ILLINOIS JOURNAL OF MATHEMATICS
Volume 40, Number 3, Fall 1996

APPROXIMATION BY ENTIRE FUNCTIONS AND ARAKELYAN-TYPE EXAMPLES FOR MOVING TARGETS

ALEXANDER RUSSAKOVSKII

Introduction

In this note we apply the approximation techniques developed in [17], together with explicit formulas for the solution of $\bar{\partial}$-equations, to construct examples in value distribution theory of one and several complex variables. From our point of view the suggested method has three main advantages: it is very natural, it is simple and explicit and it is powerful enough to derive new results. The general approximation scheme is described in Section 2.

Our aim is to construct examples of entire functions with given deficient values. It is well-known [16], [5] (see also [4], [6] for the case of meromorphic functions) that if the sum of deficiencies of an entire function is maximal, i.e.,

$$\sum_{a \in \mathbf{C}} \delta_f(a) = 1,$$

then f has integral order ρ and at most ρ deficient values $a_1, \ldots, a_\rho$, each having deficiency $1/\rho$.

Suppose that an integer ρ and the values $a_1, \ldots, a_\rho$ are given and one would like to have an example of an entire function of order ρ with these deficient values and maximal sum of deficiencies. Examples of such type have been constructed first by R. Nevanlinna [14]. The above mentioned approximation scheme permits an explicit formula to construct such examples. Moreover, the scheme is valid not only for constant deficient values, but also for small deficient functions (the so called slowly moving targets); that is, instead of just a-points, one studies the solutions of

$$f(z) = a(z),$$

for such $a(z)$ that

$$T_a(r) = o(T_f(r)), \quad r \to \infty.$$

In [13] (see [9] for the case of meromorphic functions) it was proved that the above properties of an entire function with maximal sum of deficient values take place also in the case of small functions. However, to the best of our knowledge,

Received May 5, 1995.
1991 Mathematics Subject Classification. Primary 30D35; Secondary 30E15, 32A22, 32H30.
Partially supported by NATO Linkage Grant LG 930171.

examples of entire functions with a given set of deficient functions and maximal sum of deficiencies are not known. Theorem 1 below provides such an example.

THEOREM 1. *Let ρ be a positive integer and let $a_1(z), \dots, a_\rho(z)$ be entire functions of at most minimal type with respect to order ρ. Then there exists such an entire function $f(z)$ of order ρ and normal type that $a_1, \dots, a_\rho$ are its deficient functions with deficiency $1/\rho$ each.*

One can consider another problem. Suppose that an infinite sequence of values $a_1, a_2, \dots$ is given, and we would like to have an entire function of order $\rho < \infty$ having them as deficient values. Such examples were first constructed by N. U. Arakelyan [1]. Later on his construction was generalized and strengthened by A. E. Eremenko [7, 8]. We call examples of such kind *Arakelyan-type examples*. An example for the case of infinite order was constructed in [11].

Arakelyan and Eremenko use quite sophisticated tools, particularly Eremenko's example is based on quasiconformal techniques. However it is possible to revise Eremenko's construction and to apply our approach instead of quasiconformal mappings. Our construction not only provides another Arakelyan-type example but also gives the possibility to treat the case of small functions, which is a new result.

THEOREM 2. *Let a number $\rho > 1/2$ and a sequence of entire functions $a_1, a_2, \dots,$ of at most minimal type with respect to order ρ, be given.*

Then there exists an entire function of order ρ and normal type, such that $a_1, a_2, \dots$ are its only deficient functions.

Remark. Eremenko's example [8] disproves Arakelyan's conjecture:

$$\sum \frac{1}{\log \frac{1}{\delta_f(a)}} < \infty.$$

Eremenko's construction gives deficiencies with lower bound c^n, $c < 1$. In our proof of Theorem 2 we reproduce, up to a certain stage, the construction of [7]; if one replaces it with that of [8], then the same estimates of deficiencies will also take place.

The above methods apply also to entire curves and to the multidimensional situation; see Section 6.

I would like to thank A. E. Eremenko for attracting my attention to the questions considered here and for numerous fruitful discussions.

1. Characteristics of growth and value distribution

We recall some basic definitions from the Nevanlinna theory. The reader familiar with value distribution theory can skip this section.

The counting function of a-points of a meromorphic function $f(z)$, $f(0) \neq a$, is defined as

$$N_f(r, a) = \int_0^r n(t)\frac{dt}{t},$$

where $n(t) = n_f(t, a)$ is the number of solutions of the equation $f(z) - a = 0$ in the disk of radius t.

