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SPECTRAL INTEGRATION FROM DOMINATED ERGODIC ESTIMATES

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Dedicated to Alexandra Bellow Calderón and Alberto Calderón

ABSTRACT. Suppose that $(\Omega, \mathcal{M}, \mu)$ is a σ-finite measure space, $1 < p < \infty$, and $T\colon L^p(\mu) \to L^p(\mu)$ is a bounded, invertible, separation-preserving linear operator such that the two-sided ergodic means of the linear modulus of T are uniformly bounded in norm. Using the spectral structure of T, we obtain a functional calculus for T associated with the algebra of Marcinkiewicz multipliers defined on the unit circle...

1. Introduction

Let $(\Omega, \mathcal{M}, \mu)$ be a σ-finite measure space and let $T\colon L^p(\mu) \to L^p(\mu)$ be an invertible, separation-preserving, bounded linear operator, where $1 < p < \infty$. In a previous paper [7], the dominated ergodic theorem of [17] (see Theorem 3.2 below) was combined with the Banach space spectral theory of [3] and [4] to develop a spectral representation for T in the case when the linear modulus $|T|$ of T is mean-bounded. More precisely, it was shown in [7], Theorem (4.2), that if the ergodic averages

$$(2N+1)^{-1} \sum_{n=-N}^{N} |T|^n, \quad N = 0, 1, 2, \ldots,$$

are uniformly bounded in norm, then T is *trigonometrically well-bounded*; that is, T can be represented as

$$T = \int_{0-}^{2\pi} e^{it}\, dE(t). \tag{1.1}$$

Here $E(\cdot)$ is a projection-valued function defined on $\mathbb{R}$ with certain additional properties weaker than those arising from a spectral measure, and the integral exists as a

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Riemann-Stieltjes integral in the strong operator topology. In the setting of a general Banach space $\mathfrak{X}$, an integral representation of the form (1.1) gives rise to a $BV(\mathbb{T})$-functional calculus for T. More precisely, if φ is a complex-valued function defined and of bounded variation on the unit circle $\mathbb{T}$, then the integral $\int_0^{2\pi} \varphi(e^{it})\, dE(t)$ exists strongly as a Riemann-Stieltjes integral, and the mapping

$$\varphi \to \varphi(T) \equiv \varphi(1)E(0) + \int_0^{2\pi} \varphi(e^{it})\, dE(t)$$

is a norm-continuous representation of the Banach algebra $BV(\mathbb{T})$ in $\mathfrak{X}$ which coincides with the natural mapping when φ is a trigonometric polynomial.

The aim of the present paper is to show that, in the special circumstances described above (namely, when $1 < p < \infty$, and T: $L^p(\mu) \to L^p(\mu)$ is invertible, separation-preserving, and has mean-bounded modulus), this functional calculus can be extended by the same formula to the larger algebra $\mathfrak{M}(\mathbb{T})$ of Marcinkiewicz multipliers on $\mathbb{T}$, thereby providing an ergodic analogue of the Marcinkiewicz multiplier theorem. A similar result was obtained for an arbitrary invertible power-bounded operator acting on an L^p-subspace (where $1 < p < \infty$) without any separation-preserving hypothesis in [2], Theorem 1.4 (or, more generally, for such operators acting on UMD spaces in [6], Theorem (1.1)(ii)). However, in the present context, where power-boundedness is replaced by the weaker notion of modulus mean-boundedness on $L^p(\mu)$, the more detailed structure associated with the separation-preserving hypothesis is required. In particular, the theory of A_p weights plays a crucial role. These considerations are inspired by the dominated ergodic theorem of [17], and can be viewed as a continuation of the spirit fostered for dominated ergodic estimates by the earlier work on positive L^p contractions in [1] and [14]. It should be remarked that the analogue for $E(\cdot)$ of the Littlewood-Paley property for $\ell^p(\mathbb{Z})$ was established in [7], and this also plays an important role in the present paper.

Throughout, symbols such as $C(\alpha, \beta, \ldots)$ will be used to denote a constant which depends only on the exhibited parameters $\alpha, \beta, \ldots$ The value of $C(\alpha, \beta, \ldots)$ may vary from one occurrence to another.

2. Trigonometrically well-bounded operators

In this section, the background material from Banach space spectral theory is outlined. Denote by $\mathfrak{B}(X)$ the Banach algebra of all bounded linear operators from a Banach space X into X, and let I be the identity operator on X. A *spectral family* in X is a projection-valued function $E(\cdot)$: $\mathbb{R} \to \mathfrak{B}(X)$ with the following properties:

(i) $E(\lambda)E(\tau) = E(\tau)E(\lambda) = E(\lambda)$ if $\lambda \le \tau$;
(ii) $\sup\{\|E(\lambda)\|: \lambda \in \mathbb{R}\} < \infty$;
(iii) $E(\cdot)$ is right continuous and has a left-hand limit $E(\lambda^-)$ with respect to the strong operator topology at each point $\lambda \in \mathbb{R}$;

(iv) $E(\lambda) \to I$ as $\lambda \to \infty$ and $E(\lambda) \to 0$ as $\lambda \to -\infty$, each limit being with respect to the strong operator topology.

If, in addition, there exist $a, b \in \mathbb{R}$ with $a \leq b$ such that $E(\lambda) = 0$ for $\lambda < a$ and $E(\lambda) = I$ for $\lambda \geq b$, $E(\cdot)$ is said to be *concentrated on* $[a, b]$.

Given a spectral family $E(\cdot)$ in X concentrated on a compact interval $J = [a, b]$, an associated theory of spectral integration can be developed as follows. For each bounded function φ: $J \to \mathbb{C}$ and each partition $\mathcal{P} = (\lambda_0, \lambda_1, \dots, \lambda_n)$ of J, where we take $\lambda_0 = a$ and $\lambda_n = b$, set

$$\mathcal{S}(\mathcal{P}; \varphi, E) = \sum_{k=1}^{n} \varphi(\lambda_k)\{E(\lambda_k) - E(\lambda_{k-1})\}.$$

If the net $\{\mathcal{S}(\mathcal{P}; \varphi, E)\}$ converges in the strong operator topology of $\mathfrak{B}(X)$ as $\mathcal{P}$ increases with respect to refinement through the set of partitions of J, then the limit is called the *spectral integral of* φ with respect to $E(\cdot)$ and is denoted by $\int_J \varphi(\lambda)\, dE(\lambda)$. In this case, we define $\int_J^{\oplus} \varphi(\lambda)\, dE(\lambda)$ by writing

$$\int_J^{\oplus} \varphi(\lambda)\, dE(\lambda) \equiv \varphi(a)E(a) + \int_J \varphi(\lambda)\, dE(\lambda).$$

Denote by $BV(J)$ the Banach algebra of functions φ: $J \to \mathbb{C}$ of bounded variation on J, with norm

$$\|\varphi\|_J = |\varphi(b)| + \mathrm{var}_J \varphi.$$

It can be shown that the spectral integral $\int_J \varphi(\lambda)\, dE(\lambda)$ exists for each $\varphi \in BV(J)$ and that the mapping

$$\varphi \to \int_J^{\oplus} \varphi(\lambda)\, dE(\lambda)$$

is an identity-preserving algebra homomorphism of $BV(J)$ into $\mathfrak{B}(X)$ satisfying

$$\left\| \int_J^{\oplus} \varphi(t)\, dE(t) \right\| \leq \|\varphi\|_J \sup\{\|E(\lambda)\| : \lambda \in \mathbb{R}\}.$$

(See [12],Chapter 17 or the simplified account in [5], §2.) We shall also consider the Banach algebra $BV(\mathbb{T})$, which consists of all ψ: $\mathbb{T} \to \mathbb{C}$ such that the function $\widetilde{\psi}(t) \equiv \psi(e^{it})$ belongs to $BV([0, 2\pi])$, and which is furnished with the norm $\|\psi\|_{BV(\mathbb{T})} = \|\widetilde{\psi}\|_{[0,2\pi]}$.

