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PRODUCT TUBE FORMULAS

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1. Introduction

Let $P \subset M$ be an embedding of a compact p-dimensional manifold P to an m-dimensional Riemannian manifold M. We denote by $V_P^M(r)$ the m-dimensional volume of a solid tube of radius r about P and by $A_P^M(r)$ the $(m-1)$-dimensional volume of its boundary. Throughout this paper we assume that $r > 0$ is less than or equal to the distance from P to its nearest focal point. Then it is easy to see that

$$A_P^M(r) = \frac{d}{dr} V_P^M(r).$$

The well-known Weyl's tube formula [8] for $P \subset \mathbf{R}^m$ can be written as (see for example [3])

$$V_P^{\mathbf{R}^m}(r) = \sum_{c=0}^{[p/2]} \frac{\pi^{(m-p)/2} k_{2c}(R^P)}{2^c \Gamma\left(\frac{m-p}{2} + c + 1\right)} r^{m-p+2c}, \tag{1}$$

where $k_{2c}(R^P)$ are integrals over P of scalar invariants $I_{2c}(R^P)$ constructed from the Riemannian curvature tensor R^P of the submanifold P. Specifically for an even integer e satisfying $0 \le e \le p$, $k_e(R^P)$ is defined by

$$k_e(R^P) = \int_P I_e(R^P)\, dP, \tag{2}$$

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where dP is the volume element of P and $I_e(R^P)$ is given by

$$I_e(R^P) = \frac{1}{2^e\left(\frac{e}{2}\right)!}\sum\delta\begin{pmatrix}\alpha\\ \beta\end{pmatrix}R^P_{\alpha_1\alpha_2\beta_1\beta_2}\cdots R^P_{\alpha_{e-1}\alpha_e\beta_{e-1}\beta_e}, \tag{3}$$

where $\delta\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$ is equal to 1 or -1 according as $\alpha_1,\ldots,\alpha_e$ are distinct and an even or odd permutation of $\beta_1,\ldots,\beta_e$, and otherwise is equal to zero. The summation is taken over all α and β running from 1 to p.

In this article we derive the following product formula for the volume of a tube about a compact product submanifold of a product Riemannian manifold.

THEOREM 1. *Let $P \subset M$ and $Q \subset N$ be two embeddings, and $P \times Q \subset M \times N$ be the corresponding embedding of the product. Then*

$$A^{M\times N}_{P\times Q}(r) = r\int_0^{\pi/2} A^M_P(r\cos\theta)A^N_Q(r\sin\theta)\,d\theta. \tag{4}$$

When we combine Weyl's tube formula (1) with (4) we obtain several interesting formulas. Let $p = \dim P$, $q = \dim Q$, $m = \dim M$, and $n = \dim N$.

THEOREM 2. *Let $P \subset M = \mathbf{R}^m$. If either $p = 0$, $m = 2$ or $p = 1$, $m = 3$, then for any $Q \subset N$ we have*

$$A^{M\times N}_{P\times Q}(r) = A^M_P(r)V^N_Q(r). \tag{5}$$

On the other hand two or three dimensional locally Euclidean space can be characterized by the product formula (5).

THEOREM 3. *Let $P \subset M$ be an embedding with $p = 0$ or $p = 1$. Assume $P \subset M$ satisfies* (5) *for any $Q \subset N$. Then when $p = 0$, M is locally Euclidean space of dimension* 2, *and when $p = 1$, M is locally Euclidean space of dimension* 3.

We also derive product formulas for $V^{\mathbf{R}^m\times\mathbf{R}^n}_{P\times Q}(r)$ and $A^{\mathbf{R}^m\times\mathbf{R}^n}_{P\times Q}(r)$. Specifically we have Theorem 4 below. But before stating the theorem we make the observation that (1) can be regarded as a *definition*. For any integer n, $V^n_P(r)$ and $A^n_P(r)$ are defined by the right-hand side of (1) and its derivative with respect to r respectively. Here P may be any compact manifold. For our purposes it will turn out to be irrelevant if P actually lies in $\mathbf{R}^n$, although we shall have that interpretation in mind.