The proximity function is

$$m_f(r, a) = \frac{1}{2\pi} \int_{|z|=r} \log^+ \frac{1}{|f(z) - a|} d \arg z.$$

Finally, the Nevanlinna characteristic function of f is defined as

$$T_f(r) = \frac{1}{2\pi} \int_{|z|=r} \log^+ |f(z)| d \arg z.$$

By the First Main Theorem of Nevanlinna,

$$N(r, a) + m(r, a) = T(r) + O(1),$$

and in the average the leading term on the left is $N(r, a)$. Those values for which

$$\delta_f(a) = 1 - \limsup_{r\to\infty} \frac{N_f(r, a)}{T_f(r)} = \liminf_{r\to\infty} \frac{m_f(r, a)}{T_f(r)} > 0,$$

are called the deficient values and $\delta_f(a)$ is called the deficiency.

It is well known that the set of Nevanlinna deficient values is at most countable and that

$$\sum_{a\in\overline{\mathbf{C}}} \delta_f(a) \leq 2.$$

The Nevanlinna theory studies value distribution of meromorphic functions. However at this moment it is not clear whether our method can give precise results for meromorphic functions. So in this paper we restrict ourselves to the case of entire functions; i.e., we always have

$$\delta_f(\infty) = 1.$$

If $f(z)$ is an entire function of finite order ρ, its indicator is defined as

$$H_f(z) = \limsup_{r\to\infty} \frac{\log |f(rz)|}{r^\rho}.$$

The indicator has the property $H(tz) = t^\rho H(z)$.

An entire function $f(z)$ is of completely regular growth if lim sup in the definition of the indicator can be replaced by $\lim_{r\to\infty,\ r\notin E}$, where E has zero density.

To determine the deficiency, we have to compare the L^1-norms of $\log^+|f|$ and $\log^-|f-a|$ on circles of radius r, for functions of completely regular growth, so the relation between the deficiency and the indicator of $f-a$ is obvious.

If the deficiencies of a function are known, much can be said about its behavior. The examples that we construct below seem very easy because they rest on the descriptions of the behavior and properties of the corresponding functions. It took years and efforts of many people to derive this information, but now we know precisely what to construct.

We will use substantially the information concerning the required form of the indicator in our examples. For instance, for the case when the sum of deficiencies is maximal, this information was provided in [6] and [9].

In the simplest possible case the plane splits into 2ρ angular sectors, and the indicator of $f(z)$ equals $|\cos(\rho\arg z)|$ in the even sectors while the indicator of $f(z)-a_j$ equals $-|\cos(\rho\arg z)|$ in the corresponding odd sectors.

We would like to note that once the corresponding indicators are as above, all deficiencies are determined (the deficiency of a_j in this particular case is the number of negative waves in the indicator of $f(z)-a_j$ divided by the number of positive waves in the indicator of f, which is ρ in this case). The most elementary example is the exponential function e^{z^ρ} for which all $a_j=0$ and $\delta(0)=1$.

Similar considerations play their role in the multidimensional case, too. The characteristics of growth and value distribution are defined in the same way with circles replaced by spheres, moduli by norms, number of points by area of analytic set, etc. The definition of the (radial) indicator in several variables involves upper regularization. However, in the multidimensional case we will just outline how the suggested scheme may be applied, hence there is no reason to go deeply into details here. The reader is referred, for instance, to [19, 20].

2. Approximation by entire functions on unbounded sets

Let $f(z)$ be a function holomorphic on an unbounded set $\Omega\subset\mathbf{C}^n$. It is a classical problem to find an entire function $g(z)$ which approximates f with certain rate on some subset of Ω, and to estimate the minimal possible growth of g. It is reasonable to expect that the rate of growth of g is determined by Ω, the growth of f itself, the rate of approximation, and the characteristics of the subset Ω_1 on which approximation takes place.

There is a very natural way to approach the problem. Suppose one takes a cut-off function $\chi(z)$ equal to zero outside of Ω and equal to one on a neighborhood of Ω_1, and looks at $g(z)$ of the form

$$g(z)=\chi(z)f(z)-\beta(z).$$

In order that g be entire one has to solve the $\bar\partial$-equation

$$\bar\partial\beta = \alpha = \bar\partial\chi \cdot f$$

with appropriate estimates. "Appropriate" means here that β has to be "small" (i.e., tending to zero with given rate) on Ω_1 and not very "large" elsewhere.