Definition. An operator $U \in \mathfrak{B}(X)$ is said to be *trigonometrically well-bounded* if there is a spectral family $E(\cdot)$ in X concentrated on $[0, 2\pi]$ such that $U = \int_{[0,2\pi]}^{\oplus} e^{i\lambda}\, dE(\lambda)$. In this case, it is possible to arrange that $E(2\pi^-) = I$, and with this additional property the spectral family $E(\cdot)$ is uniquely determined by U, and is called *the spectral decomposition* of U.

This class of operators was introduced in [3] and their theory further developed in [4]. We single out one result ([4], Corollary (2.10)) which has a crucial bearing on the background to the present paper and illustrates the connection between trigonometric well-boundedness and ergodic Hilbert transforms.

PROPOSITION 2.1. *Let X be a reflexive Banach space and let $U \in \mathfrak{B}(X)$ be invertible. If*

$$\rho \equiv \sup\left\{\left\|\sum_{0<|k|\le N} k^{-1}e^{ikt}U^k\right\| : N \in \mathbb{N} \text{ and } t \in [0, 2\pi]\right\} < \infty,$$

then U is trigonometrically well-bounded and the spectral decomposition $E(\cdot)$ of U satisfies $\|E(\lambda)\| \le 3\{1 + (2\pi)^{-1}\rho\}$ for all $\lambda \in \mathbb{R}$.

3. Modulus mean-bounded operators

We now describe how the theory of the previous section applies to the mean-boundedness condition discussed in [17]. Let $(\Omega, \mathcal{M}, \mu)$ be a measure space, let $1 \le p < \infty$, and let $T: L^p(\mu) \to L^p(\mu)$ be a bounded linear mapping. Then T is said to be *separation-preserving* (or *Lamperti*, or *disjoint*) if, whenever $f, g \in L^p(\mu)$ and $fg = 0$ μ-a.e., it follows that $(Tf)(Tg) = 0$ μ-a.e. This property can be characterised by the existence of a (necessarily unique) positive bounded linear operator $|T|: L^p(\mu) \to L^p(\mu)$ with the property that, for every $f \in L^p(\mu)$,

$$|Tf| = |T|(|f|) \ \mu\text{-a.e. on } \Omega.$$

Further, if T is invertible and separation-preserving, then T^{-1} is also separation-preserving, $|T|$ is invertible, and $|T|^{-1} = |T^{-1}|$ (see [7], Scholium (2.3)).

We begin by describing briefly some structural properties of separation-preserving mappings shown by Kan [15] (see [7], §2, for a more detailed summary than that presented here). The results in [15] are stated under the assumption that the underlying measure space is σ-finite. Accordingly, suppose from now on that $(\Omega, \mathcal{M}, \mu)$ is σ-finite. Also, denote by χ_E the characteristic function of a subset E of Ω.

In order to state the results needed from [15], we shall require one further concept, namely that of a σ-endomorphism. A *σ-endomorphism* of the measure algebra $(\Omega, \mathcal{M}, \mu)$ is a mapping $\Phi: \mathcal{M} \to \mathcal{M}$ (modulo μ-null sets) such that:

(i) $\Phi(\bigcup_{n=1}^{\infty} E_n) = \bigcup_{n=1}^{\infty} \Phi(E_n)$ for every disjoint sequence $\{E_n\}_{n=1}^{\infty}$ in $\mathcal{M}$;
(ii) $\Phi(\Omega\backslash E) = \Phi(\Omega)\backslash\Phi(E)$ for all $E \in \mathcal{M}$;
(iii) if $E \in \mathcal{M}$ and $\mu(E) = 0$, then $\mu(\Phi E) = 0$.

As noted in [15], §4, a σ-endomorphism Φ on $\mathcal{M}$ gives rise to a unique linear operator, also denoted by Φ, from the space of all complex-valued measurable functions on Ω into itself with the following properties:

(i) $\Phi(\chi_E) = \chi_{\Phi E}$ for all $E \in \mathcal{M}$;
(ii) if $f_n \to f$ μ-a.e., then $\Phi(f_n) \to \Phi(f)$ μ-a.e.

This associated mapping has the important property that $\Phi(fg) = \Phi(f)\Phi(g)$ for all measurable functions $f, g \colon \Omega \to \mathbb{C}$. The content of the following theorem is established in [15], Theorem 4.1 and Proposition 4.1.

THEOREM 3.1. *Let $(\Omega, \mathcal{M}, \mu)$ be a σ-finite measure space, let $1 \le p < \infty$ and let $T\colon L^p(\mu) \to L^p(\mu)$ be a separation-preserving bounded linear operator. Then there exists a unique σ-endomorphism Φ of $(\Omega, \mathcal{M}, \mu)$ with the following property:*

(i) *for each $E \in \mathcal{M}$ with $\mu(E) < \infty$, $\Phi(E) = \{\omega \in \Omega\colon (T\chi_E)(\omega) \ne 0\}$.*

Moreover, there is a unique measurable function $h\colon \Omega \to \mathbb{C}$ satisfying the following conditions (ii) and (iii):

(ii) *$h = 0$ a.e. on $\Omega \backslash \Phi(\Omega)$;*
(iii) *for each $f \in L^p(\mu)$, Tf equals the pointwise product $h \cdot \Phi(f)$ a.e. on Ω.*

If, in addition, T is invertible, then:

(iv) *$|h| > 0$ a.e. on Ω;*
(v) *Φ is a bijection, both as a σ-endomorphism on $\mathcal{M} \to \mathcal{M}$ and as a mapping on the space of all complex-valued measurable functions on Ω.*

For the remainder of the section, assume that T is separation-preserving and invertible. For each $j \in \mathbb{Z}$, T^j is also separation-preserving and invertible and so has the form

$$T^j f = h_j \cdot \Phi_j(f) \text{ a.e. on } \Omega \text{ for all } f \in L^p(\mu),$$

where h_j and Φ_j are associated with T^j as in Theorem 3.1. It is straightforward to check that, for $j, k \in \mathbb{Z}$,

$$\Phi_{j+k}(f) = \Phi_j(\Phi_k f) \text{ for all } f \in L^p(\mu)$$

and

$$h_{j+k} = h_j \cdot (\Phi_j h_k) \text{ a.e. on } \Omega.$$

By the Radon-Nikodým theorem, for each $j \in \mathbb{Z}$ we obtain a unique non-negative measurable function J_j on Ω such that

$$\int_\Omega J_j(\omega)(\Phi_j f)(\omega)\, d\mu(\omega) = \int_\Omega f(\omega)\, d\mu(\omega) \text{ for all } f \in L^1(\mu).$$

Each J_j is strictly positive a.e on Ω and

$$J_{j+k} = J_j \cdot (\Phi_j J_k) \text{ a.e. on } \Omega$$

for $j, k \in \mathbb{Z}$. Finally, notice that, for each $j \in \mathbb{Z}$,

$$|T|^j(f) = \left|T^j\right|(f) = \left|h_j\right| .(\Phi_j f)$$

for all $f \in L^p(\mu)$.

