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WEIGHTED SPHERICAL RESTRICTION THEOREMS FOR THE FOURIER TRANSFORM

BY

STEVEN BLOOM[1] AND GARY SAMPSON

1. Introduction

Spherical restriction theorems are of the form

$$(1.1)\qquad \left(\int_{\Sigma_{n-1}} |\hat{f}(t)|^q \, d\sigma(t)\right)^{1/q} \le C\left(\int_{\mathbf{R}^n} |f(x)|^p \, dx\right)^{1/p}$$

where Σ_{n-1} is the surface of the unit ball in $\mathbf{R}^n$, with the usual surface measure $d\sigma$, and where f belongs to a suitably nice class of test functions.

The first such result was discovered by Stein and published by C. Fefferman in [4]. Here $q = 2$ and $1 \le p \le 4n/(3n+1)$. That such an inequality could hold was fairly surprising. After all, why should the L^p-norm of f control the behavior of $\hat{f}$ on a set of measure zero? Certainly, (1.1) must be absurd when $p = q = 2$, since f can be obtained from $\hat{f}$ by Fourier inversion, and this is unaffected if we change $\hat{f}$ on a set of measure zero.

Actually, this reasoning is rather näive. (1.1) does fail when $p = q = 2$, but for entirely different reasons. Indeed, weighted $(2,2)$ results do hold, as we will see in Theorem 2.1. Consider a small arc Γ in $\mathbf{R}^2$ parametrized as $(1-t^2, t)$, $0 \le t \le \delta$. Obviously this approximates the circle. We consider the possibility of a Restriction theorem to Γ,

$$(1.2)\qquad \|\hat{f}_{|\Gamma}\|_q \le C\|f\|_p.$$

Let D be the rectangle

$$\left\{\frac{1}{2\delta^2} \le x \le \frac{1}{\delta^2}, 0 \le y \le \frac{1}{\delta}\right\},$$

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and put

$$f(x, y) = e^{ix} g(x, y)$$

where $g \geq 0$ is supported in D. Then

$$\int |\hat{f}_{|\Gamma}|^q = \int_0^\delta \left| \iint_D e^{-i(x,y)\cdot(1-t^2,t)} e^{ix} g(x, y) \right|^q dt.$$

Now in $[0, \delta] \times D$, $|xt^2 - yt| \leq 1$ and so $e^{i(xt^2 - yt)}$ is essentially a constant. Hence,

$$\int |\hat{f}_{|\Gamma}|^q \approx \delta \left(\int \int g \right)^q$$

and if (1.2) holds, we'd have

$$\delta^{1/q} \left(\int g \right) \leq C \|g\|_p.$$

By Hölder's inequality,

$$\sup \left\{ \frac{\left(\int g \right)}{\|g\|_p} : g \geq 0 \text{ and supported in } D \right\} = |D|^{1-p^{-1}} = \delta^{3(1-p^{-1})}$$

and so, letting $\delta \to 0$, (1.2) forces

$$\frac{1}{q} \geq 3\left(1 - \frac{1}{p}\right) \tag{1.3}$$

and of course $p = q = 2$ is ruled out. (1.3) is sharp. In 1974, Zygmund [10] proved:

THEOREM 1.4. *Let* $1 \leq p < 4/3$ *and* $1/q \geq 3(1 - 1/p)$. *Then*

$$\|\hat{f}_{|\Sigma_1}\|_q \leq C \|f\|_{L^p(\mathbf{R}^2)}$$

for suitably nice functions f (*say* $f \in C_0^\infty$).

From our perspective, a weighted version of (1.1),

$$\|\hat{f}_{|\Sigma_{n-1}}\|_q \leq C \|f\|_{L^p(v, \mathbf{R}^n)} \tag{1.5}$$

would lead to a corresponding weighted Hölder's calculation, and depending on v, (1.3) could vary considerably. (1.5) does hold for $p = q = 2$ and a wide range of weights v, the zero measure objection notwithstanding.

Zygmund's proof of Theorem 1.4 was very clever. Let us outline his argument: Consider $\hat{f}_{|\Gamma}$ where Γ is a small piece of the unit circle, say $(\cos t, \sin t)$, for $0 \leq t \leq 1$. Let T be the dual Restriction operator,

$$Tg(x, y) = \int_0^1 e^{-i(x,y)\cdot(\cos t, \sin t)} g(t)\, dt.$$

It suffices to show $\|Tg\|_{p'} \leq C\|g\|_{q'}$. For this,

(1.6)
$$|Tg(x, y)|^2 = 2\,\mathrm{Re} \int_0^1 \int_0^1 e^{-i(x,y)\cdot(\cos t - \cos s, \sin t - \sin s)} g(s) g(t)\, ds\, dt.$$

Now change variables: Let $u = \cos t - \cos s$, and $v = \sin t - \sin s$. Then (1.6) is the Fourier transform of some function h in the $u - v$ plane,

$$|Tg(x, y)|^2 = 2\,\mathrm{Re}\, \hat{h}(x, y)$$

and by Hausdorff-Young,

$$\|Tg\|_{p'} = c\|\hat{h}\|_{p'/2}^2 \leq C\|h\|_{(p'/2)'}^2.$$

Now change variables back to the (t, s) plane. Straightforward estimates finish the proof.

The splendid observation, that the square of the dual operator is essentially a Fourier transform works only because the dimension of Σ_{n-1} is half the dimension of $\mathbf{R}^n$, in other words, because $n = 2$. At present, there is no sharp analog of Theorem 1.4 known in dimension 3 or higher.

An entirely different argument, starting with Parseval's theorem, was discovered by Tomas. This yields:

THEOREM 1.7.

$$\|\hat{f}_{|\Sigma_{n-1}}\|_2 \leq C\|f\|_p \quad \textit{if } 1 \leq p \leq \frac{2n+2}{n+3}.$$

Actually, Tomas proved this for $p < (2n + 2)/(n + 3)$. Stein adapted his argument to cover the endpoint $p = (2n + 2)/(n + 3)$, and Tomas described this sharper proof in his paper [7]. The same argument that led to (1.3) shows that $p = (2n + 2)/(n + 3)$ is sharp.

Our concern in this paper is with the weighted Restriction theorem (1.5). When v is essentially a power weight,

$$v(x) = \begin{cases} |x|^\alpha & \text{if } |x| \le 1 \\ |x|^\beta & \text{if } |x| > 1, \end{cases} \tag{1.8}$$

we find necessary and sufficient conditions on α and β for (1.5) to hold over a wide range of p and q. The best results, naturally, are obtained in two dimensions, although we do obtain sharp results in higher dimensions, but for a more restrictive range of p's and q's.

Deriving weighted versions of the Restriction theorem seems to have caught several mathematicians' fancies lately. In [3], Chanillo and Sawyer showed that (1.5) holds when $p = q = 2$ and

$$v^{-(n+1)/2} \in L^1(\mathbf{R}^n). \tag{1.9}$$

Applied to power weights, this allows weights when $\beta > 2n/(n+1)$ and $\alpha < 2n/(n+1)$.[2]

This is fairly good, although not sharp. The sharp conditions are $\alpha < n$ and $\beta > 1$, and this is shown in Section 2, where we analyze the (2, 2) point.