THEOREM 4. *Let P and Q be compact Riemannian manifolds and let $r_1 \leq r_2$. Write $r = \sqrt{r_1^2 + r_2^2}$.*

(i) *If p and $n - q$ are both even, then*

$$(6)\qquad V_{P\times Q}^n(r) = \begin{cases} \displaystyle\sum_{d=0}^{\infty} V_P^{2d}(r_1)V_Q^{n-2d}(r_2) \\ \displaystyle\frac{1}{2\pi r_2}\sum_{d=0}^{\infty} V_P^{2d}(r_1)A_Q^{n-2d+2}(r_2) \\ \displaystyle\frac{1}{2\pi r_1}\sum_{d=1}^{\infty} A_P^{2d}(r_1)V_Q^{n-2d+2}(r_2) \\ \displaystyle\frac{1}{4\pi^2 r_1 r_2}\sum_{d=1}^{\infty} A_P^{2d}(r_1)A_Q^{n-2d+4}(r_2) \end{cases}$$

and

$$(7)\qquad A_{P\times Q}^n(r) = \begin{cases} \displaystyle 2\pi r\sum_{d=0}^{\infty} V_P^{2d}(r_1)V_Q^{n-2d-2}(r_2) \\ \displaystyle\frac{r}{r_2}\sum_{d=0}^{\infty} V_P^{2d}(r_1)A_Q^{n-2d}(r_2) \\ \displaystyle\frac{r}{r_1}\sum_{d=1}^{\infty} A_P^{2d}(r_1)V_Q^{n-2d}(r_2) \\ \displaystyle\frac{r}{2\pi r_1 r_2}\sum_{d=1}^{\infty} A_P^{2d}(r_1)A_Q^{n-2d+2}(r_2). \end{cases}$$

(ii) *If p and $n - q$ are both odd, then*

$$(8)\qquad V_{P\times Q}^n(r) = \begin{cases} \displaystyle\sum_{d=0}^{\infty} V_P^{2d+1}(r_1)V_Q^{n-2d-1}(r_2) \\ \displaystyle\frac{1}{2\pi r_2}\sum_{d=0}^{\infty} V_P^{2d+1}(r_1)A_Q^{n-2d+1}(r_2) \\ \displaystyle\frac{1}{2\pi r_1}\sum_{d=1}^{\infty} A_P^{2d+1}(r_1)V_Q^{n-2d+1}(r_2) \\ \displaystyle\frac{1}{4\pi^2 r_1 r_2}\sum_{d=1}^{\infty} A_P^{2d+1}(r_1)A_Q^{n-2d+3}(r_2) \end{cases}$$

and

$$(9)\qquad A^n_{P\times Q}(r)=\begin{cases} 2\pi r\sum_{d=0}^{\infty} V_P^{2d+1}(r_1)V_Q^{n-2d-3}(r_2)\\ \frac{r}{r_2}\sum_{d=0}^{\infty} V_P^{2d+1}(r_1)A_Q^{n-2d-1}(r_2)\\ \frac{r}{r_1}\sum_{d=1}^{\infty} A_P^{2d+1}(r_1)V_Q^{n-2d-1}(r_2)\\ \frac{r}{2\pi r_1 r_2}\sum_{d=1}^{\infty} A_P^{2d+1}(r_1)A_Q^{n-2d+1}(r_2).\end{cases}$$

(iii) *If p is even and $n-q$ is odd, then* (6) *and* (7) *hold either for $r_1<r_2$ or for $r_1=r_2$ with $n-p-q-1\geq 0$.*

(iv) *If p is odd and $n-q$ is even, then* (8) *and* (9) *hold either for $r_1<r_2$ or for $r_1=r_2$ with $n-p-q-1\geq 0$.*

Remarks. (1) The invariants $k_e(R^P)$ are among the most important integral invariants. In fact

$$k_0(R^P)=\text{volume of } P,\quad k_2(R^P)=\frac{1}{2}\int_P \tau(R^P)\,dP,$$

where $\tau(R^P)$ denotes the scalar curvature of R^P. If p is even, then the Gauss-Bonnet theorem says

$$k_e(R^P)=(2\pi)^{p/2}\chi(P),$$

where $\chi(P)$ is the Euler characteristic of P.