Properties of the functions h_j and J_j were used by Martín-Reyes and de la Torre [17], [18] to investigate when $|T|$ satisfies a dominated ergodic estimate. Their work involves the notion of a discrete A_p weight and applies in the case when $1 < p < \infty$. Accordingly, suppose that $1 < p < \infty$, that $w = \{w_k\}_{k=-\infty}^{\infty}$ is a weight sequence (in other words, each w_k is a strictly positive real number), and that C is a positive constant. Recall that w is said to satisy the A_p condition with constant C if

$$\left[\sum_{k=K}^{L} w_k\right]\left[\sum_{k=K}^{L} (w_k)^{-1/(p-1)}\right]^{p-1} \le C(L-K+1)^p$$

whenever $K, L \in \mathbb{Z}$ with $K \le L$. Notice that this implies that $C \ge 1$. The basic result of Martín-Reyes and de la Torre is as follows (see [17] and [18], Theorem (2.4), together with the reasoning described in the remarks immediately following Theorem (3.2) in [7]).

THEOREM 3.2. *Suppose that $1 < p < \infty$ and that, as above, T is an invertible, separation-preserving, bounded linear mapping of $L^p(\mu)$ onto $L^p(\mu)$, where $(\Omega, \mathcal{M}, \mu)$ is a σ-finite measure space. The following statements are equivalent.*

(i) *There is a constant $B > 0$ such that, for all $f \in L^p(\mu)$,*

$$\int_\Omega (Mf)^p \, d\mu \le B \int_\Omega |f|^p \, d\mu,$$

where M is the ergodic maximal operator defined on $L^p(\mu)$ by

$$Mf = \sup_{N \ge 0} (2N+1)^{-1} \sum_{k=-N}^{N} \left|T^k f\right|.$$

(ii) *The bilateral ergodic averages of $|T|$ are uniformly bounded; that is, there is a real constant $\mathfrak{s}$ such that*

$$\sup_{N \ge 0} \left\|(2N+1)^{-1} \sum_{k=-N}^{N} |T|^k\right\| \le \mathfrak{s}.$$

(iii) *There is a constant $C > 0$ such that, for μ-almost all $\omega \in \Omega$, the weight sequence $\{|h_k(\omega)|^{-p} J_k(\omega)\}_{k=-\infty}^{\infty}$ satisfies the A_p condition with constant C.*

Furthermore, assuming that (ii) holds, the constant C in (iii) can be chosen to depend only on p and $\mathfrak{s}$.

Using this result and a transference argument involved in its proof, together with the boundedness of the maximal discrete Hilbert transform on a discrete weighted ℓ^p-space when the weight sequence satisfies the A_p condition (see [13], Theorem 10), R. Sato [19] was able to establish a maximal theorem for the ergodic Hilbert transform associated with an operator T satisfying any of the equivalent conditions (i)–(iii) in Theorem 3.2. In turn, Sato's result was then combined in [7] with Propostion 2.1 to establish the trigonometric well-boundedness of such an operator (see [7], Theorem (4.2) and the reasoning indicated in the remarks immediately after its statement). The outcome is stated in the following theorem (for examples thereof where power-boundedness is absent, see [7], §4).

THEOREM 3.3. *Suppose that $(\Omega, \mathcal{M}, \mu)$ is a σ-finite measure space, $1 < p < \infty$, and $T\colon L^p(\mu) \to L^p(\mu)$ is an invertible, separation-preserving bounded linear operator such that for some real constant $\mathfrak{s}$,*

$$\sup_{N \geq 0} \left\| (2N+1)^{-1} \sum_{k=-N}^{N} |T|^k \right\| \leq \mathfrak{s}.$$

Then T is trigonometrically well-bounded. Furthermore, there is a constant $C(p, \mathfrak{s})$ such that the spectral decomposition $E(\cdot)$ of T satisfies the estimate $\|E(\lambda)\| \leq C(p, \mathfrak{s})$ for all $\lambda \in \mathbb{R}$.

The spectral decomposition occurring in this result gives rise to a strongly countably additive spectral measure on a σ-algebra associated with the dyadic decomposition of $\mathbb{T}$. To be more precise, for $j \in \mathbb{Z}$, let t_j be the j^{th} dyadic point in $(0, 2\pi)$ defined by

$$t_j = \begin{cases} 2^{j-1}\pi & \text{if } j \leq 0, \\ 2\pi - 2^{-j}\pi & \text{if } j > 0 \end{cases}$$

and put

$$\omega_j = e^{it_j}, \Gamma_j = \{e^{it}\colon t_j < t < t_{j+1}\}.$$

Define the *dyadic σ-algebra Σ_d of $\mathbb{T}$* to be the σ-algebra of subsets of $\mathbb{T}$ generated by the family $\mathcal{D}_{\mathbb{T}}$ of subsets of $\mathbb{T}$ consisting of the arcs Γ_j ($j \in \mathbb{Z}$) and the singleton sets $\{\omega_j\}$ ($j \in \mathbb{Z}$) and $\{1\}$. Note that each element of Σ_d can be expressed in a unique way as a (countable) union of mutually disjoint members of $\mathcal{D}_{\mathbb{T}}$. We have the following result ([7], Theorem (5.10) and Corollary (5.11), together with the proof of Lemma (5.5)).

THEOREM 3.4. *Suppose that T satisfies the hypotheses of Theorem* 3.3 *and that $E(\cdot)$ is the spectral decomposition of T. Then there is a uniquely determined strongly countably additive spectral measure $\mathcal{E}(\cdot)$ defined on the σ-algebra Σ_d such that, for $j \in \mathbb{Z}$,*

$$\mathcal{E}(\Gamma_j) = E(t_{j+1}^-) - E(t_j),$$

$$\mathcal{E}(\{\omega_j\}) = E(t_j) - E(t_j^-),$$

and

$$\mathcal{E}(\{1\}) = E(0).$$

Furthermore, there is a constant $C(p, \mathfrak{s}) > 0$ such that, whenever $\{\sigma_j\}_{j\geq 1}$ is a sequence of mutually disjoint elements of Σ_d satisfying $\mathbb{T} = \cup_{j\geq 1}\sigma_j$ and $f \in L^p(\mu)$,

$$C(p, \mathfrak{s})^{-1} \|f\|_{L^p(\mu)} \leq \left\| \left\{ \sum_{j\geq 1} |\mathcal{E}(\sigma_j) f|^2 \right\}^{1/2} \right\|_{L^p(\mu)} \leq C(p, \mathfrak{s}) \|f\|_{L^p(\mu)}.$$

As explained in [7], this can be viewed as an ergodic analogue of the classical Littlewood-Paley theorem for the (unweighted) bilateral sequence space $\ell^p(\mathbb{Z})$. Its proof hinges on a version of the Marcinkiewicz multiplier theorem applicable to a weighted space $\ell^p(w)$ which, in turn, is a consequence of Kurtz's weighted version [16] of the corresponding classical theorem on $\mathbb{R}$ (see below for further details).

