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NATURAL SHEAVES

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Natural vector bundles have been investigated, with some variation in the exact definition, by various people in the last few years. (For example, [3], [4], [7], [1].) A natural vector bundle is a functor, V, on some category of manifolds which assigns to each manifold, M, a vector bundle, $V(M)$, over M and to each map f: $M \to N$ a vector bundle map $V(f)$: $V(M) \to V(N)$ such that the diagram

$$\begin{array}{ccc} V(M) & \xrightarrow{V(f)} & V(N) \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & N \end{array}$$

commutes. The basic examples are tensor bundles and bundles formed from them by taking k-jets of sections.

In this paper, following [7], we will fix an integer n and denote by $\mathfrak{M}^n$ the category of n-dimensional C^∞ manifolds and maps between them which are diffeomorphisms onto their images. A natural vector bundle (NVB) will always mean a functor on this category with the properties listed above. We also assume that the fiber dimension of all vector bundles is finite. It is known that any such NVB has a smooth structure: $V(M)$ is a C^∞ vector bundle and the maps $V(f)$ are C^∞. From now on, all maps and manifolds will be assumed to be in the category $\mathfrak{M}^n$ unless specifically stated otherwise.

The category $\mathfrak{M}^n$ includes all transition functions between charts on a manifold. To the extent that physics is the study of coordinate-independent quantities, NVB's provide a natural language for doing physics. This point of view is discussed in [1], which deals with a generalization of NVB's that involves gauge-invariance as well as coordinate-invariance. Most of what is said below about natural sheaves can be similarly generalized.

There is a very neat classification of NVB's and the natural morphisms between them. The study of these objects can be reduced to the study of

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representations of certain Lie groups and equivariant maps between such representations. (See [7].) Unfortunately, these methods are, by their nature, capable of dealing only with local properties: Although an NVB is defined for all n-manifolds, it is completely determined by what goes on in an infinitesimal neighborhood of 0 in $\mathbf{R}^n$.

Nevertheless, important global questions are closely related to NVB's. For example, the most basic example of a morphism between NVB's is the exterior derivative d, which is intimately connected with the deRahm cohomology. To form the cohomology, one must take the image and the kernal of d. Each of these is a subset of the set of sections of a NVB, but there is no way to deal with such subsets in the Lie group approach mentioned above. We need to expand the category of NVB's to include such objects.

Our solution is to forget about the bundle and concentrate on the set of sections. This will lead to the concept of natural sheaves. For the definitions of sheaves and pre-sheaves, see [8]. We will use the definition of a sheaf as a certain kind of pre-sheaf, rather than a topological space. The definition of natural sheaves is an obvious extension of this definition of sheaves.

DEFINITION. A natural pre-sheaf (on n-manifolds) is a contravariant functor from the category $\mathfrak{M}^n$, defined above, to the category of $\mathbf{R}$-vector spaces. A natural sheaf is a natural pre-sheaf whose restriction to each manifold is a sheaf.

For example, the constant sheaf $\mathscr{K}$, such that $\mathscr{K}(M) = \mathbf{R}$ for each manifold M and $\mathscr{K}(f)$ is the identity for each map f, is a natural sheaf. The same example with $\mathscr{K}(f) = 0$ for each f is a natural pre-sheaf which is not a sheaf. The sheaf C^∞ of $\mathbf{R}$-valued C^∞ functions is also a natural sheaf if for a function $f\colon M \to N, C^\infty(f)\colon C^\infty(N) \to C^\infty(M)$ is the induced map $f^*(g) = g \circ f$. Since $C^\infty(M)$ is a ring and $C^\infty(f)$ is a ring homomorphism, C^∞ is actually a natural sheaf of rings.

Less trivial examples can be obtained from NVB's. Let V be a NVB. We define the associated natural sheaf, $\mathscr{V}$, by

$$\mathscr{V}(M) = \{C^\infty \text{ sections of } V(M)\}$$

and

$$\mathscr{V}(f) = V(f^{-1}) \circ s \circ f$$

for $s \in \mathscr{V}(N)$ and $f\colon M \to N$. (Note that $V(f^{-1})$ is well-defined since f is a diffeomorphism onto its image and the effect of V is locally determined.) Now, $\mathscr{V}$ has some additional structure: $\mathscr{V}(M)$ is a module over the ring $C^\infty(M)$ and $\mathscr{V}(f)$ is a (relative) homomorphism. That is, for $g \in C^\infty(N)$,

$$\mathscr{V}(f)(g \cdot s) = (f^*g) \cdot \mathscr{V}(f)(s).$$

We summarize this by saying that $\mathscr{V}$ is a C^∞-module, and we note that $\mathscr{V}(\mathbf{R}^n)$ is actually a finitely-generated $C^\infty(\mathbf{R}^n)$-module. The next proposition shows that this property characterizes those natural sheaves that are associated to NVB's.