Chanillo and Sawyer found a fascinating application of this result to an analytic continuation problem. They also unveiled this little gem: Suppose $p = (2n+2)/(n+3)$, the critical endpoint in the Tomas-Stein theorem, and $\|f\|_p = 1$. Then

$$\int |f|^p = \int |f|^2 v \quad \text{where } v = |f|^{p-2}.$$

Now, $-\frac{1}{2}(n+1)(p-2) = p$, and so (1.9) holds, and hence

$$\|\hat{f}_{|\Sigma_{n-1}}\|_2^2 \le C \int |f|^2 v = C\|f\|_p^p = C,$$

proving the Tomas-Stein result. The unweighted $(p, 2)$ Restriction theorem is a special case of the weighted (2, 2) Restriction theorem. This is a special case of the following proposition.

PROPOSITION 1.10. *Let $1 < p < q < \infty$. A linear operator T maps $L^p \to L^q$ boundedly if and only if $T: L^q(w) \to L^q$ boundedly whenever $w^{-1} \in L^{p/(q-p)}$.*

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[2]Chanillo and Sawyer actually obtain a wide range of conditions sufficient to (1.5), but for power weights this L^1 condition yields the sharpest result.

Proof. One direction is the Chanillo-Sawyer argument. For the other direction, it will suffice to show that the adjoint operator $T^* : L^{q'} \to L^{q'}(w^{-q'/q})$. But Hölder's inequality gives

$$\int |T^*f|^{q'} w^{-q'/q} \le \left(\int |T^*f|^{p'}\right)^{q'/p'} \left(\int w^{-p(q-p)}\right)^{1-q'/p'}.$$

In [1], Benedetto and Heinig derive weighted norm inequalities for the Fourier transform of functions with vanishing moments, when the weights can include radial measures. As an application, they view the Restriction operator as a weighted Fourier transform, and obtain sufficient conditions for (1.8) to imply (1.5), specifically

$$n(p-1) < \beta < n(p-1) + p. \tag{1.11}$$

This is a very clever idea. Unfortunately, in the setting without vanishing moments, condition (1.11) is very far removed from the sharp conditions.

Section 3 is devoted to the description of the various necessary conditions. In Section 4, we analyze the general (p, q) point in two dimensions, and Section 5 deals with higher dimensions.

2. The (2, 2) point

The main result of this section is:

THEOREM 2.1. *Let*

$$v(x) = \begin{cases} |x|^\alpha & \text{if } |x| \le 1 \\ |x|^\beta & \text{if } |x| > 1. \end{cases}$$

Then

$$\|\hat{f}|_{\Sigma_{n-1}}\|_2 \le C\|f\|_{L^2(v;\mathbf{R}^n)}$$

holds for all $f \in C_0^\infty(\mathbf{R}^n)$ if and only if $\alpha < n$ and $\beta > 1$.

We will prove this by studying the dual Restriction operator T,

$$Tg(x) = \int_{\Sigma_{n-1}} e^{-ix\cdot t} g(t)\, d\sigma(t).$$

We will also use a damping factor. Most any one will do, but to be specific (in this section—in Section 3, a different damping factor will prove useful), take

$$\varphi : \mathbf{R}_+ \to \mathbf{R}_+$$

given by

$$\varphi(x) = (1 + x^N)^{-1}$$

where N is a large integer depending on the dimension n appropriately chosen for the proof. Set $\varphi_R(x) = \varphi(x/R)$. Observe that the derivatives of φ_R satisfy

$$\varphi_R^{(k)}(x) \leq C_k x^{-k} \varphi_R(x) \quad \text{for } k = 1, 2, 3, \ldots . \tag{2.2}$$

LEMMA 2.3. *$\|\hat{f}_{|\Sigma_{n-1}}\|_2 \leq C\|f\|_{L^2(v)}$ for all $f \in C_0^\infty(\mathbf{R}^n)$ if and only if*

$$\|Tg\|_{L^2(v^{-1}(x)\varphi_R(|x|))} \leq C\|g\|_{L^2(\Sigma_{n-1})}$$

for all $R > 0$, with C independent of R.

Proof. Fix $f \in C_0^\infty$ and let $\{|x| \leq R\}$ contain the support of f. Then

$$\begin{aligned}\|\hat{f}_{|\Sigma_{n-1}}\|_2 &= \sup\left\{\left|\int_{\Sigma_{n-1}} g(t)\hat{f}(t)\, d\sigma(t)\right| : \|g\|_{L^2(\Sigma_{n-1})} \leq 1\right\} \\ &\leq \sup\left\{\int_{\mathbf{R}^n} |Tg(x)|\,|f(x)|\, dx : \|g\| \leq 1\right\}.\end{aligned}$$

Now $\varphi_R(|x|) \geq \frac{1}{2}$ on the support of f, and so

$$\begin{aligned}\|\hat{f}_{|\Sigma_{n-1}}\|_2 &\leq 2\sup\left\{\int_{\mathbf{R}^n} |Tg(x)|\varphi_R(|x|)|f(x)|\, dx : \|g\| \leq 1\right\} \\ &\leq 2\|f\|_{L^2(\varphi_R v)} \sup\|Tg\|_{L^2(\varphi_R v^{-1})} \\ &\leq 2\|f\|_{L^2(v)} \sup\|Tg\|_{L^2(\varphi_R v^{-1})}\end{aligned}$$

and one direction follows. The argument for the other direction is virtually identical.

We will make heavy use of Bessel functions of order ν, $J_\nu(r)$. Several facts will be pertinent.

$$\int_{\Sigma_{n-1}} e^{ix\cdot t}\, d\sigma(t) = \frac{J_{n/2-1}(|x|)}{|x|^{n/2-1}}. \tag{2.4}$$

$$J_{n/2-1}(r) = (2/\pi r)^{-1/2} \cos(r - (n-1)\pi/4) + e(r) \tag{2.5}$$

where $e(r) = O(r^{-3/2})$ as $r \to \infty$, and

$$\frac{d}{dr} r^\nu J_\nu(r) = r^\nu J_{\nu-1}(r) \quad \text{for } \nu > 0. \tag{2.6}$$

See Watson [9], for these, and our other assertions about Bessel functions.

Proof of the necessity in 2.1. In Lemma 2.3, take $g \equiv 1$. By (2.4),

$$|Tg(x)| = J_{n/2-1}(|x|)|x|^{1-n/2},$$

and so the lemma gives

$$\int_0^\infty r^{n-1} w(r)\varphi_R(r)\left|J_{n/2-1}(r)\right|^2 r^{2-n}\, dr \le C \tag{2.7}$$

where $w(|x|) = v^{-1}(x)$, and this constant C is independent of R. Near zero, by (2.4),

$$J_{n/2-1}(r) \approx C r^{n/2-1},$$

and so the integrand in (2.7) is essentially $r^{n-1-\alpha}$. Hence $\alpha < n$.

Let

$$I_k = \{r \in [2\pi k, 2\pi(k+1)] : |\cos(r - (n-1)\pi/4)| \ge 1/2\}.$$

By (2.5), for $r \in I_k$, $|J_{n/2-1}(r)| \ge C r^{-1/2}$. It also follows that

$$\begin{aligned}\int_{2\pi k}^{2\pi(k+1)} w(r)\varphi_R(r)\, dr &\le C \int_{I_k} w(r)\varphi_R(r)\, dr \\ &\le C \int_{I_k} r w(r)\varphi_R(r)\left|J_{n/2-1}(r)\right|^2 dr\end{aligned}$$

and (2.7) gives, for some integer K,

$$\sum_{k \ge K} \int_{2\pi k}^{2\pi(k+1)} w(r)\varphi_R(r)\, dr \le C,$$

and so

$$\int_{2\pi K}^\infty w(r)\varphi_R(r)\, dr \le C.$$

Choosing R much larger than $2\pi K$ forces $\beta > 1$.