There is also a version of the classical vector-valued M. Riesz theorem valid in the context of Theorem 3.3.

THEOREM 3.5. *Let T and $E(\cdot)$ be as in the statement of Theorem* 3.3. *Then there is a constant $C(p, \mathfrak{s})$ with the following property. For all sequences $\{a_j\}_{j=1}^{\infty}$ in $[0, 2\pi)$ and all sequences $\{g_j\}_{j=1}^{\infty}$ in $L^p(\mu)$,*

$$\left\| \left\{ \sum_{j=1}^{\infty} |E(a_j) g_j|^2 \right\}^{1/2} \right\|_{L^p(\mu)} \leq C(p, \mathfrak{s}) \left\| \left\{ \sum_{j=1}^{\infty} |g_j|^2 \right\}^{1/2} \right\|_{L^p(\mu)}.$$

Proof. Apart from the stated dependence of the constant $C(p, \mathfrak{s})$ used here, this is [7], Theorem (6.7). However, the proof of [7], Theorem (6.7), does in fact establish the dependence of the constant used here on p, $\mathfrak{s}$ alone (by taking account of the final sentence in the statement of each of the present Theorems 3.2 and 3.3).

As mentioned above, the proof of Theorem 3.4 relies on a weighted version of the discrete Marcinkiewicz multiplier theorem. We end this section by recording this and a useful convolution theorem for multipliers in the setting of discrete weights, since both results will be needed in the next section. For each $j \in \mathbb{Z}$, let Δ_j denote the closure in $\mathbb{T}$ of the dyadic arc Γ_j and let $\mathfrak{M}(\mathbb{T})$ denote the Banach algebra of functions $\varphi\colon \mathbb{T} \to C$ such that

$$\|\varphi\|_{\mathfrak{M}(\mathbb{T})} = \sup_{z\in\mathbb{T}} |\varphi(z)| + \sup_{j\in\mathbb{Z}} \operatorname{var}(\varphi, \Delta_j) < \infty.$$

The classical Marcinkiewicz multiplier theorem states that, for $1 < p < \infty$, each $\varphi \in \mathfrak{M}(\mathbb{T})$ is a multiplier for $\ell^p(\mathbb{Z})$, with multiplier norm dominated by $C(p)\|\varphi\|_{\mathfrak{M}(\mathbb{T})}$. Elements of $\mathfrak{M}(\mathbb{T})$ are therefore referred to as Marcinkiewicz multipliers.

Now let $w = \{w_k\}_{k=-\infty}^{\infty}$ be a weight sequence, and, for $1 < p < \infty$, let $\ell^p(w)$ denote the corresponding weighted sequence space consisting of all complex-valued sequences $x = \{x_k\}_{k=-\infty}^{\infty}$ such that

$$\|x\|_{\ell^p(w)} \equiv \left\{\sum_{k=-\infty}^{\infty} |x_k|^p w_k\right\}^{1/p} < \infty.$$

A function $\psi \in L^\infty(\mathbb{T})$ with inverse Fourier transform $\psi^\vee$ is said to be a p-multiplier for $\ell^p(w)$ if:

(i) for each $x = \{x_k\}_{k=-\infty}^{\infty}$ in $\ell^p(w)$ and each $j \in \mathbb{Z}$, the series

$$(\psi^\vee * x)(j) \equiv \sum_{k=-\infty}^{\infty} \psi^\vee(j-k)x_k$$

converges absolutely;

(ii) the mapping $S_\psi\colon x \in \ell^p(w) \to \psi^\vee * x$ is a bounded linear maping of $\ell^p(w)$ into itself.

The required weighted version of the Marcinkiewicz multiplier theorem, essentially due to Kurtz [16] (see also [7], Theorem (5.1) and [8], Theorem (5.5)), can now be stated as follows.

THEOREM 3.6. *Suppose that $1 < p < \infty$, w is a weight sequence satisfying the A_p condition with constant C, and $\varphi \in \mathfrak{M}(\mathbb{T})$. Then φ is a multiplier for $\ell^p(w)$, and the corresponding operator S_φ on $\ell^p(w)$ satisfies*

$$\|S_\varphi\| \le K\|\varphi\|_{\mathfrak{M}(\mathbb{T})},$$

where $K > 0$ is a constant depending only on p and C.

The following result will also be needed.

PROPOSITION 3.7. ([7], Theorem (5.2).) *Supose that $1 < p < \infty$, w is a weight sequence, ψ is a multiplier for $\ell^p(w)$, and $\mathfrak{k} \in L^1(\mathbb{T})$. Then $\mathfrak{k} * \psi$ is a multiplier for $\ell^p(w)$ and*

$$\|S_{\mathfrak{k}*\psi}\| \le \|\mathfrak{k}\|_{L^1(\mathbb{T})}\|S_\psi\|.$$

4. Spectral integration of Marcinkiewicz multipliers

We are now in a position to establish the main result of the paper.

THEOREM 4.1. *Let T, $\mathfrak{s}$ and $E(\cdot)$ be as in the statement of Theorem 3.3. Then, for each $\varphi \in \mathfrak{M}(\mathbb{T})$, the spectral integral $\int_0^{2\pi} \varphi(e^{it})\,dE(t)$ exists. Furthermore, the mapping*

$$\varphi \to \varphi(T) \equiv \int_{[0,2\pi]}^{\oplus} \varphi(e^{it})\,dE(t)$$

is an identity-preserving algebra homomorphism of $\mathfrak{M}(\mathbb{T})$ into $\mathfrak{B}(L^p(\mu))$ and there is a constant $C(p,\mathfrak{s})$ such that

$$\|\varphi(T)\| \le C(p,\mathfrak{s})\,\|\varphi\|_{\mathfrak{M}(\mathbb{T})}$$

for all $\varphi \in \mathfrak{M}(\mathbb{T})$.

The proof of Theorem 4.1 hinges on the following basic estimate.

LEMMA 4.2. *Under the hypotheses of Theorem 4.1, there is a constant $C(p,\mathfrak{s})$ such that*

$$\left\| \int_{[0,2\pi]} \varphi(e^{it})\,dE(t) \right\| \le C(p,\mathfrak{s})\,\|\varphi\|_{\mathfrak{M}(\mathbb{T})}$$

for all $\varphi \in BV(\mathbb{T})$.

Once Lemma 4.2 has been proved, Theorem 4.1 is a consequence of the following Banach space result, together with the bound for the spectral decomposition given in Theorem 3.3.