(2) The product formula (4) was obtained by Howard [5] when P and Q are compact oriented symmetrically embedded submanifolds of oriented symmetric spaces M and N respectively. Nijenhuis [7] also stated (4) when M and N are Euclidean spaces. But Theorem 1 is much more general.

(3) There is also a product formula for the coefficients:

$$(10)\qquad k_{2c}(R^{P\times Q})=\sum_{a=0}^{c} k_{2a}(R^P)k_{2c-2a}(R^Q).$$

This is equivalent to the formula of Nijenhuis [7]. We give a proof of (10) as an application of (1) and (4) (see §3). A direct proof of (10) from the definition (2) is given in [3].

(4) The sums are actually finite sums in the cases (i) and (ii) of Theorem 4. But in the cases (iii) and (iv) they are not finite sums.

2. Preliminaries and proof of Theorem 1

Before proving Theorem 1 we summarize some basic facts and formulas.

Let M be a complete Riemannian manifold of dimension m and P be an embedded submanifold of dimension p which is relatively compact. Recall that

$$V_P^M(r) = m\text{-dimensional volume of } \{m \in M \mid d(m, P) \leq r\}$$

and

$$A_P^M(r) = (m-1)\text{-dimensional volume of } \{m \in M \mid d(m, P) = r\}.$$

We assume that $r > 0$ is not larger than the distance from P to its nearest focal point. Let ω be a Riemannian volume form near P with $\|\omega\| = 1$, and let $(x_1, \ldots, x_m)$ be a system of Fermi coordinates of P (cf. [2]) such that

$$\omega\left(\frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_m}\right) > 0.$$

For $u \in P_p^{\perp}$ we put

$$\theta(u) = \omega\left(\frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_m}\right)(\exp_p u). \tag{11}$$

Proposition 1. *We have*

$$A_P^M(r) = \int_P \int_{S^{m-p-1}(r)} \theta(u)\, du\, dP \tag{12}$$

and

$$V_P^M(r) = \int_0^r A_P^M(r)\, dr \tag{13}$$

where $S^{m-p-1}(r)$ denotes the sphere of radius r in $P_p^{\perp}$ with its volume element du, and dP denotes the volume element of P.

For a proof see [2].

Let $P \subset M$ and $Q \subset N$ be embeddings and $P \times Q \subset M \times N$ the corresponding embedding of the product. Let $\dim P = p$, $\dim Q = q$, $\dim M = m$, and $\dim N = n$. Let ω_1 (resp. ω_2) be the Riemannian volume form of M (resp. N) near P (resp. Q) with $\|\omega_1\| = 1$ (resp. $\|\omega_2\| = 1$), and let $(x_1, \ldots, x_m)$ (resp. $(y_1, \ldots, y_n)$) be a system of Fermi coordinates such that

$$\omega_1\left(\frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_m}\right) > 0 \quad \left(\text{resp. } \omega_2\left(\frac{\partial}{\partial y_1} \wedge \cdots \wedge \frac{\partial}{\partial y_n}\right) > 0\right).$$

Then $\omega_1 \wedge \omega_2$ is the volume form of $M \times N$ near $P \times Q$ with $\|\omega_1 \wedge \omega_2\| = 1$ and $(x_1, \ldots, x_m, y_1, \ldots, y_n)$ is a system of Fermi coordinates such that

$$(\omega_1 \wedge \omega_2)\left(\frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_m} \wedge \frac{\partial}{\partial y_1} \wedge \cdots \wedge \frac{\partial}{\partial y_n}\right) > 0.$$

LEMMA 1. *For $u = (u_1, u_2) \in (P \times Q)^{\perp}_{(p,q)} = P_p^{\perp} \oplus Q_q^{\perp}$ we have*

$$\theta(u) = \theta_1(u_1)\theta_2(u_2) \tag{14}$$

where

$$\theta_1(u_1) = \omega_1\left(\frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_m}\right)(\exp_p u_1)$$

and

$$\theta_2(u_2) = \omega_2\left(\frac{\partial}{\partial y_1} \wedge \cdots \wedge \frac{\partial}{\partial y_n}\right)(\exp_q u_2).$$

Next we need the following lemma essentially due to Howard [5].