PROPOSITION. *If $\mathscr{V}$ is a natural sheaf such that $\mathscr{V}(\mathbf{R}^n)$ is a finitely-generated $C^\infty(\mathbf{R}^n)$-module, then there is a NVB V, unique up to equivalence, such that $\mathscr{V}$ is equivalent to the associated sheaf of sections of V.*

Proof. Let $\{s_1, \ldots, s_m\}$ be a generating set of $\mathscr{V}(\mathbf{R}^n)$ of minimal cardinality. We will show that $\mathscr{V}(\mathbf{R}^n)$ is a free module on these generators.

Let U be an open subset of $\mathbf{R}^n$ and let $i\colon U \to \mathbf{R}^n$ be the inclusion map. We will write $s|U$ for $\mathscr{V}(i)(s)$. We first show that the set $\{s_1|U, \ldots, s_m|U\}$ generates $\mathscr{V}(U)$ over $C^\infty(U)$. Choose a locally finite cover of U by open sets with compact closure in U, and let $\varphi_\alpha\colon U \to \mathbf{R}$ be a C^∞ partition of unity subordinate to this cover. Let $s \in \mathscr{V}(U)$.

Now, there is an $s_\alpha \in \mathscr{V}(\mathbf{R}^n)$ for each α such that $s_\alpha|U = \varphi_\alpha \cdot s$. (By the properties of a sheaf, we can define s_α by this equation and the equation $s_\alpha = 0$ on $\mathbf{R}^n - \operatorname{supp}(\varphi_\alpha)$.) Thus, we can write

$$\begin{aligned}\varphi_\alpha \cdot s &= \sum_{j=1}^{m} \left(g_\alpha^j \cdot s_j\right) \quad \text{where } g_\alpha^j \in C^\infty(\mathbf{R}^n)\\ &= \sum_{j=1}^{m} \left(g_\alpha^j \circ i\right) \cdot s_j|U.\end{aligned}$$

We can easily arrange that the supports of the functions g_α^j for all j and α, form a locally finite set. But then

$$\begin{aligned}s &= \sum_\alpha \varphi_\alpha \cdot s\\ &= \sum_{j,\alpha} \left(g_\alpha^j \circ i\right) \cdot s_j|U\\ &= \sum_j \left(\sum_\alpha g_\alpha^j \circ i\right) \cdot s_j|U,\end{aligned}$$

which shows that $\{s_j|U,\ j = 1, \ldots, m\}$ generates $\mathscr{V}$(U). (The infinite sums are admissable since elements of a sheaf may be defined locally, and locally the sums are finite.)

We can now show that $\mathscr{V}(\mathbf{R}^n)$ is in fact free on the generators $\{s_1, \ldots, s_m\}$. For suppose not. Then there are functions $g_j \in C^\infty(\mathbf{R}^n)$ such that $\Sigma g_j \cdot s_j = 0$ and some g_j, say g_m, is not identically zero. Choose $a \in \mathbf{R}^n$ such that $g_m(a) \neq 0$. Let U be an ε-ball around a on which g_m does not vanish, and i:

$U \to \mathbf{R}^n$ the inclusion map. Restricting to U, we have $0 = \Sigma(g_j \circ i) \cdot \mathscr{V}(i)(s_j)$. By the above, $\mathscr{V}(i)(s_1), \ldots, \mathscr{V}(i)(s_m)$ generate $\mathscr{V}(U)$. But $g_m \circ i$ is invertible in $C^\infty(U)$, so in fact, $\mathscr{V}(U)$ is generated by $\mathscr{V}(i)(s_1), \ldots, \mathscr{V}(i)(s_{m-1})$. Now, U is diffeomorphic to $\mathbf{R}^n$, so $\mathscr{V}(U)$ is isomorphic to $\mathscr{V}(\mathbf{R}^n)$. Thus, we have a contradiction to the assumed minimality of m. So, $\mathscr{V}(\mathbf{R}^n)$ must in fact be free.

