoid.*

(ii) *In general, $w_{\Lambda,\Gamma}\colon B^+\, {}^f\mathcal{M}(\Lambda, \Gamma) \to \mathcal{T}(B^+\Lambda, B^+\Gamma)$ is an isomorphism in* HoCW.

Let us start by observing that since both functors take coproducts in the contravariant variable to products in HoCW it is sufficient to consider connected Λ in (ii). But then both preserve coproducts in the covariant variable. Thus we may also, without loss of generality, suppose Γ connected, and indeed, just as in 12.6, that $\Lambda = K$, $\Gamma = G$ are topogenic groups.

It follows immediately from 12.6 that $w_{\Lambda,\Gamma}$ maps the components of $B^+\, {}^f\mathcal{M}(K, G)$ bijectively onto those of $\mathcal{T}(B^+K, B^+G)$. We may pick basepoints by choosing an injective $\phi\colon K \to G$; and complete the proof of the theorem by showing that ${}^f\mathcal{M}(K, G)(\phi, \phi) = ZG$ is a Kan group and that for $q = 1, 2 \ldots,$

$$\pi_q B^+ZG \approx \pi_q(\mathcal{T}(B^+K, B^+G), \bar{\phi})$$

where $\bar{\phi}$ is the basepoint corresponding to ϕ, i.e. the map $B^+\phi\colon B^+K \to B^+G$.

Suppose first that $U \to V$ is a test morphism. Then by (5.4),

$${}^i\mathcal{G}p\mathcal{P}(V, ZG) \to {}^i\mathcal{G}p\mathcal{P}(U, ZG)$$

becomes

$${}^i\mathcal{G}p\mathcal{P}(\text{rel } K)(V \times K, G) \to {}^i\mathcal{G}p\mathcal{P}(\text{rel } K)(U \times K, G).$$

But $U \times K \to V \times K$ is also a test morphism, so that this is surjective. This completes the proof of assertion (i).

By 12.2, 13.2 on the other hand

$$\begin{aligned}\pi_q B^+ZG &= \mathrm{Ho}_c\,{}^i\mathcal{G}p\mathcal{P}(S_q, ZG)\\ &\approx \mathrm{Ho}_c\,{}^i\mathcal{G}p\mathcal{P}(\text{rel } K)(S_q \times K, G) \quad \text{by 5.4}\\ &\approx \mathrm{HoCW}^\bullet(\text{rel } B^+K)(S^q \times B^+K, B^+G)\\ &\approx \mathrm{HoCW}^\bullet(S^q, \mathcal{T}(B^+K, B^+G))\\ &= \pi_q(\mathcal{T}(B^+K, B^+G), \bar{\phi}).\end{aligned}$$

This gives a reasonably perspicuous construction of a large class of unpointed function spaces within $\mathcal{G}d\mathcal{P}$. A similarly functorial construction of pointed function spaces does not seem easily attainable; indeed the same difficulty already arises for the smash product. However the pointed function space is just the homotopy fibre of the canonical map of the unpointed function space to the codomain, so that it may nevertheless be expressed "within" $\mathcal{G}p\mathcal{P}$ as the homotopy fibre of the inclusion $ZG \to G$.

Appendix A. Central extensions

Kervaire's paper [7] on the Steinberg group and the Schur multiplicator is a standard source for the homology theory of central extensions, and in particular for the statement cited above (Section 3). We should like here to sketch an alternate route to some of these results, and to comment, in the light of subsequent developments, on some of the terminology.

Suppose that $A \to E \to G$ is a central group extension with G perfect. Applying the standard classifying-space functor B we get a fibration $BA \to BE \to BG$ whose Serre spectral sequence is just the Hochschild-Serre spectral sequence of the extension, providing Kervaire's principal tool in investigation the homology of the extension.

But if we recall that B preserves products we may conclude that BA is an abelian topological group and that (since $A \times E \to E$ is a homomorphism) $BA \to BE \to BG$ is a principal fibre bundle. Thus $BE \to BG$ is induced by a homotopy class of maps $f\colon BG \to B^2A \approx K(A, 2)$, and BE is the homotopy fibre of f. The E^2-term of the Serre spectral sequence associated with *this* fibration is, in low degrees,

$$\begin{array}{ccccc} H_4E & & & & \\ H_3E & 0 & & & \\ H_2E & 0 & A\otimes H_2E & & \\ H_1E & 0 & A\otimes H_1E & 0 & \\ Z & 0 & A & 0 & H_4(A,2). \end{array}$$

From this we can read off the exact sequence

$$A\otimes H_1E\to H_2E\to H_2G\to A\to H_1E\to 0.$$

In particular if E is perfect then $H_2E\to H_2G$ is injective. The "universal" central extension has $A=H_2G$ and $\alpha=1$, giving $H_1E=0$, $H_2E=0$ the further exact sequence

$$H_4E\to H_4G\to H_4(A,2)\to H_3E\to H_3G\to 0.$$

Kervaire proposes to call this universal central extension the "universal covering" of the perfect group G and write accordingly $\pi_1G=H_2G$, $\pi_2G=\pi_2E=H_3E$. In our present context it seems clear that this numbering is unfortunate: a perfect group G ought rather to be regarded as a simply connected topogenic groupoid so that $\pi_1G=1$, $\pi_2G=\pi_2B^+G=H_2G$ and H_3E, in the universal central extension, is $\pi_3G=\pi_3B^+G$. The interpretation of E, then, is not the universal covering, but the 2-connected covering.