For the sufficiency, we will use a surface measure version of Minkowski's inequality, whose proof, which is quite standard, we will omit:

LEMMA 2.8. *If $1 \leq p \leq \infty$, and if*

$$\sup_{s \in \Sigma_{n-1}} \int_{\Sigma_{n-1}} |\alpha(s-t)|\, d\sigma(t) < \infty,$$

then

$$\left[\int_{\Sigma_{n-1}} \left| \int_{\Sigma_{n-1}} g(t)\alpha(s-t)\, d\sigma(t)\right|^p d\sigma(s)\right]^{1/p} \leq C\left[\int_{\Sigma_{n-1}} |g(t)|^p\, d\sigma(t)\right]^{1/p}.$$

Proof of the sufficiency of Theorem 2.1. Fix a $\varphi = \varphi_R$ and consider

$$\begin{aligned}
&\int |Tg|^2 \varphi(|x|) v^{-1}(x)\, dx \\
&= \int \varphi(|x|) v^{-1}(x) \int_{\Sigma_{n-1}} g(s) e^{ix\cdot s}\, d\sigma(s) \int_{\Sigma_{n-1}} g(t) e^{-ix\cdot t}\, d\sigma(t)\, dx \\
&= \int_{\Sigma_{n-1}} g(s) \int_{\Sigma_{n-1}} g(t) \int_0^\infty r^{n-1} \varphi(r) w(r) \int_{\Sigma_{n-1}} e^{ir\tau\cdot(s-t)}\, d\sigma(\tau)\, dr\, d\sigma(t)\, d\sigma(s)
\end{aligned}$$

where again $w(|x|) = v^{-1}(x)$

$$= \int\int g(s) g(t) \int_0^\infty r^{n-1} w(r) \varphi(r) \frac{J_{n/2-1}(r|s-t|)}{(r|s-t|)^{n/2-1}}\, dr\, d\sigma(t)\, d\sigma(s)$$

by (2.4)

$$= \int\int g(s) g(t) |s-t|^{-n/2} I(|s-t|)$$

where

$$I(t) = t\int_0^\infty r^{n/2} J_{n/2-1}(rt) \varphi(r) w(r)\, dr.$$

It will suffice to show that

$$|I(t)| \leq C t^{\beta - n/2} \tag{2.9}$$

with C independent of R. For if (2.9) holds, then

$$\int |Tg|^2 \varphi v^{-1} \le C \int \int |g(s)|\,|g(t)|\,|s-t|^{\beta-n}\, d\sigma(s)\, d\sigma(t).$$

Now,

$$\sup_{s\in\Sigma_{n-1}} \int_{\Sigma_{n-1}} |s-t|^{\beta-n}\, d\sigma(t) \le C\int_0^2 r^{\beta-n} r^{n-2}\, dr < \infty,$$

since $\beta > 1$. So, by Lemma 2.8 and a Cauchy-Schwarz,

$$\int |Tg|^2 \varphi v^{-1} \le C\int_{\Sigma_{n-1}} |g(t)|^2\, d\sigma(t)$$

and the theorem follows from Lemma 2.3.

$$\left|\int_0^1 tr^{n/2} J_{n/2-1}(rt)\varphi(r) w(r)\, dr\right| \le C\int_0^1 tr^{n/2}(rt)^{n/2-1} r^{-\alpha}\, dr \le Ct^{n/2}$$

since $\alpha < n$. For the term from 1 to ∞, let D be the operator

$$D = \frac{1}{r}\cdot\frac{d}{dr}.$$

Using (2.6), we have

$$\int_1^\infty \varphi(r) r^{-\beta}\, d\left[r^{n/2} J_{n/2}(rt)\right]$$

$$= \varphi(r) r^{-\beta} J_{n/2}(rt)|_1^\infty - \int_1^\infty D(\varphi r^{-\beta}) r^{n/2+1} J_{n/2}(rt)\, dr$$

and the term at infinity vanishes. Continuing this integration by parts, we obtain

$$I(t) = c(t) + (-1)^k t^{1-k} \int_1^\infty r^{n/2+k} J_{n/2+k-1}(rt) D^k\left[\varphi r^{-\beta}\right] dr.$$

Because of (2.2), each of the infinity terms vanish (so long as N is much larger than k), and so $c(t)$ is a bounded term. Thus (2.9) will follow if we can demonstrate that

$$t^{1-k}\left|\int_1^\infty r^{n/2+k} J_{n/2+k-1}(rt) D^k\left[\varphi r^{-\beta}\right] dr\right| \le Ct^{\beta-n/2} \tag{2.10}$$

for some positive integer k. We split up this integral into the pieces from zero to $1/t$, and from $1/t$ on, calling them $A + B$. From (2.2), we get

$$\left|D^k\left[\varphi r^{-\beta}\right]\right| \leq c_k r^{-\beta-2k}\left[1 + (r/R)^N\right]^{-1}$$

and from (2.4),

$$\left|J_{n/2+k-1}(rt)\right| \leq C(rt)^{n/2+k-1} \quad \text{when } r \leq 1/t.$$

So,

$$\begin{aligned} A &\leq Ct^{1-k}\int_1^{1/t} r^{n/2+k}(rt)^{n/2+k-1} r^{-\beta-2k}\left[1 + (r/R)^N\right]^{-1} dr \\ &\leq Ct^{n/2}\int_1^{1/t} r^{n-\beta-1}\, dr \leq Ct^{\beta-n/2} \end{aligned}$$

(so long as $\beta < n$. If $\beta \geq n$, the estimates change, but the proof does not).

For B, we use the asymptotic estimate $|J_\nu(r)| \leq Cr^{-1/2}$ from (2.5). This gives

$$B \leq Ct^{1-k}\int_{1/t}^{\infty} r^{n/2+k}(rt)^{-1/2} r^{-\beta-2k}\, dr \leq Ct^{\beta-n/2}$$

so long as $k > (n + 1 - 2\beta)/2$, and the proof is complete.

We close off our study of the (2, 2) point by investigating power weights v that are no longer radial. That is, v will have the form

$$v(x) = \left(|x_1| \cdot |x_2| \cdots |x_n|\right)^{\alpha}.$$

The analysis of such weights can get very complicated, and for simplicity, we will consider this case in two dimensions only. Here,

$$v(x, y) = w(x)w(y)$$

where

$$w(x) = \begin{cases} |x|^{\alpha/2} & \text{if } |x| \leq 1 \\ |x|^{\beta/2} & \text{if } |x| > 1. \end{cases}$$

This is, essentially, a smaller weight than the radial weight considered in Theorem 2.1. Still, one would reasonably expect the weights to behave identically, and for the Restriction operator to be bounded if and only if $\alpha < 2$ and $\beta > 1$. And so, the actual result, that $\beta \geq 4/3$, seems surprising.