This may be achieved when the sets Ω and Ω_1 have special form. Suppose u and v are two plurisubharmonic functions in $\mathbf{C}^n$, $v \geq 0$, and suppose that our sets are given by

$$\Omega = \{z\colon u(z) < 0\}, \quad \Omega_1 = \{z\colon u(z) < -v(z)\}.$$

Then the function $\varphi(z) = u(z) + \frac{1}{2}v(z)$ is plurisubharmonic, negative in Ω_1 and positive near the boundary of Ω. If we solve our $\bar\partial$-equation by Hörmander's method with estimation of weighted L_2-norms using a multiple of φ as a weight, then one can obtain the required approximating functions. Their growth will thus be estimated in terms of u and v. This construction has been presented in [17]. The precise formulation of the corresponding result looks as follows:

Let $\omega(z)$ and $\varphi(z)$ will be plurisubharmonic functions in $\mathbf{C}^n$, both possessing the property

$$(u)^{[1]}(z) \leq -A(-u)^{[1]}(z) + B,$$

where $u^{[r]}(z)$ denotes $\sup\{u(w)\colon |z-w| < r\}$. Also, assume that $\varphi(z) \geq 0$, $\log|z| = o(\varphi(z))$, $|z| \to \infty$. For $\varepsilon \geq 0$ let

$$\Omega_\varepsilon = \{z \in \mathbf{C}^n\colon \omega(z) < -\varepsilon\varphi(z)\}$$

and suppose that

$$\inf\{|z_1 - z_2|\colon z_1 \in \Omega_{\varepsilon_1}, z_2 \in \mathbf{C}^n \setminus \Omega_{\varepsilon_2}\} > 0, \quad \forall \varepsilon_1 > \varepsilon_2,$$

which is a kind of smoothness condition on ω and φ.

THEOREM [17]. *Let $f(z)$ be an analytic function in Ω_0 satisfying the estimate*

$$|f(z)| \leq C_f e^{C_f\varphi(z)}, \quad z \in \Omega_0.$$

Then for each $\varepsilon > 0$ and each $N \geq 1$ there exists such an entire function $g(z)$ that

$$|f(z) - g(z)| \leq Ce^{-N\varphi(z)}, \quad z \in \Omega_\varepsilon,$$

$$|g(z)| \leq Ce^{C max(N,C_f)\cdot(\frac{1}{\varepsilon}\cdot\omega^+ + \varphi)(z)}, \; z \in \mathbf{C}^n,$$

where C does not depend on N.

This result was applied in [12] to the study of real determination and uniqueness sets for functions holomorphic in cones of $\mathbf{C}^n$. We show below how it may be applied to the construction of examples of entire functions with given deficient values. However, instead of using the approximation result directly (which would require certain work anyway) we apply the same scheme with explicit formulas for solving the $\bar\partial$-equation.

3. A formula for the solution of $\bar{\partial}$-equation in C

Suppose that we are given a function $\alpha(z)$ and that we have been able to find an entire function $h(z)$ such that

$$I_h(\alpha; w) = \int_{\mathbf{C}} \frac{\alpha(z)}{h(z)(w-z)} \frac{i}{2} dz \wedge d\bar{z} \tag{3.1}$$

is uniformly bounded.

Then the function

$$\beta(z) = \frac{h(z)}{\pi} I_h(\alpha, z) \tag{3.2}$$

solves the equation

$$\frac{\partial \beta}{\partial \bar{z}} = \alpha \tag{3.3}$$

and satisfies the estimate

$$|\beta(z)| \leq C|h(z)|. \tag{3.4}$$

The formula (3.2) is known. Apparently it is due to I. N. Vekua (see for instance [21] where several formulas of this kind are given); it has been applied to interpolation problems by A. F. Grishin and the author [10]. Its very powerful multidimensional analogues were constructed by B. Berndtsson and M. Andersson [3].

4. Proof of Theorem 1

Let

$$H_\rho(z) = \operatorname{Re} z^\rho.$$

Then the plane is a union of 2ρ angular sectors, with the origin as vertex, in each of which the function H_ρ has constant sign. Denote the sectors in which $H_\rho < 0$ by $G_1, \ldots, G_\rho$.

The question is reduced to the construction of an entire function $f(z)$ whose indicator equals $\max(H_\rho, 0)$ and which has the property that $f(z) - a_j(z)$ has indicator H_ρ in G_j.