It follows immediately that $\mathscr{V}(\mathbf{R}^n)$ is isomorphic as a sheaf over $\mathbf{R}^n$ to the sheaf of sections of a vector bundle of fiber dimension m. We can construct the NVB V directly by setting the fiber

$$V(M)_x = \mathscr{V}(M)/\mathfrak{m}_x \cdot \mathscr{V}(M)$$

where $x \in M$ and $\mathfrak{m}_x$ is the ideal of $C^\infty(M)$ of functions vanishing at x. Given f: $M \to N$, there is an induced isomorphism

$$\mathscr{V}(N)/\mathfrak{m}_{f(x)} \cdot \mathscr{V}(N) \to \mathscr{V}(M)/\mathfrak{m}_x \cdot \mathscr{V}(M)$$

and we take $V(f)|M_x$ to be the inverse of this map. All considerations being local, it is easy to check that this defines a NVB and that $\mathscr{V}$ is equivalent to its sheaf of sections. And since a vector bundle can always be recovered, as above, from its sheaf of sections, it is clear that V is unique up to equivalence. ∎

Remark. If V is a NVB and $k < \infty$, there is a C^∞-module $\mathscr{V}^k$ given by $\mathscr{V}^k(M) = \{C^k$ sections of $V(M)\}$. $\mathscr{V}^k(\mathbf{R}^n)$ is not finitely generated over $C^\infty(\mathbf{R}^n)$. More generally, any of the section functors discussed in [6] can be applied to a NVB to give a C^∞-module. If $\mathscr{V}$ is a natural sheaf obtained in this way, then $\mathscr{V}(M)$ will be a topological space, and $\mathscr{V}(\mathbf{R}^n)$ will contain a dense sub-module which is finitely generated over $C^\infty(\mathbf{R}^n)$, namely the set of C^∞ sections of the corresponding NVB. Conversely, if $\mathscr{W}$ is a "topological C-module" such that $\mathscr{W}(\mathbf{R}^n)$ has a dense, finitely-generated sub-module, the proposition shows that $\mathscr{W}$ can be obtained by applying some sort of "section functor" to the usual associated bundle of some NVB.

From now on, we will be concerned with sub-sheaves of sheaves associated to NVB's. Such sheaves can be given the C^∞ topology, and when we mention continuity, we will be implicitely using this topology.

The morphisms we will deal with are equivalent in the case of associated sheaves to linear natural differential operators. (See [7] for the definition.) The use of the name in the more general case will be justified below.

DEFINITION. Let V and W be NVB's and let $\mathscr{S}$ and $\mathscr{T}$ be natural sub-sheaves of the associated sheaves of sections. A linear natural differential operator (LNDO), D, from $\mathscr{S}$ to $\mathscr{T}$ is a collection $\{D(M)\colon \mathscr{S}(M) \to \mathscr{T}(M)\}$

of continuous, $\mathbf{R}^n$-linear maps such that for any $f\colon M \to N$, the diagram

$$\begin{tikzcd}
\mathscr{S}(N) \arrow[r, "D(N)"] \arrow[d, "\mathscr{S}(f)"] & \mathscr{T}(N) \arrow[d, "\mathscr{T}(f)"] \\
\mathscr{S}(M) \arrow[r, "D(M)"] & \mathscr{T}(M)
\end{tikzcd}$$

commutes.

PROPOSITION. *Let V and W be NVB's. Let $\mathscr{S}$ and $\mathscr{T}$ be natural sub-sheaves of the associated sheaves of sections. Let D be a LNDO from $\mathscr{S}$ to $\mathscr{T}$. Then there is a $k < \infty$ such that for all M, for all $s \in \mathscr{S}(M)$, and for all $x \in M$, $(D(M)(s))(x)$ depends only on the k-jet, $j_k(s)_x$, of s at x. Thus D really is a differential operator.*