THEOREM 2.11. *Let*

$$w(x) = \begin{cases} |x|^\alpha & \text{if } |x| \leq 1 \\ |x|^\beta & \text{if } |x| > 1 \end{cases}$$

and let $v(x, y) = w(x)w(y)$. *Then*

$$\|\hat{f}_{|\Sigma_1}\|_2 \leq C\|f\|_{L^2(v;\mathbf{R}^2)}$$

if and only if $\alpha < 1$ *and* $\beta \geq 2/3$.

Proof. We will consider only the portion of the circle $(\cos t, \sin t)$ with $0 \leq t \leq 1$. With T the usual dual operator,

$$|Tg(x, y)|^2 = 2\int_0^1\int_0^t g(t)g(s)\cos[x(\cos t - \cos s) + y(\sin t - \sin s)]\, ds\, dt.$$

Now apply the addition law for cosines. Since w is an even function, the sine terms vanish, and we have

$$\int |Tg|^2 v^{-1} = 8\int_0^1\int_0^t g(t)g(s)\int_0^\infty w^{-1}(x)\cos[x(\cos t - \cos s)]\, dx$$
$$\times \int_0^\infty w^{-1}(y)\cos[y(\sin t - \sin s)]\, dy\, ds\, dt.$$

Clearly for these inside integrals to be well-behaved near zero, $\alpha < 1$. Consider integrals of the form

$$\int_0^\infty w^{-1}(x)\cos(\tau x)\, dx, \quad \text{with } \tau \text{ small.} \tag{2.12}$$

These are clearly bounded when $\beta \geq 1$. When $\beta < 1$, (2.12) is essentially a constant times $|\tau|^{\beta-1}$, and so, up to a constant factor,

$$\int |Tg|^2 v^{-1} \approx \int_0^1 \int_0^t g(t) g(s) |\cos s - \cos t|^{\beta-1} |\sin s - \sin t|^{\beta-1}\, ds\, dt$$
$$\approx \int_0^1 \int_0^t g(t) g(s) t^{\beta-1} (t-s)^{2(\beta-1)}\, ds\, dt.$$

And so, assuming that $\alpha < 1$, the Restriction problem is entirely equivalent to the behavior of the bilinear form

$$\int_0^1 \int_0^t g(t) g(s) t^{\beta-1} (t-s)^{2(\beta-1)}\, ds\, dt \tag{2.13}$$

where $g \in L^2([0,1])$. We need to determine when (2.13) is bounded by $C\|g\|_2^2$. Take $g = t^{-p}$, where $0 < p < 1/2$. So $g \in L^2$, and

$$\int_0^t g(s)(t-s)^{2(\beta-1)}\, ds = \int_0^{t/2} s^{-p}(t-s)^{2(\beta-1)}\, ds + \int_0^{t/2} (t-s)^{-p} s^{2(\beta-1)}\, ds$$
$$\approx c_p t^{2\beta-p-1}$$

and (2.13) is essentially

$$\int_0^1 t^{3\beta-2p-2}.$$

Since this must be integrable, we must have $2p + 2 - 3\beta < 1$, or $\beta > (2p+1)/3$, and since this holds for all $p < 1/2$, $\beta \geq 2/3$ is necessary.

For the sufficiency, the bilinear form in (2.13) is bounded on L^2 if the operator S, given by

$$Sg(t) = t^{\beta-1} \int_0^t (t-s)^{2(\beta-1)} g(s)\, ds$$

is bounded on L^2, and this holds, by Schur's Lemma, if there exists a positive function h for which

$$Sh(t) + S^*h(t) \leq Ch(t) \tag{2.14}$$

where

$$S^*h(t) = \int_t^1 s^{\beta-1}(s-t)^{2(\beta-1)} h(s)\, ds$$

is the adjoint operator of S.[3] Let $\gamma > 0$ and take $h(t) = t^{-\gamma}$. Then

$$\begin{aligned} Sh(t) &= t^{\beta-1}\left[\int_0^{t/2} s^{-\gamma}(t-s)^{2(\beta-1)}\,ds + \int_{t/2}^{t} s^{-\gamma}(t-s)^{2(\beta-1)}\,ds\right] \\ &\le Ct^{3(\beta-1)+1-\gamma} \le Ct^{-\gamma} \end{aligned}$$

if $\beta \ge 2/3$ and if $\gamma < 1$.

Likewise,

$$S^*h(t) \le \int_t^{2t} s^{\beta-1}(s-t)^{2(\beta-1)} s^{-\gamma}\,ds + \int_{2t}^{1} s^{\beta-1-\gamma}(s-t)^{2(\beta-1)}\,ds$$

where this last integral should be taken to be zero if $t \ge 1/2$,

$$\le Ct^{\beta-1-\gamma}\int_0^t s^{2(\beta-1)}\,ds + C\int_t^1 s^{3(\beta-1)-\gamma}\,ds.$$

When $\beta < 1$, we choose γ so that $3(\beta-1)-\gamma < -1$, to obtain

$$S^*h(t) \le Ct^{3(\beta-1)+1-\gamma}$$

here as well. If $\beta \ge 1$, then the first integral above dominates, and the same estimate prevails. Hence, the Schur condition (2.14) holds, and the proof is complete.

3. Necessary conditions

Suppose that $\|\hat{f}_{|\Sigma_{n-1}}\|_q \le C\|f\|_{L^p(v)}$ where $1 < p,\ q < \infty$, with

$$v(x) = \begin{cases} |x|^\alpha & \text{if } |x| \le 1 \\ |x|^\beta & \text{if } |x| > 1. \end{cases}$$

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[3]It is easy to see why (2.14) is sufficient. Clearly the operator $f \to h^{-1}S(hf)$ is bounded on L^∞, and by duality, $f \to hS(h^{-1}f)$ is bounded on L^1. Complex interpolation establishes the boundedness of S on L^2.

Although we need only the sufficiency of (2.14) in our argument, every step so far, including Schur's Lemma, is reversible, and (2.14) is also a necessary condition for the boundedness of the Restriction operator.

Then the dual operator $T: L^{q'}(\Sigma_{n-1}) \to L^{p'}(v^{1-p'})$. Taking $g \equiv 1$,

$$Tg(x) = J_{n/2-1}(|x|)|x|^{1-n/2},$$

and so a necessary condition is

$$\int v(x)^{1-p'}\left(\frac{J_{n/2-1}(|x|)}{|x|^{n/2-1}}\right)^{p'} dx \leq C. \tag{3.1}$$

From the estimates (2.4), this tells us that $v^{1-p'}$ is integrable at zero, or

$$\alpha < n(p-1) \tag{3.2}$$

while (2.5) and (3.1) show that

$$r^{\beta(1-p')+(1-n)p'/2+n-1}$$

is integrable at infinity, or

$$\beta > \frac{n+1}{2}p - n. \tag{3.3}$$

Next, fix $\varepsilon > 0$ and let $e_1 = (1, 0, \ldots, 0)$ be the standard basis vector for $\mathbf{R}^n$. Set

$$D = \{rt : 0 \leq r \leq 1/(n\varepsilon^2),\ t \in \Sigma_{n-1}, \text{ and } |e_1 - t| \leq \varepsilon\}.$$