THEOREM 4.3. *Let U be a trigonometrically well-bounded operator on a Banach space X with spectral decomposition $F(\cdot)$. Suppose that there is a constant $C > 0$ such that*

$$\left\| \int_{[0,2\pi]} \varphi(e^{it})\,dF(t) \right\| \le C\,\|\varphi\|_{\mathfrak{M}(\mathbb{T})}$$

for all $\varphi \in BV(\mathbb{T})$. Then, for each $\varphi \in \mathfrak{M}(\mathbb{T})$, the spectral integral $\int_0^{2\pi} \varphi(e^{it})\,dF(t)$ exists. Furthermore, the mapping

$$\varphi \to \varphi(U) \equiv \int_{[0,2\pi]}^{\oplus} \varphi(e^{it})\,dF(t)$$

is an identity-preserving algebra homomorphism of $\mathfrak{M}(\mathbb{T})$ into $\mathfrak{B}(X)$ satisfying

$$\|\varphi(U)\| \le \{3C + \|F(0)\|\}\,\|\varphi\|_{\mathfrak{M}(\mathbb{T})}$$

for all $\varphi \in \mathfrak{M}(\mathbb{T})$.

Notation. Suppose that $F(\cdot)$ is a spectral family in a Banach space concentrated on $[0, 2\pi]$, φ: $\mathbb{T} \to C$ is bounded, and $\mathcal{P} = (\lambda_0, \ldots, \lambda_n)$ is a partition of $[0, 2\pi]$. Write

$$\mathcal{S}(\mathcal{P}; \varphi, F) = \sum_{k=1}^{n} \varphi(e^{i\lambda_k}) \{F(\lambda_k) - F(\lambda_{k-1})\}$$

for the corresponding Riemann-Stieltjes sum of the formal expression $\int_{[0,2\pi]} \varphi(e^{it})\, dF(t)$. Thus, in the notation of §2, $\mathcal{S}(\mathcal{P}; \varphi, F) = \mathcal{S}(\mathcal{P}; \tilde{\varphi}, F)$, where $\tilde{\varphi}(t) = \varphi(e^{it})$ for $0 \le t \le 2\pi$. It is also convenient to define

$$\varphi_{\mathcal{P}} = \sum_{k=1}^{n} \varphi(e^{i\lambda_k}) \chi_k,$$

where χ_k is the characteristic function relative to $\mathbb{T}$ of the arc $\{e^{it}\colon \lambda_{k-1} < t \le \lambda_k\}$. Notice that $\varphi_{\mathcal{P}} \in BV(\mathbb{T})$, $\| \varphi_{\mathcal{P}} \|_{\mathfrak{M}(\mathbb{T})} \le 3\|\varphi\|_{\mathfrak{M}(\mathbb{T})}$ if $\varphi \in \mathfrak{M}(\mathbb{T})$, and that

$$\mathcal{S}(\mathcal{P}; \varphi, F) = \int_{[0,2\pi]} \varphi_{\mathcal{P}}(e^{it})\, dF(t).$$

Proof of Theorem 4.3. Fix $x \in X$, $\varphi \in \mathfrak{M}(\mathbb{T})$ and $\epsilon > 0$. Choose dyadic points t_N and t_M in $(0, 2\pi)$ with $t_N < t_M$ such that

$$\|\{F(t_N) - F(0)\}x + \{I - F(t_M)\}x\| \le \epsilon$$

and write $x = y + z + F(0)x$, where

$$y = \{F(t_M) - F(t_N)\}x \text{ and } z = \{F(t_N) - F(0)\}x + \{I - F(t_M)\}x.$$

Thus $\|z\| \le \epsilon$. Let $\mathcal{P}$ be a partition of $[0, 2\pi]$. With the notation introduced before the start of the present proof, we have

$$\begin{aligned} \|\mathcal{S}(\mathcal{P}; \varphi, F)\| &= \left\| \int_{[0,2\pi]} \varphi_{\mathcal{P}}(e^{it})\, dF(t) \right\| \\ &\le C\, \|\varphi_{\mathcal{P}}\|_{\mathfrak{M}(\mathbb{T})} \le 3C\, \|\varphi\|_{\mathfrak{M}(\mathbb{T})}, \end{aligned} \tag{4.1}$$

and so

$$\|\mathcal{S}(\mathcal{P}; \varphi, F)z\| \le 3C\epsilon\, \|\varphi\|_{\mathfrak{M}(\mathbb{T})}. \tag{4.2}$$

Also,

$$\mathcal{S}(\mathcal{P}; \varphi, F)F(0)x = 0. \tag{4.3}$$

Furthermore, if $\mathcal{P}$ contains the points t_N and t_M, then

$$\mathcal{S}(\mathcal{P}; \varphi, F)y = \mathcal{S}(\mathcal{W}; \psi, F_0)y,$$

where $\psi(t) = \varphi(e^{it})$ for $t_N \le t \le t_M$, $\mathcal{W}$ is the partition $\mathcal{P} \cap [t_N, t_M]$ of $[t_N, t_M]$, and F_0 is the spectral family obtained by restricting $F(\cdot)$ to $\{F(t_M) - F(t_N)\}X$. The spectral family F_0 is concentrated on $[t_N, t_M]$. Since ψ has bounded variation on $[t_N, t_M]$, the integral $\int_{[t_N, t_M]} \psi(t)\, dF_0(t)$ exists in the strong operator topology. It follows that

$$\mathcal{S}(\mathcal{P}; \varphi, F)y \to \int_{[t_N, t_M]} \psi(t)\, dF_0(t)y \text{ in norm as } \mathcal{P} \text{ increases} \\ \text{by refinement through the partitions of } [0, 2\pi]. \tag{4.4}$$

Since $x = y + z + F(0)x$, it can be seen from (4.2), (4.3) and (4.4) that

$$\{\mathcal{S}(\mathcal{P}; \varphi, F)x\colon\ \mathcal{P} \text{ is a partition of } [0, 2\pi]\} \text{ is a Cauchy net in } X.$$

Hence in view of (4.1), the spectral integral $\int_{[0,2\pi]} \varphi(e^{it})\, dF(t)$ exists and satisfies

$$\left\| \int_{[0,2\pi]} \varphi(e^{it})\, dF(t) \right\| \le 3C\, \|\varphi\|_{\mathfrak{M}(\mathbb{T})},$$

which immediately yields

$$\begin{aligned} \left\| \int_{[0,2\pi]}^{\oplus} \varphi(e^{it})\, dF(t) \right\| &= \left\| \int_{[0,2\pi]} \varphi(e^{it})\, dF(t) + \varphi(1)F(0) \right\| \\ &\le \{3C + \|F(0)\|\}\, \|\varphi\|_{\mathfrak{M}(\mathbb{T})}. \end{aligned}$$

Finally, the stated algebraic properties of the mapping $\varphi \to \int_{[0,2\pi]}^{\oplus} \varphi(e^{it})\, dF(t)$ are easily deduced. In particular, for the multiplicativity property, we utilize the defining properties of a spectral family listed in §2 together with the uniform boundedness of Riemann-Stieltjes sums in the present context, as exhibited in (4.1). This completes the proof of Theorem 4.3.

