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A REMARK ON FIBRE HOMOTOPY EQUIVALENCES

BY

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1. Introduction

Let $\mathscr{L}(E)$ denote the group of all (non-based) fibre homotopy classes of fibre homotopy equivalences of the fibration (E, p, B). Y. Nomura [7] has studied this group when (E, p, B) is a principal fibre space in the restricted sense, and established some exact sequences.

In this note we study this group for the relative principal fibration

$$(P_f, p, B) \quad (p^{-1}(*) = \Omega Y),$$

and generalize Nomura's result. We obtained the following result.

THEOREM. *The following sequence of groups and maps is exact:*

$$1 \to \operatorname{Ker} i^* \xrightarrow{\Delta} \mathscr{L}_0(P_f) \xrightarrow{j_0} \mathscr{E}(\Omega Y), \text{ where } i^*: [P_f, \Omega_D Z]_D \to [\Omega Y, \Omega Y].$$

In particular, if $Y = K(G, n+1)$ $(n \geq 1)$, *the sequence of groups and homomorphisms*

$$1 \to H^n(B; G) \xrightarrow{\Delta} \mathscr{L}_0(P_f) \xrightarrow{j_0} \mathscr{E}(\Omega Y) = \operatorname{Aut} G$$

is exact, where $H^n(B; G)$ *is the local coefficient cohomology induced by* $\phi: \pi_1(B) \to \operatorname{Aut} G$ *and* $\mathscr{L}_0(P_f) = \mathscr{L}(P_f)$ *if* $n \geq 2$.

These results apply to a fairly large class of fibrations (cf. [6]), especially to the stage of Postnikov-systems of non-simple spaces and fibrations. Specific examples are worked out in Section 5.

All spaces have well-pointed base points. Maps and homotopies preserve base points unless otherwise stated. All spaces are assumed to have the homotopy type of a connected CW-complex.

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2. Based fibre homotopy equivalences

Let $\mathscr{L}_0(E)$ denote the group of based fibre homotopy classes of based fibre homotopy equivalences of the fibration (E, p, B) $(p^{-1}(*) = F)$. Then we have the following:

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PROPOSITION 2.1. *$\pi_1(F)$ operates on $\mathscr{L}_0(E)$ on the right, and $\mathscr{L}(E)$ is isomorphic to the quotient.*

Proof. This follows from standard arguments along the lines of [9, p. 380].

3. Relative principal fibrations

Let $Z \to D$ be the arbitrary Hurewicz fibration; let $Z \in \mathrm{Top}\,(D = D)$ and $P_D Z \to Z \in \mathrm{Top}\,(D = D)$ be the canonical path-loop fibration in the category. Let $B \in \mathrm{Top}\,(* \to D)$ and $f\colon B \to Z \in \mathrm{Top}\,(* \to D)$. Then f induces a fibration $p\colon P_f = B \times_Z P_D Z \to B$, which is called a relative principal fibration. This is a fibration with fibre ΩY, where Y is the fibre of $Z \to D$ (cf. [4], [5], [15] and [1]).

$$\begin{array}{ccccc}
\Omega Y & \longrightarrow & \Omega Y & & \\
\Big\downarrow i & & \Big\downarrow & & \\
P_f & \longrightarrow & P_D Z & \supset & \Omega_D Z \\
\Big\downarrow p & & \Big\downarrow & & \\
B & \xrightarrow{f} & Z & \longrightarrow & D
\end{array}$$

LEMMA 3.1 (cf. [13, Lemma 1.7]). *P_f is well-pointed.*

Proof. If Y is well-pointed, the mapping space Y^I is well-pointed [11, Lemma 4]. Also the fibre of the fibration

$$Y^I \to Y \times Y\colon \lambda \to (\lambda(0), \lambda(1)),$$

that is, ΩY is well-pointed since $Y \times Y$ is well-pointed [11, Lemma 6]. Since ΩY and B are well-pointed, P_f is well-pointed [10, Theorem 12]. Q.E.D.