Let $g \geq 0$ be supported in D and put

$$f(x) = e^{ie_1 \cdot x} g(x).$$

Then for $s \in \Sigma_{n-1}$ with $|e_1 - s| \leq \varepsilon$ and for $x \in D$, we have

$$\begin{aligned} |(e_1 - s) \cdot x| &= r|(e_1 - s) \cdot t| \\ &= r|(1 - s_1)t_1 - s_2 t_2 - \cdots - s_n t_n| \\ &\leq r(1 - s_1) + (n-1)\varepsilon^2 r. \end{aligned}$$

But

$$\varepsilon^2 \geq (1 - s_1)^2 + s_2^2 + \cdots + s_n^2 = (1 - s_1)^2 + (1 - s_1^2) = 2(1 - s_1),$$

and so

$$|(e_1 - s) \cdot x| \leq r\varepsilon^2(\tfrac{1}{2} + n - 1) \leq 1.$$

Hence,

$$|\hat{f}(s)| = \left|\int_D e^{i(e_1-s)\cdot x} g(x)\,dx\right| \approx \int_D g(x)\,dx.$$

Therefore,

$$\begin{aligned}\int_{\Sigma_{n-1},|e_1-s|\le\varepsilon}\left(\int_D g\right)^q d\sigma(s) &\le C\int_{\Sigma_{n-1}} |\hat{f}(s)|^q\,d\sigma(s)\\ &\le C\left(\int |f|^p v(x)\,dx\right)^{q/p}\\ &= C\left(\int_D g^p(x)v(x)\,dx\right)^{q/p},\end{aligned}$$

which means that

$$\varepsilon^{n-1}\left(\int_D g\right)^q \le C\left(\int_D g^p v\right)^{q/p}.$$

If we take $g = v^{1-p'}$, we obtain

$$\varepsilon^{n-1}\left(\int_D v^{1-p'}\right)^{q/p'} \le C$$

(this is justified, since $v^{1-p'}$ is integrable, i.e. by (3.2)).

On the other hand,

$$\begin{aligned}\int_D v^{1-p'} &\ge \int_{\Sigma_{n-1},|e_1-s|\le\varepsilon}\int_1^{1/n\varepsilon^2} r^{(1-p')\beta} r^{n-1}\,dr\\ &\ge C\varepsilon^{n-1}\int_1^{1/n\varepsilon^2} r^{(1-p')\beta+n-1}\,dr.\end{aligned}$$

We gain no information from this if $\beta \ge n(p-1)$, but if $\beta < n(p-1)$, we have

$$\int_D v^{1-p'} \ge C\varepsilon^{n-1}\varepsilon^{2\beta(p'-1)-2n}$$

which forces

$$\beta \ge n(p-1) - \frac{n-1}{2}\left(p-1+\frac{p}{q}\right). \tag{3.4}$$

Notice that (3.4) is more restrictive than (3.3) when $p < q$. Otherwise, (3.3) prevails.

Finally, suppose that $\beta < 0$. Let $g_k(t) = e^{ike_1 \cdot t}$ and let $x = ke_1 + y$. Then

$$\begin{aligned} Tg_k(x) &= \int_{\Sigma_{n-1}} e^{ike_1 \cdot t} e^{-i(ke_1+y)\cdot t}\, d\sigma(t) = \int_{\Sigma_{n-1}} e^{-iy\cdot t}\, d\sigma(t) \\ &= J_{n/2-1}(|y|)|y|^{1-n/2}. \end{aligned}$$

In particular, $|T_k g(x)| \geq c_1 > 0$ if $|y| \leq 1$. So,

$$\int |Tg_k(x)|^{p'} v^{-p'/p} \geq c_1^{p'} \int_{|y|\leq 1} |ke_1 + y|^{-\beta p'/p}\, dy$$

which tends to infinity as k tends to infinity. Yet each $\|g_k\|_{q'} \leq C$, so that v cannot be a good weight. We've shown

$$\beta \geq 0. \tag{3.5}$$

Let us summarize all of these results:

THEOREM 3.6. *Let*

$$v(x) = \begin{cases} |x|^\alpha & \text{if } |x| \leq 1, \\ |x|^\beta & \text{if } |x| > 1. \end{cases}$$

If $1 < p,\ q < \infty$, *and if*

$$\|\hat{f}_{|\Sigma_{n-1}}\|_q \leq C\|f\|_{L^p(v)}$$

then $\beta \geq 0$ *and* $\alpha < n(p-1)$. *Moreover*:

(a) *if* $p < q$ *then*

$$\beta \geq n(p-1) - \frac{n-1}{2}\left(p - 1 + \frac{p}{q}\right);$$

(b) *if* $p \geq q$ *then*

$$\beta > \frac{n+1}{2}p - n.$$

In some circumstances, we can improve on condition (a) above.

THEOREM 3.7. *Under the hypotheses of Theorem* 3.6, *if in addition*, $p \leq 2$ *and* $1/p + 1/q \leq 1$, *then*

$$\beta \geq n(p-1) - (n-1)\frac{p}{q}. \tag{3.8}$$

Proof. We will show that (3.8) is necessary whenever $p \le 2$. The other restriction, that $1/p + 1/q \le 1$ is simply because (3.8) is weaker than (3.6) (a) otherwise.

We can assume without loss of generality that $\beta < n(p-1)$ (and α as well). By the argument of Lemma 2.3, with T the usual dual operator,

$$\|Tg\|_{L^{p'}(\varphi v^{-p'/p})} \le C\|g\|_{q'}$$

where φ is a nice cutoff function and C is independent of φ. For this argument, we will take

$$\varphi = \chi_{[|x| \le R]}.$$

Let $w(r) = v^{-p'/p}(x)$ when $|x| = r$. Then, as usual, we have

$$\|Tg\|^2_{L^2(\varphi v^{-p'/p})} = \int_{\Sigma_{n-1}} \int_{\Sigma_{n-1}} g(s) g(t) \int_0^R r^{n-1} w(r) \frac{J_{n/2-1}(r|s-t|)}{(r|s-t|)^{n/2-1}} \times dr\, d\sigma(s)\, d\sigma(t).$$

From (2.4),

$$\lim_{x \to 0} \frac{J_{n/2-1}(|x|)}{|x|^{n/2-1}} = c > 0,$$

and so, if $|x| \le k$,

$$\frac{J_{n/2-1}(|x|)}{|x|^{n/2-1}} \text{ is essentially a constant.}$$

Let D be a disk on Σ_{n-1} of diameter k/R. Let $g = \chi_D$. Then, ignoring constants,

$$\|Tg\|^2_{L^2(\varphi v^{-p'/p})} \approx \int_D \int_D \int_0^R r^{n-1} w(r)\, dr \approx |D|^2 R^{n-\beta p'/p} \approx R^{n-\beta p'/p-2(n-1)}$$

and so

$$\begin{aligned} R^{n-\beta p'/p-2(n-1)} &\approx \int |Tg|^2 \varphi v^{-p'/p} \\ &\le \left(\int |Tg|^{p'} \varphi v^{-p'/p}\right)^{2/p'} \left(\int \varphi v^{-p'/p}\right)^{1(2/p')} \\ &\qquad\qquad \text{by Hölder's Inequality} \\ &\approx \|Tg\|^2_{L^{p'}(\varphi v^{-p'/p})} \left(R^{n-\beta p'/p}\right)^{1(2/p')}. \end{aligned}$$

And so,

$$R^{(n-\beta p'/p)2/p'-2(n-1)/q} \leq C,$$

and letting $R \to \infty$ gives the result.