Fix a number $\delta \in (0, 1/2)$. Let Ω_j, $j = 1, \ldots, \rho$, be the truncated angular sectors

$$\{z\colon\ H_\rho(z) < 2\delta |z|^\rho,\ |z| > 2\}.$$

It is clear that Ω_j are disjoint, and each one intersects only one of the sectors G_k. We enumerate Ω_j and G_j in concordance.

Let $\chi_j(z)$ be a smooth function with the properties: $\chi_j = 0$ outside of Ω_j, $\chi_j = 1$ on the set $\{z \in \Omega_j\colon \operatorname{dist}(z, \partial\Omega_j) > c > 0\}$, $\quad |\partial \chi_j / \partial \bar{z}| \leq C < \infty$.

Let

$$\alpha(z) = \sum a_j(z)\partial\chi_j/\partial\bar{z}.$$

Note that for R large enough,

$$(\operatorname{supp}\alpha)\bigcap\{|z| > R\} \subset \{z\colon\ H_\rho(z) > \delta|z|^\rho\}.$$

Therefore the integral

$$I(w) = \int\limits_{\mathbf{C}} \frac{\alpha(z)}{e^{z^\rho}(w-z)}\frac{i}{2}dz\wedge d\bar{z}$$

is uniformly bounded independent of w, so that we can apply (3.1)–(3.4).

We claim that the function

$$f(z) = \sum \chi_j(z)a_j(z) - e^{z^\rho}I(z)/\pi$$

is the one we are looking for.

Indeed, f is entire by (3.2) and (3.3), the sum does not affect the indicator where it is positive while by (3.4) the second term has indicator not exceeding H_ρ. Also, if $z \in G_j$ and $|z|$ is large enough, then

$$f(z) - a_j(z) = e^{z^\rho}I(z)/\pi$$

and hence

$$|f(z) - a_j(z)| \leq Ce^{H_\rho(z)}.$$

Thus the function $f(z)$ has the required properties. The theorem is proved.

Remark. If $f(z)$ is the function constructed above and if $g(z)$ is any entire function of minimal type order ρ, then the function $f(z)+g(z)e^{z^\rho}$ has the same deficiencies as $f(z)$.

5. Proof of Theorem 2

The construction of the function $f(z)$ is similar to the previous one. The difference is that now we have an infinite number of a_j and that the exponential function has to be replaced by another entire function, $h(z)$, of normal type order ρ, with properties of essentially the same nature. However, all the deficient values can no longer be asymptotic, so one has to replace the angles by "local sectors" (the idea going back to Arakelyan). Namely, we need to construct a set consisting of disjoint components which are truncated sectors. These components have to cross each circumference in an infinite number of components. A sequence of components, each chosen so that its radial projection is a ray going to infinity and its circular width is bounded below, plays the role of the set G_j. We need the function $h(z)$ to have the properties

$$\log|h(z)| < -x_j|z|^\rho, \quad z \in G_j$$

and

$$\log|h(z)| > y_j|z|^\rho$$

on certain sets around the components of G_j. Then, in the same way as before, we will construct the function $f(z)$ of the form

$$f(z) = \sum \chi_j(z)a_j(z) - h(z)I_h(\alpha, z)/\pi,$$

so that $\log|f(z) - a_j(z)| \le C|h(z)|$ on G_j. This would yield $\delta_f(a_j) > 0$.

It turns out that a certain entire function constructed in Eremenko's paper [7] has exactly the required properties of $h(z)$. For reader's convenience we reproduce the main stages of Eremenko's construction below, preserving his notations and constants. We only give a sketch; the details may be found in [7].

Let $B(r)$ be the disk of radius r centered at the origin and let $E(\varepsilon)$ denote the ε-neighborhood of a set E.