Appendix B. An alternative existence theorem for weak fibrations

Theorem 8.3 asserts the existence of "sufficiently many" weak fibrations in $\mathcal{G}d\mathcal{P}$ and $\mathcal{G}p\mathcal{P}$. We outline here a very different argument for a variant of this theorem.

THEOREM B1. *If Λ is a topogenic groupoid and $p:X\to B^+\Gamma$ in* Ho $\mathcal{T}$ *then there is a weak fibration $\phi:\Gamma\to\Lambda$ provided with a homotopy equivalence $f: X\to B^+\Gamma$ such that $(B^+\phi)f=p$.*

We may without loss of generality suppose that Λ is a topogenic group K and that p is represented by a fibration in $\mathcal{T}$. The pullback of p along $b_K: BK^0\to B^+K$ is then of the fibre homotopy type of a fibre bundle associated with the universal covering of BK^0 and thus with structure-group K^0. We may further suppose that the fibre is the geometrical realization $|V|$ of a simplicial complex V on which K^0 operates.

The universal covering of BK^0, on the other hand, is just $B\hat{K}^0$, where $\hat{K}^0$ is the indiscrete groupoid whose objects are the elements of K^0, the covering projection being $B\eta$, where η: $\hat{K}^0\to K^0$ identifies K^0 as the orbit-space in $\mathcal{G}d$ of $\hat{K}^0$ under the obvious action of K^0. Thus the bundle in question is $(B\hat{K}^0\times |V|)/K^0\to BK^0$.

Now in [1] there was constructed a functor T^0: ${}^iK\to\mathcal{G}d$ (there denoted by L), together with a natural transformation $BT^0\to|\quad|$ in CW which is always a homological equivalence. If we set $\Phi^0 = T^0 V$ then Φ^0 has a K^0 action and we may construct the "associated bundle"

$$(\hat{K}^0 \times \Phi^0)/K^0 = \Gamma^0 \xrightarrow{\phi^0} K^0.$$

The natural transformation $BT^0 \to |\quad|$ provides a K^0-equivariant map $v\colon B\Phi^0 \to |V|$ and we have thus a fibre map g,

$$\begin{tikzcd} B\Gamma^0 \arrow[r, "g"] \arrow[d, "B\phi^0"'] & X \arrow[d, "p"] \\ BK^0 \arrow[r, "b_K"] & B^+K \end{tikzcd}$$

which induces a homological equivalence on the fibre and thus, since b_K is a homological equivalence, must itself be a homological equivalence, by the comparison theorem for the Serre spectral sequences.

Now let Γ^1 and Φ^1 be the congruences in Γ^0 and Φ^0 induced by the compositions

$$\Gamma^0 \approx \pi B\Gamma^0 \xrightarrow{\pi g} \pi X$$

$$\Phi^0 \approx \pi B\Phi^0 \xrightarrow{\pi v} \pi|V|.$$

Then $\Gamma = (\Gamma^0, \Gamma^1)$, $\Phi = (\Phi^0, \Phi^1)$ are topogenic groupoids and g, v induce homotopy equivalences $B^+\Gamma \to X$, $B^+\Phi \to |V|$.

Furthermore $\phi^0(\Gamma^1) \subset K^0$ since $\Gamma^0 \to K^0 \to K^0/K^1 = \pi_1 K$ factors through πg, so that ϕ^0 induces $\phi\colon \Gamma \to K$. We shall have completed our argument when we show that ϕ is a weak fibration in $\mathscr{G}d\mathscr{P}$. Since ϕ^0 is evidently a fibration of groupoids, we need only show that for any $x \in \mathrm{ob}\,\Gamma^0$,

$$\phi_x^1\colon \Gamma^1(x, x) \to K^1$$

is a fibration in $\mathscr{P}$.

In the diagram

$$\begin{tikzcd} & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & & \\ & \Phi^1(x, x) \arrow[r] \arrow[d] & \Gamma^1(x, x) \arrow[r] \arrow[d] & K^1 \arrow[d] & & \\ 1 \arrow[r] & \Phi^0(x, x) \arrow[r] \arrow[d] & \Gamma^0(x, x) \arrow[r] \arrow[d] & K^0 \arrow[r] \arrow[d] & \pi_0\Phi \arrow[r] & * \\ & \pi\Phi(x, x) \arrow[r] \arrow[d] & \pi\Gamma(x, x) \arrow[r] \arrow[d] & K/K^1 \arrow[r] \arrow[d] & \pi_0\Phi \arrow[r] & * \\ & 1 & 1 & 1 & & \end{tikzcd}$$

the columns and the second and third rows are exact. A diagram-chase shows that ϕ_x^1 is surjective.

Now $B^+\phi_x^1$ is, up to homotopy, the universal covering of

$$(B^+\Gamma, x) \to B^+K.$$

Thus its homotopy fibre is the covering space of $B^+(\Phi^0(x, x))$ corresponding to the kernel of $\Phi^0(x, x)/\Phi^1(x, x) \to \Gamma^0(x, x)/\Gamma(x, x)$, which is to say that it is just $B^+(\Gamma^1(x, x) \cap \Phi^0(x, x), \Phi^1(x, x))$. The conclusion now follows from 7.2.

COROLLARY B2. *If K is a topogenic group, $p: X \to B^+K$ in* HoCW$^\bullet$ *and π_1 f is surjective then there is a weak fibration $\phi: G \to K$ in $\mathscr{G}p\mathscr{P}$ provided with a homotopy equivalence $f: X \to B^+G$ such that $(B^+\phi)f = p$.*

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