We shall present two different proofs of the basic estimate given in Lemma 4.2, one based on the transference of weighted multiplier inequalities and the other on more operator-theoretic considerations. Both approaches have as their basis the discrete version of Kurtz's Marcinkiewicz multiplier theorem for A_p weights (Theorem 3.6). The first transfers this directly, whilst the second obtains the result from Theorem 3.4 (whose proof ultimately relies on Kurtz's result) and Theorem 3.5.

The transference approach to Lemma 4.2. Suppose that T, $\mathfrak{s}$ and $E(\cdot)$ are as in the statement of Theorems 3.3 and 4.1. The aim is to transfer norm estimates for convolution operators from appropriate weighted spaces $\ell^p(w)$ to $L^p(\mu)$ via the representation $n \to T^n$ of $\mathbb{Z}$ in $L^p(\mu)$. This representation is not necessarily uniformly bounded, and so the classical transference approach of Calderón [9] and Coifman-Weiss (see [11], Theorem 2.4) is not applicable. However, the special structural features arising from the separation-preserving property of T do permit an adaptation of the Coifman-Weiss technique, as was first observed in [17] and has also been used in [18], [19] and [7].

Given $\varphi \in BV(\mathbb{T})$, define $\varphi^{\ddagger}\colon \mathbb{R} \to C$ by

$$\varphi^{\ddagger}(t) = 2^{-1}\left\{\lim_{s\to t+}\varphi(e^{is}) + \lim_{s\to t-}\varphi(e^{is})\right\}.$$

LEMMA 4.4. ([5], Theorem (3.10)(i).) *For each* $\varphi \in BV(\mathbb{T})$,

$$\sum_{k=-n}^{n}\left(1 - |k|\,(n+1)^{-1}\right)\widehat{\varphi}(k)T^k \to \int_{[0,2\pi]}^{\oplus} \varphi^{\ddagger}(t)\,dE(t)$$

in the strong operator topology of $\mathfrak{B}(L^p(\mu))$, *as* $n \to \infty$.

The key transference estimate is as follows.

LEMMA 4.5. *There is a constant* $C(p, \mathfrak{s})$ *such that*

$$\left\|\int_{[0,2\pi]}^{\oplus} \varphi^{\ddagger}(t)\,dE(t)\right\| \le C(p,\mathfrak{s})\|\varphi\|_{\mathfrak{M}(\mathbb{T})}$$

for all $\varphi \in BV(\mathbb{T})$.

Proof. For each non-negative integer n, let κ_n denote the n^{th} Fejér kernel for $\mathbb{T}$, so that $\widehat{\kappa_n}(k) = 1 - (n+1)^{-1}|k|$ for $|k| \le n$ and $\widehat{\kappa_n}(k) = 0$ for $|k| > n$. By Lemma 4.4, it suffices to establish the existence of a constant $C(p, \mathfrak{s})$ such that

$$\left\|\sum_{k=-n}^{n} \widehat{\kappa_n}(k)\widehat{\varphi}(k)T^k\right\| \le C(p,\mathfrak{s})\ \|\varphi\|_{\mathfrak{M}(\mathbb{T})} \tag{4.5}$$

for all $\varphi \in BV(\mathbb{T})$ and all $n \ge 0$.

Fix $\varphi \in BV(\mathbb{T})$ and $n \ge 0$, and let $f \in L^p(\mu)$. For $L \in \mathbb{N}$, denote by $\chi_{L,n}$ the characteristic function relative to $\mathbb{Z}$ of $\{k \in \mathbb{Z}\colon |k| \le L + n\}$. Using the structural properties of T and the associated notation discussed in §3, we have

$$\begin{aligned}
&\int_{\Omega}\left|\sum_{k=-n}^{n} \widehat{\kappa_n}(k)\widehat{\varphi}(k)T^k f\right|^p d\mu \\
&\quad = (2L+1)^{-1}\int_{\Omega}\sum_{j=-L}^{L}\left|\Phi_j\sum_{k=-n}^{n}\widehat{\kappa_n}(k)\widehat{\varphi}(k)T^k f\right|^p J_j\,d\mu \\
&\quad = (2L+1)^{-1}\int_{\Omega}\sum_{j=-L}^{L}\left|\sum_{k=-n}^{n}\widehat{\kappa_n}(k)\widehat{\varphi}(k)\Phi_j(h_k)\Phi_{j+k}f\right|^p J_j\,d\mu
\end{aligned}$$

$$= (2L+1)^{-1} \int_\Omega \sum_{j=-L}^{L} \left| \sum_{k=-n}^{n} \widehat{\kappa_n}(k)\widehat{\varphi}(k) h_{j+k} \Phi_{j+k} f \right|^p J_j \left|h_j\right|^{-p} d\mu$$

$$= (2L+1)^{-1} \int_\Omega \sum_{j=-L}^{L} \left| \sum_{k=-n}^{n} \widehat{\kappa_n * \varphi}(-k) h_{j-k} \Phi_{j-k} f \right|^p J_j \left|h_j\right|^{-p} d\mu.$$

From Theorems 3.2 and 3.6, together with Proposition 3.7, we see that there is a constant $C(p, \mathfrak{s})$ such that, μ-a.e. on Ω,

$$\sum_{j=-L}^{L} \left| \sum_{k=-n}^{n} \widehat{\kappa_n * \varphi}(-k) h_{j-k} \Phi_{j-k} f \right|^p J_j \left|h_j\right|^{-p}$$

$$= \sum_{j=-L}^{L} \left| \sum_{k=-\infty}^{\infty} (\kappa_n * \varphi)^\vee(k) h_{j-k} \Phi_{j-k} f \right|^p \chi_{L,n}(j-k) J_j \left|h_j\right|^{-p}$$

$$\leq C(p, \mathfrak{s})^p \, \|\varphi\|_{\mathfrak{M}(\mathbb{T})}^p \sum_{k \in \mathbb{Z}} |h_k \Phi_k f|^p \, \chi_{L,n}(k) J_k \, |h_k|^{-p}$$

$$= C(p, \mathfrak{s})^p \, \|\varphi\|_{\mathfrak{M}(\mathbb{T})}^p \sum_{k=-L-n}^{L+n} |\Phi_k f|^p \, J_k.$$

Hence

$$\int_\Omega \left| \sum_{k=-n}^{n} \widehat{\kappa_n}(k)\widehat{\varphi}(k) T^k f \right|^p d\mu$$

$$\leq (2L+1)^{-1} C(p, \mathfrak{s})^p \, \|\varphi\|_{\mathfrak{M}(\mathbb{T})}^p \int_\Omega \sum_{k=-L-n}^{L+n} |\Phi_k f|^p \, J_k \, d\mu$$

$$= C(p, \mathfrak{s})^p \, \|\varphi\|_{\mathfrak{M}(\mathbb{T})}^p \, (2L+2n+1)\,(2L+1)^{-1} \int_\Omega |f|^p \, d\mu.$$

Letting $L \to \infty$, we conclude that

$$\left\| \sum_{k=-n}^{n} \widehat{\kappa_n}(k)\widehat{\varphi}(k) T^k f \right\|_p \leq C(p, \mathfrak{s}) \, \|\varphi\|_{\mathfrak{M}(\mathbb{T})} \, \|f\|_p \, .$$

This establishes (4.5) and completes the proof of Lemma 4.5.