Because (P_f, p, B) is a relative principal fibration, we have a D-map $v\colon \Omega_D Z \times_D P_f \to P_f$ that defines the following action for any B-space X (cf. [4] and [5]):

$$v_*\colon [X, \Omega_D Z]_D \times [X, P_f]_B \to [X, P_f]_B. \tag{3.2}$$

Also we have a D-map $h\colon P_f \times_B P_f \to \Omega_D Z$, which is called a relative primary difference (cf. [1]). The map h induces a map

$$h_*\colon [X, P_f]_B \times [X, P_f]_B \to [X, \Omega_D Z]_D. \tag{3.3}$$

Given a B-map $v\colon X \to P_f$, we define the mappings Δ_v and Γ_v by

$$\Delta_v = v_*(\ ,[v]) \quad \text{and} \quad \Gamma_v = h_*([v],\ ).$$

PROPOSITION 3.4 (cf. [1, Theorem (2.2.11)]). *Let $u\colon X \to B$ be a (based) map. If u has a lifting $v\colon X \to P_f$, the map Δ_v gives a bijection between $[X, \Omega_D Z]_D$ and $[X, P_f]_B$.*

Proof. It is easy to check that Δ_v and Γ_v are inverse bijections. Q.E.D.

Let $u = p$: $P_f \to B$ and $v = 1$: $P_f \to P_f$. Then we have the following corollary, which is a generalization to relative principal fibrations of the result of D. W. Kahn [3].

COROLLARY 3.5 (cf. [3, Lemma 2.3]). *Δ_1 gives a 1-1 correspondence between $\mathscr{L}_0(P_f)$ and a subset of $[P_f, \Omega_D Z]_D$. Also $\mathscr{L}_0(P_f)$ is equivalent to*

$$\{[w] \in [P_f, \Omega_D Z]_D \mid i^*w + 1_{\Omega Y}: \Omega Y \to \Omega Y \text{ is a homotopy equivalence}\}.$$

Consider the following sequence of groups:

$$(3.6) \qquad [\Omega Y, \Omega Y] \xleftarrow{i^*} [P_f, \Omega_D Z]_D \xleftarrow{p^*} [B, \Omega_D Z]_D.$$

We define the map Δ: Ker $[i^*: [P_f, \Omega_D Z]_D \to [\Omega Y, \Omega Y]] \to \mathscr{L}_0(P_f)$ by

$$\Delta(a) = v_*\{a, 1_{P_f}\} \quad (a \in [P_f, \Omega_D Z]_D).$$

$$\begin{tikzcd}
\Omega Y \arrow[r, "\{\bar{a}, 1\}"] \arrow[d, "i"] & \Omega Y \times \Omega Y \arrow[r, "\bar{v}"] \arrow[d, "i"] & \Omega Y \arrow[d, "i"] \\
P_f \arrow[r, "\{a, 1\}"] \arrow[dd, "p"] & \Omega_D Z \times_D P_f \arrow[r, "v"] \arrow[d] & P_f \arrow[dd, "p"] \\
& P_f \arrow[d, "p"] & \\
B \arrow[r, "1"] & B \arrow[r, "1"] & B
\end{tikzcd}$$

Since $\bar{a}$: $\Omega Y \to \Omega Y$ is null-homotopic, the restriction of the map $v_*\{a, 1_{P_f}\}$ to the fibre ΩY is homotopic to the identity. Hence $v_*\{a, 1_{P_f}\}$ is a fibre homotopy equivalence by the theorem of A. Dold [2, Theorem 6.3]. Since P_f is well-appointed by Lemma 3.1, this map is a based fibre homotype equivalence (cf. [1, Lemma (1.1.9)]). Therefore the map Δ is well defined.

We have the sequence of groups and maps

$$\text{Ker } i^* \xrightarrow{\Delta} \mathscr{L}_0(P_f) \xrightarrow{J_0} \mathscr{E}(\Omega Y),$$

where $\mathscr{E}(\Omega Y)$ is a group of based homotopy classes of based homotopy equivalences of ΩY.

PROPOSITION 3.7. *The map Δ is injective.*

Proof. By Corollary 3.5, $\mathscr{L}_0(P_f)$ is equivalent to a subset of $[P_f, \Omega_D Z]_D$. By definition, Δ is injective. Q.E.D.

4. Proof of the theorem

THEOREM 4.1. $\operatorname{Im} \Delta = \operatorname{Ker} J_0$.