Choose a number $\mu < \pi(1 - \frac{1}{2\rho})$. Let

$$U_1 = \{z \in B(2^3)\colon\ 0 < \arg z < \mu\},$$

$$U_2 = \{z \notin B(2^6)\colon\ -\mu < \arg z < 0\},$$

$$U_3 = \{z \notin B(2^9)\colon\ 0 \le \arg z < \mu\},$$

$$D_0 = B(1) \bigcup U_1 \bigcup U_2 \bigcup U_3.$$

Fix an arbitrary sequence

$$\mu = \theta_1 > \theta_1' > \theta_2 > \theta_2' > \cdots > \theta_n > \theta_n' > \cdots \to 0,$$

let $\delta_k = (\theta_k - \theta_k')/5$ and for $k = 1, 2, 3, \ldots$ consider domains

$$D_k^+ = \{z \in B(2^8) \setminus B(2^4)\colon\ \theta_k' < \arg z < \theta_k\},$$

$$D_k^- = \{z\colon \in B(2^5) \setminus B(2)\colon\ -\theta_k < \arg z < -\theta_k'\},$$

$$D_k = D_k^+ \bigcup D_k^-, \quad D = \bigcup_{k=0}^{\infty} D_k,$$

$$E_k^+ = \{z \in B(2^{7.9}) \setminus B(2^{4.1})\colon\ \theta_k' + \delta_k \le \arg z \le \theta_k - \delta_k\},$$

$$E_k^- = \{z\colon \in B(2^{4.9}) \setminus B(2^{1.1})\colon -\theta_k + \delta_k \le \arg z \le -\theta_k' - \delta_k\},$$

$$E_k = E_k^+ \bigcup E_k^-.$$

With respect to the coordinates $s = \log_2 r$, $t = \theta$, where r, θ are the usual polar coordinates, each set D_k is a union of two rectangular boxes of equal width $5\delta_k$, whose projection onto the s-axis is the interval $(1, 8)$, $E_k \subset D_k$ is a union of two boxes of equal width $3\delta_k$, whose projection onto the s-axis is $(1.1, 7.9)$.

Let $E_k' \subset D_k$ be a neighborhood of E_k.

Let w be a subharmonic function in $\mathbf{C}$, with the following properties: (1) $w(z) > 0$, $z \in \mathbf{C} \setminus \overline{D}$; (2) $w(z) = 0$, $z \in D_0$; (3) $w(z) < 0$, $z \in D_k$, $k = 1, 2, \ldots$; (4) $\max\{w(z)\colon z \in B(r)\} \le cr^\lambda$, $\lambda < \rho$; (5) $w(z) = -x_k$, $z \in E_k$, $k = 1, 2, \ldots$, for some sequence $x_k \searrow 0$.

We omit the details of the construction of w.

Define

$$u(z) = \sum_{n=1}^{\infty} 2^{6n\rho} w\left(\frac{z}{2^{6n\rho}}\right).$$

The second property of w implies that only a finite number of terms are non-zero for fixed z (if $z \in B(2^6 m)$ then the terms with $n \ge m$ vanish). Hence u is subharmonic in $\mathbf{C}$. Next, for $z \in B(6(m+1)) \setminus B(6m)$ in view of properties (1) and (4) we have

$$u(z) \le c \sum_{n=1}^{m} 2^{6n\rho} \frac{|z|^\lambda}{2^{6n\lambda}} \le c_1 2^{6m(\rho-\lambda)} |z|^\lambda \le c_1 |z|^\rho.$$

Finally we observe that $u(z)$ has the properties

$$u(2^{6n} z) = 2^{6n\rho} u(z), \quad |\arg z| \le \mu,$$

$$u(2^{6n} z) \ge 2^{6n\rho} u(z), \quad z \in \mathbf{C},$$

which follow from property (2) of w.

Remark. At this point we could use a version of our approximation theorem from Section 2; however we follow Eremenko's construction a bit further in order to apply (3.1)–(3.4).

Choose $\varepsilon_k > 0$ such that the sets $D_k(\varepsilon_k)$ are disjoint. Let

$$E_{kn} = 2^{6n} E_k, \quad D_{kn}^\varepsilon = 2^{6n} D_k(\varepsilon).$$

By a theorem of Azarin [2], there exists an entire function $h(z)$ such that

$$\log |h(z)| = u(z) + o(|z|^\rho), \quad |z| \to \infty,$$

outside of some exceptional set of zero density. A standard argument involving comparison of lengths of circular projections shows that for $n > n_0(k)$ this exceptional set cannot intersect the sets E_{kn} and that $D_{kn}^{\varepsilon_k} \setminus D_{kn}^{\varepsilon_k/2}$, contains a closed curve (containing E_{kn} in its interior) whose 1-neighborhood Γ_{kn} lies outside of the exceptional set. Denote the bounded component of the complement of Γ_{kn} by Ω_{kn}.