4. The (p, q) point in two dimensions

We start by extending the (2, 2) result of Theorem 2.1 to arbitrary q.

THEOREM 4.1. *Let*

$$v(x) = \begin{cases} |x|^{\alpha} & \text{if } |x| \leq 1 \\ |x|^{\beta} & \text{if } |x| > 1, \end{cases}$$

and $1 < q < \infty$. *Then*

$$\|\hat{f}_{|\Sigma_{n-1}}\|_q \leq C\|f\|_{L^2(v)}$$

if and only if $\alpha < n$ *and*
(a) *if* $q \leq 2$ *then* $\beta > 1$,
while
(b) *if* $q > 2$, *then*

$$\beta \geq n - (n-1)\frac{2}{q}.$$

The necessity side of this theorem follows immediately from Theorem 3.6(b) and Theorem 3.7.

For the sufficiency, in case (a), we simply apply the following lemma to Theorem 2.1:

LEMMA 4.2. *If* $\|\hat{f}_{|\Sigma_{n-1}}\|_q \leq C\|f\|_{L^p(v)}$ *and if* $s < q$, *then*

$$\|\hat{f}_{|\Sigma_{n-1}}\|_s \leq C\|f\|_{L^p(v)}$$

as well.

Proof. Use Hölder's Inequality.

The sufficiency of case (b) rests on the "fractional" integration theory applied to Σ_{n-1}, which we give in the following lemma.

LEMMA 4.3. *Let* $1 < p < \infty$ *and* $0 < \alpha < (n-1)/p$. *For* $t \in \Sigma_{n-1}$, *set*

$$I_\alpha g(t) = \int_{\Sigma_{n-1}} g(s)|t-s|^{\alpha-n+1}\, d\sigma(s).$$

Then $I_\alpha : L^p \to L^q$ *is a bounded operator so long as*

$$\frac{1}{q} \geq \frac{1}{p} - \frac{\alpha}{n-1}.$$

Proof. We will assume with no loss in generality that $g \geq 0$ and that $\|g\|_p = 1$. Let Mg be the maximal function of g on Σ_{n-1}. We follow the proof given for the one-dimensional fractional integral operator in Torchinsky's text [8, p. 151], proving the estimate

$$I_\alpha g(t) \leq C[1 + Mg(t)]^{1-\alpha p/(n-1)}. \tag{4.4}$$

Hölder's Inequality and the boundedness of the maximal operator would then give $I_\alpha g \in L^q$ so long as

$$q\left[1 - \frac{\alpha p}{n-1}\right] \leq p,$$

which is the condition on α in the hypotheses.

So we must prove (4.4). Fix some η with $0 < \eta \leq 1$. Write $I_\alpha g$ as

$$\begin{aligned} I_\alpha g(t) &= \int_{|s-t|\leq\eta} g(s)|s-t|^{\alpha-n+1} + \int_{|s-t|>\eta} g(s)|s-t|^{\alpha-n+1} \\ &= A + B. \end{aligned}$$

Break A up into intervals where $|s-t| \approx 2^{-k}\eta$ to get

$$A \leq C\eta^\alpha Mg(t)$$

and apply Hölder's Inequality to B to get

$$\begin{aligned} B &\leq \|g\|_p \left(\int_{|s-t|>\eta} \left(\int_{|s-t|>\eta} |s-t|^{(\alpha-n+1)p'}\, d\sigma(s) \right)^{1/p'} \right) \\ &= C\eta^{\alpha-(n-1)/p}. \end{aligned}$$

Now we choose η so that

$$\eta^{\alpha-(n-1)/p} = \eta^\alpha Mg(t),$$

in other words,

$$\eta = [Mg(t)]^{-p/(n-1)}.$$

This $\eta \le 1$ so long as $Mg(t) \ge 1$. Otherwise, take $\eta = 1$, and then $A + B \le C$. But with this choice of η,

$$A + B \le C\eta^{\alpha} Mg(t) = C[Mg(t)]^{1-\alpha p/(n-1)}$$

and (4.4) follows.

Now we can complete the proof of Theorem 4.1. We will need to show that

$$\|Tg\|_{L^2(\varphi_R v^{-1})} \le C\|g\|_{q'}.$$

Following the proof of Theorem 2.1,

$$\begin{aligned} \|Tg\|^2_{L^2(\varphi_R v^{-1})} &\le C\int_{\Sigma_{n-1}}\int_{\Sigma_{n-1}} |g(s)|\,|g(t)|\,|s-t|^{\beta-n}\,d\sigma(s)\,d\sigma(t) \\ &\le C\|g\|_{q'}\left[\int_{\Sigma_{n-1}}\left(\int_{\Sigma_{n-1}} |g(s)|\,|s-t|^{\beta-n}\,d\sigma(s)\right)^q d\sigma(t)\right]^{1/q} \end{aligned}$$

and so we require the fractional operator $I_{\beta-1}$ to map $L^{q'}$ into L^q boundedly, and this will ensue from the last lemma, provided

$$\frac{1}{q} \ge \frac{1}{q'} - \frac{\beta-1}{n-1}$$

or

$$\beta \ge n - (n-1)\frac{2}{q}.$$

Next, we require a generalization of Zygmund's Theorem.

THEOREM 4.5. *Let* $1 < p < 4/3$. *If*

$$\frac{1}{q} \ge 3\left(1 - \frac{1}{p}\right),$$

and if

$$v(x) = \begin{cases} |x|^{\alpha} & \text{if } |x| \le 1 \\ |x|^{\beta} & \text{if } |x| > 1, \end{cases}$$

then

$$\|\hat{f}_{|\Sigma_1}\|_q \le C\|f\|_{L^p(v)}$$

if and only if $\beta \ge 0$ *and* $\alpha < 2(p-1)$.

Proof. For the sufficiency, we follow Zygmund's proof as outlined in the introduction. From (1.6) and the appropriate change of variables, we have

$$\|Tg\|_{L^{p'}(v^{-p'/p})} = C\|\hat{h}\|^2_{L^{p'/2}(v^{-p'/p})}$$

and we dovetail back into Zygmund's proof if we can show

$$\|\hat{h}\|_{L^{p'/2}(v^{-p'/p})} \le C\|h\|_{(p'/2)'}.$$

The h in question is supported in $\{|x| \le 2\}$. So this is a weighted Fourier transform problem,[4]

$$\|\hat{f}\|_{L^{p'/2}(v^{-p'/p})} \le C\|f\chi_{[|x|\le 2]}\|_{(p'/2)'}.$$

When v is increasing, this will hold, by the Jurkat-Sampson Theorem [5], when

$$\sup_{0<s\le 2} s^2 \int_{|x|\le 1/s} v^{-p'/p}\,dx < \infty,$$

which is true when $\beta \ge 0$ and $\alpha < 2(p-1)$.

The necessity is immediate from Theorem 3.6.

We are almost ready to interpolate. We need the conditions at the $(4/3, 4/3)$ point first, and then we can let loose the machinery.

PROPOSITION 4.6. *For* $v(x)$ *as above,*

$$\|f_{|\Sigma_1}\|_{4/3} \le C\|f\|_{L^{4/3}(v)}$$

if and only if $\alpha < 2/3$ *and* $\beta > 0$.