Proof. $J_0 \Delta = 1$ is evident. Take a based fibre homotopy equivalence $g: P_f \to P_f$ such that $J_0(g) \simeq 1$ rel $*$. By Proposition 3.4, there exists a D-map $\omega: P_f \to \Omega_D Z$ such that $v\{\omega, 1_{P_f}\}$ is based fibre homotopic to g. Then

$$v\{\omega i, i\} = v\{\omega, 1_{P_f}\} i \simeq gi = ig_0 \simeq i \quad (\ : \text{based fibre homotopic}).$$

By Proposition 3.4, $v_*(\ \ , [i]): [\Omega Y, \Omega Y] = [\Omega Y, \Omega_D Z]_D \to [\Omega Y, P_f]_B$ is a bijection. Therefore $\omega i \simeq *$ rel $*$, that is, $\omega \in \operatorname{Ker} i^*$. Q.E.D.

THEOREM 4.2. *Δ is a homomorphism of groups on the image of*

$$p^*: [B, \Omega_D Z]_D \to [P_f, \Omega_D Z]_D.$$

Proof. The proof is the same as that of Theorem 3.5 in [13] by using Lemma 1.6 in [15]. Q.E.D.

THEOREM 4.3 (cf. [7, Theorem 2.2]). *The following sequence of groups and maps is exact:*

$$1 \longrightarrow \operatorname{Ker} i^* \xrightarrow{\Delta} \mathscr{L}_0(P_f) \xrightarrow{J_0} \mathscr{E}(\Omega Y).$$

$\operatorname{Im} J_0$ *is the subgroup of all elements of $\mathscr{E}(\Omega Y)$ which extend to the fibre homotopy equivalence from P_f to P_f.*

Let $\phi: \pi_1(B) \to \operatorname{Homeo}(K(G, n+1), *)$ be the action of $\pi_1(B)$ on $K(G, n+1)$. Let $K = K(\pi_1(B), 1)$ and consider the universal covering $\tilde{K} \to K$ and the usual action of $\pi_1(B)$ on $\tilde{K}$. Then we have a Hurewicz fibration

$$K(G, n+1) \longrightarrow L = \tilde{K} \times_{\pi_1(B)} K(G, n+1) \xrightarrow{q} K$$

Since $\tilde{K} \times_{\pi_1(B)} * = K$, we have the canonical cross section $s: K \to L$. Hence $L \in \operatorname{Top}(K = K)$. Let $Z = L$ and $D = K$ in Section 3. Then we have the following:

COROLLARY 4.4. *The following sequence of groups and homomorphisms is exact for $n \geq 1$:*

$$1 \longrightarrow H^n(B; G) \xrightarrow{\Delta} \mathscr{L}_0(P_f) \xrightarrow{J_0} \mathscr{E}(\Omega K(G, n+1)) = \operatorname{Aut} G,$$

where $H^n(B; G)$ is the local coefficient cohomology induced by $\phi: \pi_1(B) \to \operatorname{Aut} G$, and $\mathscr{L}_0(P_f) = \mathscr{L}(P_f)$ if $n \geq 2$.

Proof. The following sequence is exact (cf. [4, p. 4]):

$$[\Omega Y, \Omega Y] \xleftarrow{i^*} [P_f, \Omega_K L]_K \xleftarrow{p^*} [B, \Omega_K L]_K \longleftarrow 0.$$

also $[B, \Omega_K L]_K = H^n(B; G)$. Apply Theorem 4.2. Q.E.D.

Remark. The above exact sequence seems to be closely related to the extension for the essential term $E_1^{p,-p}$ of the spectral sequence obtained by W. Shih [8].

COROLLARY 4.5 [12, Proposition 2.9]. *If $H^n(B; G) = 0$, then the map J_0 is monic.*

Let $\{X_n\}$ be the Postnikov-system of a CW-complex X. Every X_n is well-pointed, since X_n is a mapping track of the inclusion $X \to X_{n-1}$ (cf. [12, p. 219], [11, p. 439]). And every p_n: $X_n \to X_{n-1}$ ($p^{-1}(*) = F$) is a relative principal fibration (cf. [4]).