LEMMA 2. *Let g be a continuous real valued function on $\mathbf{R}^m \times \mathbf{R}^n$ defined by $g(u) = g_1(u_1)g_2(u_2)$ for $u = (u_1, u_2) \in \mathbf{R}^m \times \mathbf{R}^n$, where g_1 and g_2 are continuous real valued functions on $\mathbf{R}^m$ and $\mathbf{R}^n$ respectively. Then*

$$\begin{aligned}(15)\quad &\int_{S^{m+n-1}(r)} g(u)\, du \\ &= r\int_0^{\pi/2}\left\{\int_{S^{m-1}(r\cos\theta)} g_1(u_1)\, du_1\right\}\left\{\int_{S^{n-1}(r\sin\theta)} g_2(u_2)\, du_2\right\} d\theta,\end{aligned}$$

where du, du_1, du_2 are the volume elements of the corresponding spheres.

Proof. Let $S_1 = S^{m-1}(1)$ and $S_2 = S^{n-1}(1)$ be unit spheres in $\mathbf{R}^m$ and $\mathbf{R}^n$ respectively. Consider the product $[0, \pi/2] \times S_1 \times S_2$ and define a map

$$\phi\colon [0, \pi/2] \times S_1 \times S_2 \to S^{m+n-1}(r)$$

by

$$\phi(\theta, u_1, u_2) = r\cos\theta\, u_1 + r\sin\theta u_2.$$

Since ϕ is bijective except on a set of measure zero we have

$$\int_{S^{m+n-1}(r)} g(u)\, du = \int_{[0,\pi/2]\times S_1\times S_2} g\circ\phi\,\phi^*(du).$$

A straightforward computation shows that

$$\phi^*(du) = r\,d\theta\wedge(r\cos\theta)^{m-1}\,du_1\wedge(r\sin\theta)^{n-1}\,du_2.$$

By the change of variable

$$\int_{S^{m-1}(1)} r^{m-1}g_1(ru_1)\,du_1 = \int_{S^{m-1}(r)} g_1(u_1)\,du_1,$$

we obtain (15).

Now we can prove Theorem 1.

Proof of Theorem 1. From Lemmas 1 and 2 we have

$$\begin{aligned} A_{P\times Q}^{M\times N}(r) &= \int_{P\times Q}\int_{S^{m+n-p-q-1}(r)} \theta(u)\,du\,d(P\times Q) \\ &= \int_{P\times Q} r\int_0^{\pi/2}\left\{\int_{S^{m-p-1}(r\cos\theta)} \theta_1(u_1)\,du_1\right\} \\ &\quad\times\left\{\int_{S^{n-q-1}(r\sin\theta)} \theta_2(u_2)\,du_2\right\} d\theta\,d(P\times Q) \\ &= r\int_0^{\pi/2}\left\{\int_P\int_{S^{m-p-1}(r\cos\theta)} \theta_1(u_1)\,du_1\,dP\right\} \\ &\quad\times\left\{\int_Q\int_{S^{n-q-1}(r\sin\theta)} \theta_2(u_2)\,du_2\,dQ\right\} d\theta \\ &= r\int_0^{\pi/2} A_P^M(r\cos\theta)A_Q^N(r\sin\theta)\,d\theta. \end{aligned}$$

3. Product formulas

In this section we prove Theorems 2, 3 and 4 which give various product formulas.