Proof. Define $f\colon \mathscr{S}(\mathbf{R}^n) \to W(\mathbf{R}^n)_0$ by $f(s) = (D(\mathbf{R}^n)(s))(0)$. Then f completely determines D, since for any $\varphi\colon \mathbf{R}^n \to M$ and $s \in \mathscr{S}(M)$,

$$(Ds)(\varphi(0)) = W(\varphi^{-1})(f(\mathscr{S}(\varphi)(s))).$$

Now, clearly $f(s)$ depends only on the germ, $[s]_0$, of s at 0, so we might as well extend f to be defined on the set

$$\mathscr{S}^\infty = \{s \in \mathscr{V}(\mathbf{R}^n) \mid [s]_0 = [\bar{s}]_0 \text{ for some } \bar{s} \in \mathscr{S}(\mathbf{R}^n)\}.$$

Let $\mathscr{S}^k = \{s \in \mathscr{S}^\infty \mid j_k(s)_0 = 0\}$ for $k < \infty$. Then

$$f(\mathscr{S}^1) \supseteq f(\mathscr{S}^2) \supseteq \cdots \supseteq f(\mathscr{S}^k) \supseteq \cdots$$

is a decreasing sequence of subspaces of $W(\mathbf{R}^n)_0$. It must stabilize. Say $f(\mathscr{S}^k) = f(\mathscr{S}^{k+1}) = \cdots$. We will prove the theorem if we show that $f(\mathscr{S}^k) = \{0\}$.

Let $w \in f(\mathscr{S}^k)$. Now, for $i = k, k+1, \ldots,$ there is an $s_i \in \mathscr{S}^i$ such that $w = f(s_i)$. But we can find $\mathbf{R}$-valued functions g_i on $\mathbf{R}^n$ such that $g_i = 1$ on a neighborhood of 0 and $g_i s_i \to 0$ in the C^∞ topology. (See the proof of Lemma 1.4 in [7].) Since $[s_i]_0 = [g_i s_i]_0$, we see that

$$w = \lim f(s_i) = \lim f(g_i s_i) = f(\lim g_i s_i) = 0. \qquad \blacksquare$$

We can now turn to the reasons for introducing natural sheaves. The following obvious remark is important enough to state as a proposition.

PROPOSITION. *If D is a LNDO from $\mathscr{S}$ to $\mathscr{T}$, then* ker(D) *is a natural sub-sheaf of $\mathscr{S}$, and* image(D) *is a natural sub-presheaf of $\mathscr{T}$, where*

$$(\ker(D))(M) = \{s \in \mathscr{S}(M) | Ds = 0\}$$

and

$$(\mathrm{image}(D))(M) = \{t \in \mathscr{T}(M) | t = Ds \text{ for some } s \in \mathscr{S}(M)\}.$$

Furthermore, the sheaf generated by image(D) is a natural sub-sheaf of $\mathscr{T}$. ■

The sheaf ker(D) of solutions of D is the object that we often want to study. One way to try to study it is to introduce sheaf cohomology with coefficients in ker(D). We now explain why cohomology with coefficients in a natural sheaf is especially useful.

Recall that given any sheaf $\mathscr{S}$ over a manifold M, the cohomology with coefficients in $\mathscr{S}$ is defined as follows. Let

$$0 \to \mathscr{S} \to \mathscr{S}_1 \to \mathscr{S}_2 \to \cdots$$

be a fine, torsionless resolution of $\mathscr{S}$. (See [8] for definitions.) Then $H^*(M, \mathscr{S})$ is the cohomology of the sequence of global sections

$$0 \to \mathscr{S}(M) \to \mathscr{S}_1(M) \to \mathscr{S}_2(M) \to \cdots.$$

The cohomology is independent of the resolution chosen, and for a sheaf over a C^∞ manifold, there is a particularly useful resolution. Let $\mathscr{E}^p$, $p = 0, 1, \ldots, n$, be the sheaves of p-forms on M. If $\mathscr{S}$ is a sheaf over M, then

$$0 \to \mathscr{S} \otimes \mathscr{E}^0 \to \mathscr{S} \otimes \mathscr{E}^1 \to \cdots \to \mathscr{S} \otimes \mathscr{E}^n \to 0$$

is the resolution in question. Now, the $\mathscr{E}^i$ are themselves natural sheaves, so that if $\mathscr{S}$ is natural and $f\colon M \to N$, we get a commutative diagram