From the properties of $u(z)$ it follows that $h(z)$ has order ρ, and that, for some $c > 0$ and $n > n_0(k)$,

$$\log |h(z)| > c y_k |z|^\rho, \quad z \in \Gamma_{kn},$$

$$\log |h(z)| \leq -c x_k |z|^\rho, \quad z \in E_{kn}.$$

By increasing $n_0(k)$ if necessary we can assume that

$$\log |a_j(z)| < \frac{c}{2} y_k |z|^\rho, \quad |z| > 2^{6n_0(k)} \tag{5.1}$$

and that

$$\int_{2^{6n_0(k)}}^{\infty} e^{-\frac{c}{2} y_k x^\rho} dx < \frac{1}{2^k}. \tag{5.2}$$

Finally, let

$$G_j = \bigcup_{n=n_0(j)}^{\infty} E_{jn}, \quad \Gamma_j = \bigcup_{n=n_0(j)}^{\infty} \Gamma_{jn}, \quad \Omega_j = \bigcup_{n=n_0(j)}^{\infty} \Omega_{jn} \bigcup \Gamma_{jn}.$$

We have constructed the required entire function $h(z)$ and the sets G_j. So we proceed in the same way as in Section 4.

Choose cut-off functions χ_j, with bounded derivatives, that take the value 1 on G_j, vanish outside Ω_j, and such that the carrier of $\partial \chi / \partial \bar{z}$ is contained in Γ_j.

Let

$$\alpha(z) = \sum a_j(z) \partial \chi_j(z) / \partial \bar{z}.$$

Note that in the series $\sum \chi(z) a_j(z)$ at most one term is non-zero at z.

In view of (5.1)–(5.2), the integral

$$I(w) = \int_{\mathbf{C}} \frac{\alpha(z)}{h(z)(w-z)} \frac{i}{2} dz \wedge d\bar{z},$$

is uniformly bounded independent of w.

Thus the entire function

$$f(z) = \sum \chi_j(z) a_j(z) - h(z) I(z)/\pi$$

is well-defined. As we mentioned in the beginning of the section, from the properties of $h(z)$ it follows that $\delta_f(a_j) > 0$. The theorem is proved.

6. Further results

In this section we provide simple examples of the application of Theorem 1 to entire curves.

An entire curve is a mapping $f\colon \mathbf{C} \to \mathbf{P}^m$ whose components are linearly independent entire functions. It is possible to study preimages of analytic sets of various codimensions in $\mathbf{P}^m$, that is investigate solutions of systems of equations

$$Q(f(z)) = 0,$$

where $Q\colon \mathbf{P}^m \to \mathbf{C}^q$ is an analytic mapping (actually, a homogeneous polynomial mapping).

However, in this note we restrict ourselves to the simplest case when Q is a homogeneous linear function. In other words, if $Z = [Z_0, \dots, Z_m] \in \mathbf{P}^m$, then $Q(Z) = \sum_{j=0}^{m} a_j Z_j$ and we study zeros of linear combinations of the form

$$\sum a_j f_j(z) = \langle a, f(z)\rangle,$$

where $a = (a_0, a_1, \dots, a_m)$ is a non-zero vector. As before, the coefficients a_j may be functions of small growth (with no common zeros).

One defines the counting function $N_f(r, a)$ and the Nevanlinna characteristic function for entire curves in the usual way. Hence the deficiencies $\delta_f(a)$ are defined, too.

In what follows we will identify hyperplanes in $\mathbf{P}^m$ and their normal vectors when there is no danger of misunderstanding. The normal vectors will be considered as points of $\mathbf{P}^m$.

Each system of m hyperplanes in $\mathbf{P}^m$ always has a common point. A system of q hyperplanes in $\mathbf{P}^m$, $q > m$, is said to be *in general position* if no subsystem of $m + 1$ hyperplanes has a common point. In other words, each system of p equations

$$\langle a^{(k)}, Z\rangle = 0, \quad k = 1, \dots, m + 1,$$

has no solution in $\mathbf{P}^m$.

In our examples we will always have $f_0 = 1$. Thus the hyperplane $Z_0 = 0$ will always have deficiency 1, and we exclude it from further consideration. Such curves are direct analogues of entire functions. Another way to think of this is to consider a mapping to $\mathbf{C}^m$ instead of $\mathbf{P}^m$ (operate only in one chart of $\mathbf{P}^m$) and study preimages of affine hyperplanes.

The defect relation for such entire curves looks similar to the one mentioned before: for each system of hyperplanes in general position one has

$$\sum \delta_f(a^{(k)}) \le m.$$

If no general position is assumed, the sum of deficiencies may be larger than m. However it seems natural to think of m as the "maximal sum of deficiencies".