Proof of Theorem 2. From (1) and (4) we find

$$\begin{aligned} A_{P\times Q}^{\mathbf{R}^m\times N}(r) &= r\int_0^{\pi/2} 2\pi k_0(R^P) r\cos\theta A_Q^N(r\sin\theta)\,d\theta \\ &= A_P^{\mathbf{R}^m}(r)\int_0^{\pi/2} A_Q^N(r\sin\theta) r\cos\theta\,d\theta \\ &= A_P^{\mathbf{R}^m}(r)\int_0^{r} A_Q^N(s)\,ds \\ &= A_P^{\mathbf{R}^m}(r) V_Q^N(r). \end{aligned}$$

Proof of Theorem 3. Let $Q \subset N$ be a point $q \subset \mathbf{R}^2$ or $S^1(\rho) \subset \mathbf{R}^3$, where $S^1(\rho)$ is a one-dimensional sphere of radius ρ. It follows from the hypothesis and from Theorem 2 that

$$A_{P\times Q}^{M\times N}(r) = A_P^M(r)V_Q^N(r) = A_Q^N(r)V_P^M(r)$$

which implies

$$rA_P^M(r) = V_P^M(r). \tag{16}$$

From (16) we obtain

$$\frac{d^2}{dr^2}A_P^M(r) = 0. \tag{17}$$

Then we have the conclusion of the theorem according to results of [4] and [6]. In fact, Gray and Vanhecke [4, p. 195] showed that two-dimensional Riemannian manifolds of constant curvature equal to c are characterized by the equation

$$\frac{d^2}{dr^2}A_P^M(r+s) + cA_P^M(r+s) = 0, \tag{18}$$

for sufficiently small $r \geq 0$, $s > 0$, and for each 0-dimensional submanifold P. Similarly the author [6] characterized Riemannian manifolds of constant curvature c of dimension 2 or 3 by (18) for sufficiently small $r \geq 0$, $s > 0$, and for each one-dimensional submanifold P.

Proof of Theorem 4. We prove (7) and (9). The proofs of (6) and (8) are similar. Applying (1) and (13) to (4) we obtain for $P \subset \mathbf{R}^m$ and $Q \subset \mathbf{R}^n$,

$$\begin{aligned} &A_{P\times Q}^{\mathbf{R}^m\times\mathbf{R}^n}(r) \\ &\quad= \sum_{a=0}^{[p/2]}\sum_{b=0}^{[q/2]} \frac{\pi^{(m+n-p-q)/2}k_{2a}(R^P)k_{2b}(R^Q)}{2^{a+b-1}\Gamma\left(\frac{m+n-p-q}{2}+a+b\right)} r^{m+n-p-q+2a+2b-1}. \end{aligned} \tag{19}$$

Comparing (19) with Weyl's formula for $A_{P\times Q}^{m+n}(r)$ we have the product formula (10).

Let P and Q be any compact manifolds and let n be any integer. Then by (10), we can write $A_{P\times Q}^{n}(r)$ as

(20)
$$A_{P\times Q}^{n}(r)=\sum_{a=-\infty}^{\infty}\sum_{b=-\infty}^{\infty}\frac{\pi^{(n-p-q)/2}k_{2a}(R^P)k_{2b}(R^Q)}{2^{a+b-1}\Gamma\left(\frac{n-p-q}{2}+a+b\right)}r^{n-p-q+2a+2b-1},$$

because the $k_{2a}(R^P)$ are different from zero only in the range $0\leq a\leq[p/2]$. Applying the binomial expansion

$$r^s=\sum_{c=0}^{\infty}\binom{s/2}{c}r_1^{2c}r_2^{s-2c} \tag{21}$$

with $s=n-p-q+2a+2b-2$ we have from (20),

(22) $A_{P\times Q}^{n}(r)$
$$=r\sum_{a,b}\sum_{c=0}^{\infty}\frac{\pi^{(n-p-q)/2}k_{2a}(R^P)k_{2b}(R^Q)}{2^{a+b-1}\Gamma(c+1)\Gamma\left(\frac{n-p-q+2a+2b-2c}{2}\right)}$$
$$\times r_1^{2c}r_2^{n-p-q+2a+2b-2c-2}.$$

When p is even, the substitution $c=\frac{1}{2}(2d-p+2a)$ shows the first two formulas of (7). The remaining two formulas of (7) can be obtained by the substitution $c=\frac{1}{2}(2d-p+2a-2)$. If p is odd, (9) follows from (22) by the substitution $c=\frac{1}{2}(2d-p+2a+1)$ or $c=\frac{1}{2}(2d-p+2a-1)$. In the cases (iii) and (iv), inequalities are induced from the convergence of (21).

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