---

[4]One could try more general weights at this stage of the argument in an effort to obtain more general Restriction theorems. After finishing an early draft of this paper, we received a preprint from Carton-Lebrun and Heinig [2], where they do just that to Sjölin's extension of Zygmund's Theorem [6]. They obtain sufficient conditions for weighted norm inequalities that are quite sharp in the Zygmund region, but well removed from the necessary conditions away from the region.

Proof. The necessity is Theorem 3.6(b). For the sufficiency, let $p_0 < 4/3$ and let $\mathscr{R}$ be the Restriction operator. By Theorem 4.5,

$$\mathscr{R} : L^{p_0}(v_0) \to L^{p_0}$$

where v_0 has powers $\alpha_0 < 2(p_0 - 1)$ and $\beta_0 = 0$. Likewise, from Theorem 2.1,

$$\mathscr{R} : L^2(v_1) \to L^2$$

where $\alpha_1 < 2$ and $\beta_1 > 1$. Let $0 \le t \le 1$ and put

$$\frac{1}{p} = \frac{t}{2} + \frac{1-t}{p_0}.$$

Interpolating with change of measure gives

$$\mathscr{R} : L^p(v) \to L^p$$

where

$$v = v_1^{pt/2} v_0^{p(1-t)/p_0}$$

and hence,

$$\alpha < 2(p-1)$$

as expected, and

$$\beta = \beta_1 \cdot \frac{p - p_0}{2 - p_0}. \tag{4.7}$$

For $p = 4/3$, and $\beta > 0$ will suffice, if p_0 is sufficiently close to $4/3$.

From (4.7), for any $4/3 \le p \le 2$, we obtain

$$\beta > \tfrac{3}{2}p - 2$$

which was also necessary by (3.6)(b). Thus when $p \ge q$ or in the Zygmund region, the necessary conditions are also sufficient. Straightforward interpolation extends these conditions into the remainder of the region $p \le 2$, first along the line $1/p + 1/q = 1$, and then into the rest of the region. In some of these cases, since one boundary has a condition of the form β *strictly* greater than some exponent, that is all we obtain inside. We have:

THEOREM 4.8. *Let $1 < p \le 2$, $1 \le q \le \infty$. Let $v(x)$ be a power weight in $\mathbf{R}^2$, of the form*

$$v(x) = \begin{cases} |x|^{\alpha} & \text{if } |x| \le 1 \\ |x|^{\beta} & \text{if } |x| > 1. \end{cases}$$

Then, in order to have

$$\|\hat{f}_{|\Sigma_1}\|_q \le C\|f\|_{L^p(v)}$$

it is both necessary and sufficient that $\alpha < 2(p-1)$. Regarding β:

(a) *If $1 < p < 4/3$ and if*

$$\frac{1}{q} \ge 3\left(1 - \frac{1}{p}\right),$$

then $\beta \ge 0$ is both necessary and sufficient.

(b) *If $4/3 \le p \le 2$ and if $q \le p$, then $\beta > \frac{3}{2}p - 2$ is both necessary and sufficient.*

(c) *When $p = 2$ and $q > 2$, $\beta \ge 2(1 - 1/q)$ is both necessary and sufficient.*

(d) *When $p < 2$ and $1/p + 1/q = 1$, then*

$$\beta > 2(p-1) - \frac{p}{q}$$

is sufficient, while

$$\beta \ge 2(p-1) - \frac{p}{q}$$

is necessary.

(e) *When $p < 2$ and $1/p + 1/q < 1$, then*

$$\beta \ge 2(p-1) - \frac{p}{q}$$

is both necessary and sufficient.

(f) *In the remaining triangle,*

$$\beta > \frac{3}{2}(p-1) - \frac{p}{2q}$$

is sufficient, while

$$\beta \geq \frac{3}{2}(p-1) - \frac{p}{2q}$$

is necessary.

The proof of this theorem is fairly straightforward. To illustrate, we will prove (e). The necessity for (e) is Theorem 3.7. For the sufficiency, fix a $q_0 > 2$, $\beta_0 \geq 2(1 - 1/q_0)$, and $\alpha_0 < 2$. Put

$$v_0(x) = \begin{cases} |x|^{\alpha_0} & \text{if } |x| \leq 1 \\ |x|^{\beta_0} & \text{if } |x| > 1. \end{cases}$$

By Theorem 4.1(b),

$$\|\hat{f}_{|\Sigma_{n-1}}\|_{q_0} \leq C\|f\|_{L^2(v_0)}$$

and so the operator

$$f \to \mathscr{R}\left(f v_0^{-1/2}\right)$$

maps $L^2 \to L^{q_0}$ boundedly. Trivially, $\mathscr{R}: L^1 \to L^\infty$. Let

$$\mathscr{R}_z f = \mathscr{R}\left(f v_0^{-(1-z)/2}\right) \quad \text{for } 0 \leq \operatorname{Re} z \leq 1.$$

Notice, when $\operatorname{Re} z = 0$, $\mathscr{R}_z : L^2 \to L^{q_0}$, while when $\operatorname{Re} z = 1$, $\mathscr{R}_z : L^1 \to L^\infty$. The operator norms are independent of $\operatorname{Im} z$, and so we can apply complex interpolation. Fix $0 < t < 1$, and define (p, q) by

$$\frac{1}{p} = \frac{1-t}{2} + t, \quad \frac{1}{q} = \frac{1-t}{q_0}.$$

Then

$$\mathscr{R}_t : L^p \to L^q,$$

or equivalently,

$$\mathscr{R} : L^p\left(v_0^{\frac{p(1-t)}{2}}\right) \to L^q.$$

Notice that

$$\frac{1-t}{2} = \frac{p-1}{p}$$

Hence

$$\mathscr{R}: L^p\left(v_0^{p-1}\right) \to L^q.$$

Let

$$v(x) = v_0^{p-1}(x) = \begin{cases} |x|^\alpha & \text{if } |x| \le 1 \\ |x|^\beta & \text{if } |x| > 1. \end{cases}$$

Here $\alpha = \alpha_0(p-1) < 2(p-1)$, and any such α will do. Likewise,

$$\beta = \beta_0(p-1) \ge 2\left(1 - \frac{1}{q_0}\right)(p-1).$$

But

$$\frac{2}{q_0} = -\frac{2}{(1-t)} \cdot \frac{1}{q} = \frac{p}{(p-1)q}$$

and so

$$\beta \ge 2(p-1) - \frac{p}{q}.$$

Here too, any such β is admissible.

Finally, judicious choices of q_0 and t will pick up any (p, q) in the region.

5. Higher dimensions

When $n > 2$, the Restriction theorem should still hold in the Zygmund region

$$\frac{1}{p} + \frac{n-1}{n+1} \cdot \frac{1}{q} \ge 1, \quad p < q,$$

but this has only been proven when $q \ge 2$, and so $q \ge 2$ will certainly be the limit of any sharp power weight theorems currently obtainable.

We will have to make do with a higher-dimensional analog of Theorem 4.5 when $q \ge 2$. In order to generalize the Stein-Tomas proof, we need a weighted disk-multiplier result.