If we think of the most elementary replacement of the exponential function to provide an example of a curve with maximal deficiencies, then it is

$$f(z) = (1, e^{z^\rho}, ze^{z^\rho}, z^2 e^{z^\rho}, \dots, z^{m-1} e^{z^\rho}),$$

which has m deficient hyperplanes

$$a^{(1)} = (0, 1, 0, \dots, 0), \quad a^{(2)} = (0, 0, 1, 0, \dots, 0), \dots, a^{(m)} = (0, \dots, 0, 1),$$

each with deficiency 1.

Certain interesting examples of entire curves with given deficiencies have been constructed by Ya. I. Savchuk; e.g., see [18].

In general it is not known, that if an entire curve has maximal sum of deficiencies then its order is an integer. It is no longer true that the deficiencies are rational. The corresponding example, due to A. Eremenko, has been communicated to the author by B. Shiffman. However, since our aims are different, we will not consider the corresponding problem in full generality.

We illustrate our methods by constructing an example of an entire curve of integer order ρ with maximal sum of deficiencies for a given system of deficient hyperplanes $a_1, \dots, a_{m\rho}$ in general position. We will actually construct a "componentwise" example since our method allows us to control the rate of approximation.

In our considerations, the $a^{(k)}$ can be small functions as well. The general position condition for this case which we adopt here is that for every fixed $z \in C$ the set of hyperplanes $\langle a^{(k)}(z), Z\rangle = 0$ is in general position. Actually, in the construction the general position condition is not essential; it is assumed mostly for the sake of having m as the sharp bound for the sum of deficiencies.

THEOREM 3. *Let positive integers m and ρ, and $m\rho$ vectors $a^{(k)}(z)$ whose components are functions of at most minimal type with respect to order ρ be given. Suppose that the system consisting of hypersurfaces defined by normal vectors $a^{(k)}$ and the hyperplane $Z_0 = 0$ is in general position. Then there exists an entire curve $f\colon \mathbf{C} \to \mathbf{P}^m$, of order ρ, such that each $a^{(k)}$ is a deficient vector with deficiency $1/\rho$ so that the sum of deficiencies corresponding to all $a^{(k)}$ equals m.*

Let $H_\rho(z)$ and the sets G_j, $j = 1, \dots, \rho$, be defined as in Section 3. Break arbitrarily the set of $a^{(k)}$ into m-tuples, so that to each j we associate m of vectors $a^{(k)}$ which we denote by b_{jl}, $l = 1, \dots, m$, $j = 1, \dots, \rho$. Each system of m hypersurfaces $\langle b_{jl}(z), Z\rangle = 0$, $l = 1, \dots, m$, has a common vector $d^{(j)}(z)$. From the general position condition it follows that $d^{(j)}(z)$, $j = 1, \dots, \rho$, may be written in the form $(1, d_1^{(j)}, \dots, d_m^{(j)})$ with $d_l^{(j)}$ being entire functions of small growth.

Indeed, since $d^{(j)}$ is the solution vector in $\mathbf{P}^m$ of a system of linear equations with entire coefficients, it can be written as a vector of the corresponding determinants formed from the coordinates of the b_{jl}; hence all its coordinates are entire functions of small growth. If $d_0^{(j)}$ vanished, then the hyperplane $Z_0 = 0$ would also pass through

$d^{(j)}$, which would contradict the general position condition. Hence $d_0^{(j)} \neq 0$ and we can divide through by it.

With the help of Theorem 1 construct a set of m functions $f_1, \ldots, f_m$, so that f_k corresponds to ρ deficient values $d_k^{(j)}$ in a way that $f_k \to d_k^{(j)}$ in G_j. From the estimates of Theorem 1 it follows that the indicator of $f = (1, f_1, \ldots, f_m)$ is in fact $\max(H_\rho(z), 0)$ while

$$|\langle b_{jl}, f\rangle| \leq Ce^{H_\rho(z)}, \quad z \in G_j, \ l = 1, \ldots, m,$$

which implies that each deficiency corresponding to b_{jl} (and hence to $a^{(k)}$) is precisely $1/\rho$, and thus their sum is m.

It may happen that the functions we constructed are linearly dependent. Then, according to the remark in the end of Section 4, we can modify each of f_l so that they become linearly independent while the deficiencies remain unchanged.

The theorem is proved.