$$\begin{array}{ccccccccc}
0 \to \mathscr{S}(M) & \to & \mathscr{S}(M) \otimes \mathscr{E}^0(M) & \to & \cdots & \to & \mathscr{S}(M) \otimes \mathscr{E}^n(M) & \to & 0 \\
\uparrow & & \uparrow & & & & \uparrow & & \\
0 \to \mathscr{S}(N) & \to & \mathscr{S}(N) \otimes \mathscr{E}^0(N) & \to & \cdots & \to & \mathscr{S}(N) \otimes \mathscr{E}^n(N) & \to & 0
\end{array}$$

and thus we get induced maps $f^*\colon H^*(N, \mathscr{S}(N)) \to H^*(M, \mathscr{S}(M))$. We see that cohomology with coefficients in $\mathscr{S}$ is functorial with respect to the maps in the category $\mathscr{M}^n$. Note in particular that the group of diffeomorphisms of M acts on $H^p(M, \mathscr{S}(M))$ for each p. It is also easy to see that any LNDO D from $\mathscr{S}$ to $\mathscr{T}$ induces a natural transformation of the associated cohomologies, and that each of the induced maps $H^p(M, \mathscr{S}(M)) \to H^p(M, \mathscr{T}(M))$ is a homomorphism of Diffeo(M)-spaces.

It should be noted that $H^0(M, \mathscr{S}(M)) = \mathscr{S}(M)$, and that if $f\colon M \to N$, the induced map from $H^0(N, \mathscr{S}(N))$ to $H^0(M, \mathscr{S}(M))$ is just $\mathscr{S}(f)$. In particular, if $\mathscr{S}$ is the natural sheaf associated to some NVB, then all the higher cohomology groups are trivial, and we have introduced nothing new.

Before closing, we discuss one problem with the approach outlined above: LNDO's are not very common. Most interesting natural differential operators are non-linear. However, we should also note that if D is any natural differential operator defined on a subset of a natural sheaf $\mathscr{V}$, then

$$\mathscr{K}(M) = \{s \in \mathscr{V}(M) | Ds = 0\}$$

defines a natural sheaf of *sets* which in turn generates a natural subsheaf (of vector spaces) of $\mathscr{V}$ if we take **R**-linear combinations. One could try to study D by studying this sheaf.

More important, perhaps, are applications to Riemannian and to affine manifolds. For Riemannian manifolds, there is a sort of Einsteinian equivalence principle which can be stated for our purposes as follows:

Let $\mathscr{V}$ and $\mathscr{W}$ be natural sheaves. Let D_0 be an $\mathscr{O}(n)$-invariant linear map from $\mathscr{V}(\mathbf{R}^n)$ to $\mathscr{W}(\mathbf{R}^n)$. Let $\mathscr{R}$ be the sheaf (of sets) of Riemannian metrics. Then there is a natural differential operator D from $\mathscr{V} \times \mathscr{R} \to \mathscr{W}$ such that:

(1) If μ_0 is the usual metric on $\mathbf{R}^n$, then for $s \in \mathscr{V}(\mathbf{R}^n)$, $D(s, \mu_0) = D_0(s)$.

(2) If μ is a metric on M, then the map $\mathscr{V}(M) \to \mathscr{W}(M)$ given by $s \to D(s, \mu)$ is a linear differential operator which is natural with respect to isometries of M.

(3) If μ is a metric on M, and $\varphi\colon \mathbf{R}^n \to M$ gives normal coordinates at a point $x = \varphi(0)$ of M, and $s \in \mathscr{V}(M)$, then

$$D(s, \mu)(x) = \mathscr{W}(\varphi^{-1})(D_0(\mathscr{V}(\varphi))(s)).$$

Property 2 gives us a linear differential operator between sheaves over a Riemannian manifold M, and we can form the cohomology of M with coefficients in the kernal of this operator. The resulting cohomology groups will be representations of the group of isometries of M.

Similar statements can be made about affine manifolds if the orthogonal group $\mathscr{O}(n)$ is replaced by the general linear group $\mathrm{Gl}(n)$. A possible application of these ideas would be to the "differential hyperforms" introduced recently by P. Olver ([5]), which are higher order $\mathrm{Gl}(n)$-invariant linear operators between tensor bundles on $\mathbf{R}^n$ whose extension to affine or Riemannian manifolds should be interesting.

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