COROLLARY 4.6. *There is a constant $C(p, \mathfrak{s})$ such that*

$$\left\| \int_{[0,2\pi]}^{\oplus} \varphi(e^{it}) \, dE(t) \right\| \leq C(p, \mathfrak{s}) \, \|\varphi\|_{\mathfrak{M}(\mathbb{T})}$$

whenever $\varphi \in BV(\mathbb{T})$, and φ is continuous on $\mathbb{T}\backslash\{1\}$.

Proof. If $\varphi \in BV(\mathbb{T})$ is continuous on $\mathbb{T}\backslash\{1\}$, then $\varphi(e^{it}) = \varphi^{\ddagger}(t)$ on $(0, 2\pi)$. Hence, using the left continuity of $E(\cdot)$ at 2π, we see that

$$\int_{[0,2\pi]}^{\oplus} \varphi(e^{it})\, dE(t) = \int_{[0,2\pi]}^{\oplus} \varphi^{\ddagger}(t)\, dE(t) + \alpha E(0)$$

for some $\alpha \in \mathbb{C}$ with $|\alpha| \leq 2\|\varphi\|_{\mathfrak{M}(\mathbb{T})}$. The result now follows from Lemma 4.5 and the bound for $\|E(0)\|$ given in Theorem 3.3

Completion of first proof of Lemma 4.2. Let $\varphi \in BV(\mathbb{T})$ and fix a partition $\mathcal{P} = (\lambda_0, \ldots, \lambda_n)$ of $[0, 2\pi]$. For $0 < \epsilon < \min\{\lambda_k - \lambda_{k-1}\colon 2 \leq k \leq n\}$, define $\varphi_\epsilon\colon \mathbb{T} \to \mathbb{C}$ so that:

(i) $\varphi_\epsilon(e^{it}) = \varphi(e^{i\lambda_1})$ when $0 = \lambda_0 < t \leq \lambda_1$;
(ii) $\varphi_\epsilon(e^{it}) = \varphi(e^{i\lambda_k})$ when $\lambda_{k-1} + \epsilon \leq t \leq \lambda_k$, for $2 \leq k \leq n$;
(iii) $\varphi_\epsilon(e^{it})$ is linear as a function of t on $[\lambda_{k-1}, \lambda_{k-1} + \epsilon]$, for $2 \leq k \leq n$.

Then $\{\varphi_\epsilon\}$ is a norm bounded set in $BV(\mathbb{T})$, and $\varphi_\epsilon \to \varphi_{\mathcal{P}}$ pointwise on $\mathbb{T}$ as $\epsilon \to 0$. Hence, by [12], Theorem 17.5 (see also [5], Proposition (2.10)),

$$\int_{[0,2\pi]}^{\oplus} \varphi_\epsilon(e^{it})\, dE(t) \to \int_{[0,2\pi]}^{\oplus} \varphi_{\mathcal{P}}(e^{it})\, dE(t) = \mathcal{S}(\mathcal{P}; \varphi, E) + \varphi(1)E(0)$$

in the strong operator topology as $\epsilon \to 0$. Furthermore, each φ_ϵ is continuous on $\mathbb{T}\backslash\{1\}$ and

$$\|\varphi_\epsilon\|_{\mathfrak{M}(\mathbb{T})} \leq 6\, \|\varphi\|_{\mathfrak{M}(\mathbb{T})}\,.$$

Since

$$\left\| \int_{[0,2\pi]}^{\oplus} \varphi_\epsilon(e^{it})\, dE(t) \right\| \leq C(p, \mathfrak{s})\, \|\varphi_\epsilon\|_{\mathfrak{M}(\mathbb{T})}$$

for all $\epsilon > 0$ by Corollary 4.6, we deduce that

$$\|\mathcal{S}(\mathcal{P}; \varphi, E)\| \leq C(p, \mathfrak{s})\, \|\varphi\|_{\mathfrak{M}(\mathbb{T})}\,,$$

and hence that

$$\left\| \int_{[0,2\pi]} \varphi(e^{it})\, dE(t) \right\| \leq C(p, \mathfrak{s})\, \|\varphi\|_{\mathfrak{M}(\mathbb{T})}\,.$$

This completes the proof of Lemma 4.2.

An alternative approach to Lemma 4.2. An alternative way to prove Lemma 4.2 is to apply the following result concerning trigonometrically well-bounded operators on subspaces of L^p-spaces, together with Theorems 3.4 and 3.5.

THEOREM 4.7. *Let X be a closed subpace of $L^p(\nu)$, where ν is an arbitrary measure and $1 \le p < \infty$, and let $U\colon X \to X$ be a trigonometrically well-bounded operator with spectral decomposition $F(\cdot)$. Suppose that there are positive constants α, β and γ such that:*

(i) *for all sequences $\{a_j\}_{j=1}^{\infty}$ in $[0, 2\pi)$ and all sequences $\{g_j\}_{j=1}^{\infty}$ in X,*

$$\left\| \left\{ \sum_{j=1}^{\infty} |F(a_j)g_j|^2 \right\}^{1/2} \right\|_{L^p(\nu)} \le \alpha \left\| \left\{ \sum_{j=1}^{\infty} |g_j|^2 \right\}^{1/2} \right\|_{L^p(\nu)} ; \tag{4.6}$$

(ii) *whenever $f \in X$, and $0 = u_0 < u_1 < u_2 < \cdots < u_{L-1} < u_L = 2\pi$ is a partition of $[0, 2\pi]$ with u_j a dyadic division point of $(0, 2\pi)$ for $1 \le j \le L-1$, then*

$$\begin{aligned} \beta \, \|f\|_{L^p(\nu)} &\le \left\| \left\{ |F(0)f|^2 + \sum_{j=1}^{L} |(F(u_j) - F(u_{j-1}))f|^2 \right\}^{1/2} \right\|_{L^p(\nu)} \\ &\le \gamma \, \|f\|_{L^p(\nu)} . \end{aligned} \tag{4.7}$$

Then

$$\left\| \int_{[0,2\pi]} \varphi(e^{it})\, dF(t) \right\| \le \beta^{-1}\gamma(2 + 3\alpha) \, \|\varphi\|_{\mathfrak{M}(\mathbb{T})}$$

for all $\varphi \in BV(\mathbb{T})$.