Similar reasoning may be used to prove the following counterpart of Theorem 2 for entire curves.

THEOREM 4. *Let a positive number $\rho > \frac{1}{2}$ and vectors $a^{(1)}(z), a^{(2)}(z), \ldots$, whose components are functions of at most minimal type with respect to order ρ, be given. Suppose that the system consisting of hypersurfaces defined by normal vectors $a^{(k)}$ and the hyperplane $Z_0 = 0$ is in general position. Then there exists an entire curve $f\colon \mathbf{C} \to \mathbf{P}^m$, of order ρ, such that each $a^{(k)}$ is its deficient vector.*

Clearly the next step would be to consider entire mappings $\mathbf{C}^n \to \mathbf{P}^m$ (or $\mathbf{C}^n \to \mathbf{C}^m$) and the question of constructing the corresponding examples in that case. This is also possible if we apply the above construction with respect to, say, z_1, considering the rest of the variables as parameters. Since this would be essentially the same argument, we omit the details and leave the precise formulation of the corresponding result to the reader.

REFERENCES

1. N. U. Arakelyan, *Entire functions of finite order with infinite set of deficient values*, Dokl. AN SSSR **170** (1966), 999–1002.
2. V. S. Azarin, *On asymptotic behavior of subharmonic functions of finite order*, Mat. Sbornik, **108** (1979), 147–169.
3. B. Berndtsson and M. Andersson, *Henkin-Ramirez formulas with weight factors*, Ann. Inst. Fourier, **32** (1982), 91–110.
4. D. Drasin, *Proof of a conjecture of F.Nevanlinna concerning functions which have deficiency sum two*, Acta Math. **158** (1987), 1–94.
5. A. Edrei and W. H. J. Fuchs, *Valeurs déficientes et valeurs asymptotiques des fonctions méromorphes*, Comment. Math. Helv. **33** (1959), 258–295.
6. A. E. Eremenko, *A new proof of Drasin's theorem on meromorphic functions of finite order with maximal deficiency sum*, I, II, Teor. Funk., Funk. Anal. i Pril. **51** (1989), 107–116; **52** (1989), 69–78.

7. ________, *On the set of deficient values of an entire function of finite order*, Ukr. Math. J. **39** (1987), 295–299.
8. ________, *A counterexample to Arakelyan conjecture*, Bull. Amer. Math. Soc. **27** (1992), 159–164.
9. A. E. Eremenko and M. L. Sodin, *On meromorphic functions of finite order with maximal sum of deficiencies*, Theor. Funk., Funk. Anal. i Pril. **55** (1991), 84–95.
10. A. F. Grishin and A. M. Russakovskii, *Free interpolation by entire functions*, J. Sov. Math. **48** (1990), 267–275.
11. W. H. J. Fuchs and W. K. Hayman, "An entire function with assigned deficiencies" in *Studies in mathematical analysis and related topics*, Essays in honor of George Pólya, Stanford University Press, 1962.
12. V. N. Logvinenko and A. M. Russakovskii, *Cartwright-type and Bernstein-type theorems for functions holomorphic in cones*, (to appear).
13. Q. Li and Y. Ye, *Sum of deficiencies of deficient functions and F. Nevanlinna's conjecture*, Contemp. Math. **48** (1985), 21–63.
14. R. Nevanlinna. *Analytic functions*, Springer, New York, 1970.
15. J. Noguchi, *A relation between order and defects of meromorphic mappings of* $\mathbf{C}^n$ *into* $\mathbf{P}^N(C)$, Nagoya Math. J. **59** (1975), 97–106.
16. A. Pfluger, *Zur Defektrelation ganzer Funktionen endlicher Ordnung*, Comment. Math. Helv. **19** (1946), 91–104.
17. A. M. Russakovskii, *Approximation by entire functions on unbounded sets in* $\mathbf{C}^n$, J. Approx. Theory **74** (1993), 353–358.
18. Ya. I. Savchuk, *On the set of deficient vectors of entire curves*, Ukr. Math. J. **35** (1983), 385–389.
19. B. V. Shabat, *Distribution of values of holomorphic mappings*, Amer. Math. Soc., Providence, R.I., 1985.
20. B. Shiffman, *Nevanlinna defect relations for singular divisors*, Invent. Math. **31** (1975), 155–182.
21. I. N. Vekua, *Generalized analytic functions*, Pergamon Press, London, 1962.

Institute for Low Temperature Physics
Kharkov, Ukraine