LEMMA 5.1. *Let χ be the characteristic function of the unit ball B in $\mathbf{R}^n$, and let*

$$Sf(x) = \hat{\chi} * f(x).$$

If

$$v(x) = \begin{cases} |x|^{\alpha} & \text{if } |x| \le 1 \\ 1 & \text{if } |x| > 1 \end{cases}$$

and if $\alpha < n$, *then* $S : L^2(v) \to L^2(v^{-1})$ *boundedly.*

Proof. By the Jurkat-Sampson Theorem [5], the Fourier transform $\mathcal{F}.\ L^2(v) \to L^2(w)$ provided w is radial, decreasing, and

$$\int_{|x| \le s} w(x)\, dx \int_{|x| \le 1/s} v^{-1}(x)\, dx \le C \quad \text{for all } s > 0. \tag{5.2}$$

Since

$$\int_{|x| \le 1/s} v^{-1}(x)\, dx \cong s^{-n}(1 + s^{\alpha}),$$

(5.2) will hold if w is bounded and decreases rapidly enough. In particular, (5.2) holds when $w = \chi$. Thus,

$$\int_B |\hat{f}|^2\, dx \le C \int |f|^2 v. \tag{5.3}$$

Now fix f and $g \in L^2(v)$ with $\|f\|_{L^2(v)}$ and $\|g\|_{L^2(v)} \le 1$. It will suffice to prove that

$$\left| \int (Sf) g \right| \le C.$$

But, by Plancherel's Theorem,

$$\left| \int (Sf) g \right| = c \left| \int (Sf)\ \bar{\hat{g}} \right| = c \left| \int_B \hat{f} \bar{\hat{g}} \right| \le C \left(\int_B |\hat{f}|^2 \right)^{1/2} \left(\int_B |\hat{g}|^2 \right)^{1/2} \le C$$

by (5.3).

THEOREM 5.4. *Let* $q \ge 2$, $p > 1$ *with*

$$\frac{1}{p} + \frac{n-1}{n+1} \cdot \frac{1}{q} \ge 1.$$

If

$$v(x) = \begin{cases} |x|^\alpha & \text{if } |x| \le 1 \\ |x|^\beta & \text{if } |x| > 1 \end{cases}$$

then

$$\|f_{|\Sigma_{n-1}}\|_q \le C\|f\|_{L^p(v)}$$

if and only if $\beta \ge 0$ *and* $\alpha < n(p-1)$.

Proof. The necessity has been done. For the sufficiency, since the restriction operator maps $L^1 \to L^\infty$, it suffices to prove the theorem at the endpoint $q = 2$ and $p = 2(n+1)/(n+3)$, with $\beta = 0$. Put $w = v^{1/(p-1)}$. By Lemma 5.1,

$$S : L^2(w) \to L^2(w^{-1}),$$

where

$$Sf(x) = \hat{\chi} * f(x) = \frac{J_{n/2}(|y|)}{|y|^{n/2}} * f(x).$$

Set

$$K_z(x) = \frac{J_{n/2(n+1)z/2}(|x|)}{|x|^{n/2(n+1)z/2}}$$

and

$$T_z f(x) = w^{(1-z)/2} K_z * \left(f w^{(1-z)/2}\right).$$

Since $K_0 = \hat{\chi}$, when $\operatorname{Re} z = 0$, $T_z : L^2 \to L^2$. Also, $K_1(x) = \cos(|x|)$, and so $K_z : L^1 \to L^\infty$, when $\operatorname{Re} z = 1$, with sufficient control on the operator norms to invoke complex interpolation (this is exactly the Stein-Tomas argument). Thus for $t = 2/(n+1)$,

$$T_t : L^p \to L^{p'},$$

or equivalently,

$$f \to K_t * f : L^p(v) \to L^{p'}(w^{-1}). \tag{5.5}$$

But

$$K_t(x) = \frac{J_{n/2-1}(|x|)}{|x|^{n/2-1}} = d\hat{\sigma}(x)$$

is the Fourier transform of the surface measure of the unit ball. Finally, Plancherel's Theorem gives

$$\int_{\Sigma_{n-1}} |\hat{f}|^2 \, d\sigma = \int f(d\hat{\sigma} * f) \leq \|f\|_{L^p(v)} \|d\hat{\sigma} * f\|_{L^{p'}(w^{-1})} \leq C\|f\|^2_{L^p(v)}$$

by (5.5), and the proof is complete.

So we have sharp results in the truncated Zygmund region and along the line $p = 2$. The obvious interpolations yield

THEOREM 5.6. *Let*

$$v(x) = \begin{cases} |x|^\alpha & \text{if } |x| \leq 1 \\ |x|^\beta & \text{if } |x| > 1, \end{cases}$$

$1 < p \leq 2$, *and* $2 \leq q < \infty$. *Then in order to have*

$$\|f_{|\Sigma_{n-1}}\|_q \leq \|f\|_{L^p(v)},$$

$\alpha < n(p-1)$ *is necessary and sufficient. Regarding* β:

(a) *When*

$$\frac{1}{p} + \frac{n-1}{n+1} \cdot \frac{1}{q} \geq 1,$$

$\beta \geq 0$ *is both necessary and sufficient.*

(b) *When* $p = 2$ *and* $q > 2$,

$$\beta \geq n - (n-1)\frac{2}{q}$$

is necessary and sufficient.

(c) *When* $p = q = 2$, $\beta > 1$ *is both necessary and sufficient.*

(d) *When* $p < 2$ *and* $1/p + 1/q = 1$, *then*

$$\beta > n(p-1) - (n-1)\frac{p}{q}$$

is sufficient, while

$$\beta \geq n(p-1) - (n-1)\frac{p}{q}$$

is necessary.

(e) *When* $p < 2$ *and* $1/p + 1/q < 1$, *then*

$$\beta \geq n(p-1) - (n-1)\frac{p}{q}$$

is both necessary and sufficient.

(f) *In the remaining triangle,*

$$\beta > n(p-1) - \frac{n-1}{2}\left(p - 1 + \frac{p}{q}\right)$$

is sufficient, while this condition with $\geq$ *is necessary.*

References

1. J.J. Benedetto and H. Heinig, *Fourier transform inequalities with measure weights*, Adv. Math., to appear.
2. C. Carton-Lebrun and H. Heinig, "Weighted extensions of restriction theorems for the Fourier transform," to appear in *Fourier analysis and its applications*, Kluwer Publishers, The Netherlands, 1989.
3. S. Chanillo and E. Sawyer, *Unique continuation for* $\Delta + V$ *and the C. Fefferman-Phong class*, preprint.
4. C. Fefferman, *Inequalities for strongly singular convolution operators*, Acta. Math., vol. 124 (1970), pp. 9–36.
5. W. Jurkat and G. Sampson, *On rearrangement and weight inequalities for the Fourier transform*, Indiana Univ. Math. J., vol. 33 (1984), pp. 257–270.
6. P. Sjölin, *Fourier multipliers and estimates of the Fourier transform of measures carried by smooth curves in* $\mathbf{R}^2$, Studia Math., vol. 51 (1974), pp. 169–182.
7. P. Tomas, *Restriction theorems for the Fourier transform*, Proc. Sympos. Pure Math. Amer. Math. Soc., no. 35 (part 1), 1979, pp. 111–114.
8. A. Torchinsky, *Real variable methods in harmonic analysis*, Academic Press, Orlando, Florida, 1986.
9. G.N. Watson, *A treatise on the theory of Bessel functions*. Cambridge University Press, Cambridge, 1966.
10. A. Zygmund, *On Fourier coefficients and transforms of functions of two variables*, Studia Math., vol. 50 (1974), pp. 189–201.

Siena College
Loudonville, New York

Auburn University
Auburn, Alabama