THEOREM 4.6. *The following sequence of groups and homomorphisms is exact for every $n \geq 2$:*

$$1 \longrightarrow H^n(X_{n-1}; \pi_n(X)) \xrightarrow{\Delta} \mathscr{L}(X_n) \xrightarrow{J_0} \mathscr{E}(F) = \operatorname{Aut} \pi_n(X),$$

where $\pi_1(X) = \pi_1(X_{n-1})$ acts on $\pi_n(X)$ usually. Im J_0 *is contained in the equivariant subgroup od* Aut $\pi_n(X)$ *under the action of $\pi_1(X)$ on $\pi_n(X)$, and $\mathscr{L}(X_n) = \mathscr{L}_0(X_n)$.*

Let $\mathscr{E}_\#(X)$ be the group of all based homotopy classes of based homotopy-equivalences of X inducing the identity automorphisms of all homotopy groups.

COROLLARY 4.7 [12, Theorem 1.3]. *Assume that the connected* CW*-complex X satisfies $\pi_i(X) = 0$ $(i > N)$ or* dim $X = N$, *for some integer N, and that the cohomology groups of the local coefficients are $H^n(X_{n-1}; \pi_n(X)) = 0$ $(1 < n \leq N)$. Then $\mathscr{E}_\#(X) = 1$.*

Proof. Let f: $X \to X$ be a based homotopy equivalence which induces the identity automorphisms of all homotopy groups. Then f induces a based homotopy equivalence f_n: $X_n \to X_n$ for every $n \geq 1$ up to homotopy (cf. [13, Proposition 2.3]). Every f_n induces the identity automorphisms of all homotopy groups by the definition of the Postnikov-system. Assuming $\mathscr{E}_\#(X_{n-1}) = 1$, then $f_{n-1} \simeq 1$ and f_n can be deformed to a based fibre homotopy equivalence f'_n, that is, $f'_n \in \mathscr{L}_0(X_n)$. Since $H^n(X_{n-1}; \pi_n(X)) = 0$, $f'_n \simeq 1$ by Theorem 4.6. Hence $f_n \simeq 1$. Note that $\mathscr{E}(X) = \mathscr{E}(X_N)$ and $\mathscr{E}_\#(X_1) = \mathscr{E}_\#(K(\pi_1(X), 1)) = 1$. The result is shown by induction on n. Q.E.D.

COROLLARY 4.8. *The following sequence of groups and homomorphisms is exact:*

$$1 \longrightarrow \mathscr{E}_\#(X_2) \longrightarrow \mathscr{L}(X_2) \xrightarrow{J_0} \operatorname{Aut} \pi_2(X)$$

Proof. Let $k \in H^3(X_1; \pi_2(X))$ be the Postnikov k-invariant. Then by [14, Theorem 2.2], Δ: $H^2(X_1; \pi_2(X)) \to \mathscr{E}_\#(X_2)$ is an isomorphism, where $\Delta(a) = v_*(a, 1_{P_k})$ in (3.2). Q.E.D.

5. Examples

Example 5.1. If $\pi_2(X_2) = Z_2$, then $\mathscr{L}_0(X_2) = \mathscr{L}(X_2) = H^2(\pi_1(X); Z_2)$.

Proof. Aut $Z_2 = 1$, and use Corollary 4.8. Q.E.D.

In this example, if $\pi_1(X) = Z$, then $\mathscr{L}(X_2) = \mathscr{L}_0(X_2) = 1$.

Example 5.2 (cf. [12, Example 4.1]). Let G be a finite group, which acts on the odd dimensional sphere S^{2n-1}, and let $X = S^{2n-1}/G$. Then $\mathscr{L}_0(X_{2n-1}) = \mathscr{L}(X_{2n-1})$ is a subgroup of Aut $Z = Z_2$.

Proof. Apply Corollary 4.8, since $\mathscr{E}_{\#}(X_2) = H^{2n-1}(G; Z) = 0$. Q.E.D.

Example 5.3 (cf. [12, Example 4.3]). If $\pi_1(X)$ is a free group, $\mathscr{L}_0(X_2) = \mathscr{L}(X_2)$ is a subgroup of Aut $\pi_2(X)$.

Proof. If $\pi_1(X)$ is a free group, $H^2(X_1; \pi_2(X)) = 0$. Q.E.D.

Example 5.4 (cf. [12, Example 4.4]). Let X be simple and acyclic. Then $\mathscr{L}_0(X_n) = \mathscr{L}(X_n)$ is a subgroup of Aut $\pi_n(X)$ for every n.

Proof. If X is simple and acyclic, $H^n(X_{n-1}; \pi_n(X)) = 0$ for every n. Apply Theorem 4.6. Q.E.D.

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