Proof. Let $\mathcal{P} = (\lambda_0, \lambda_1, \ldots, \lambda_n)$ be a partition of $[0, 2\pi]$, and let $\varphi \in BV(\mathbb{T})$. Choose dyadic division points t_N and t_M in $(0, 2\pi)$ such that $t_N < \lambda_1$ and $t_M > \lambda_{n-1}$, fix $g \in X$ and let $g_m = \{F(t_m) - F(t_{m-1})\}g$ for $N + 1 \le m \le M$. Then

$$\begin{aligned} \mathcal{S}(\mathcal{P}; \varphi, F)g &= \varphi(e^{i\lambda_1})\{F(t_N) - F(0)\}g + \sum_{m=N+1}^{M} \mathcal{S}(\mathcal{P}; \varphi, F)g_m \\ &\quad + \varphi(1)\{I - F(t_M)\}g. \end{aligned}$$

Since $\{F(t_N) - F(0)\}g \to 0$ as $N \to -\infty$ and $\{I - F(t_M)\}g \to 0$ as $M \to \infty$, it suffices to establish the inequality

$$\left\| \sum_{m=N+1}^{M} \mathcal{S}(\mathcal{P}; \varphi, F)g_m \right\|_{L^p(\nu)} \le \beta^{-1}\gamma(2 + 3\alpha) \, \|\varphi\|_{\mathfrak{M}(\mathbb{T})} \, \|g\|_{L^p(\nu)} . \tag{4.8}$$

If m is such that no partition point λ_k lies in the interval $[t_{m-1}, t_m]$, then $\mathcal{S}(\mathcal{P}; \varphi, F)g_m = \varphi(e^{i\lambda_j})g_m$ for some partition point λ_j. Suppose on the other hand that m is such that $\mathcal{P}\cap[t_{m-1}, t_m] = \{\lambda_{r_m}, \lambda_{r_m+1}, \ldots, \lambda_{s_m}\}$. Then a straightforward calculation gives

$$\begin{aligned}\mathcal{S}(\mathcal{P}; \varphi, F)g_m &= \varphi(e^{i\lambda_{s_m+1}})g_m \\ &\quad + \sum_{k=r_m}^{s_m}\{\varphi(e^{i\lambda_k}) - \varphi(e^{i\lambda_{k+1}})\}F(\lambda_k)g_m,\end{aligned}$$

which can be written more succinctly as

$$\mathcal{S}(\mathcal{P}; \varphi, F)g_m = \varphi(e^{i\lambda_{s_m+1}})g_m + \sum_{k=r_m}^{s_m}(\Delta_k\varphi)F(\lambda_k)g_m,$$

where $\Delta_k\varphi = \varphi(e^{i\lambda_k}) - \varphi(e^{i\lambda_{k+1}})$. Notice that $\sum_{k=r_m}^{s_m-1}|\Delta_k\varphi| \le \operatorname{var}_{[t_{m-1},t_m]}\varphi$ and $|\Delta_{s_m}\varphi| \le 2\|\varphi\|_{\mathfrak{M}(\mathbb{T})}$. We then have the chain of inequalities

$$\begin{aligned}|\mathcal{S}(\mathcal{P}; \varphi, F)g_m| &\le \|\varphi\|_{\mathfrak{M}(\mathbb{T})}\left\{|g_m| + 2\left|F(\lambda_{s_m})g_m\right|\right\} + \sum_{k=r_m}^{s_m-1}|\Delta_k\varphi|\ |F(\lambda_k)g_m| \\ &\le \|\varphi\|_{\mathfrak{M}(\mathbb{T})}\left\{|g_m| + 2\left|F(\lambda_{s_m})g_m\right|\right\} \\ &\quad + \left\{\sum_{k=r_m}^{s_m-1}|\Delta_k\varphi|\right\}^{1/2}\left\{\sum_{k=r_m}^{s_m-1}|\Delta_k\varphi|\ |F(\lambda_k)g_m|^2\right\}^{1/2} \\ &\le \|\varphi\|_{\mathfrak{M}(\mathbb{T})}\left\{|g_m| + 2\left|F(\lambda_{s_m})g_m\right|\right\} \\ &\quad + (\|\varphi\|_{\mathfrak{M}(\mathbb{T})})^{1/2}\left\{\sum_{k=r_m}^{s_m-1}|\Delta_k\varphi|\ |F(\lambda_k)g_m|^2\right\}^{1/2},\end{aligned}$$

valid pointwise on the underlying measure space. Denote by J the set of integers m in $[N+1, M]$ such that $\mathcal{P}\cap[t_{m-1}, t_m]$ is non-empty. Taking account of the fact that $\mathcal{S}(\mathcal{P}; \varphi, F)g_m = \varphi(e^{i\lambda_j})g_m$ for some partition point λ_j when $\mathcal{P}\cap[t_{m-1}, t_m]$ is empty, we now have the pointwise inequality

$$\begin{aligned}&\left\{\sum_{m=N+1}^{M}|\mathcal{S}(\mathcal{P}; \varphi, F)g_m|^2\right\}^{1/2} \\ &\quad \le 2\,\|\varphi\|_{\mathfrak{M}(\mathbb{T})}\left[\left\{\sum_{m=N+1}^{M}|g_m|^2\right\}^{1/2} + \left\{\sum_{m\in J}\left|F(\lambda_{s_m})g_m\right|^2\right\}^{1/2}\right] \\ &\quad + (\|\varphi\|_{\mathfrak{M}(\mathbb{T})})^{1/2}\left\{\sum_{m\in J}\sum_{k=r_m}^{s_m-1}|\Delta_k\varphi|\ |F(\lambda_k)g_m|^2\right\}^{1/2}.\end{aligned}$$

Using (4.6) and (4.7), we deduce from this estimate that

$$\begin{aligned}
&\left\| \left\{ \sum_{m=N+1}^{M} |\mathcal{S}(\mathcal{P}; \varphi, F) g_m|^2 \right\}^{1/2} \right\|_{L^p(\nu)} \\
&\quad \le 2 \, \|\varphi\|_{\mathfrak{M}(\mathbb{T})} (1+\alpha) \left\| \left\{ \sum_{m=N+1}^{M} |g_m|^2 \right\}^{1/2} \right\|_{L^p(\nu)} \\
&\qquad + \{\|\varphi\|_{\mathfrak{M}(\mathbb{T})}\}^{1/2} \left\| \left\{ \sum_{m\in J} \sum_{k=r_m}^{s_m-1} |\Delta_k \varphi| \; |F(\lambda_k) g_m|^2 \right\}^{1/2} \right\|_{L^p(\nu)} \\
&\quad \le 2\gamma \, \|\varphi\|_{\mathfrak{M}(\mathbb{T})} (1+\alpha) \, \|g\|_{L^p(\nu)} \\
&\qquad + (\|\varphi\|_{\mathfrak{M}(\mathbb{T})})^{1/2} \left\| \left\{ \sum_{m\in J} \sum_{k=r_m}^{s_m-1} |\Delta_k \varphi| \, |F(\lambda_k) g_m|^2 \right\}^{1/2} \right\|_{L^p(\nu)} \\
&\quad \le 2\gamma \, \|\varphi\|_{\mathfrak{M}(\mathbb{T})} (1+\alpha) \, \|g\|_{L^p(\nu)} \\
&\qquad + \alpha (\|\varphi\|_{\mathfrak{M}(\mathbb{T})})^{1/2} \left\| \left\{ \sum_{m\in J} \sum_{k=r_m}^{s_m-1} |\Delta_k \varphi| \; |g_m|^2 \right\}^{1/2} \right\|_{L^p(\nu)}
\end{aligned}$$

Hence

$$\begin{aligned}
&\left\| \left\{ \sum_{m=N+1}^{M} |\mathcal{S}(\mathcal{P}; \varphi, F) g_m|^2 \right\}^{1/2} \right\|_{L^p(\nu)} \\
&\quad \le 2\gamma \|\varphi\|_{\mathfrak{M}(\mathbb{T})} (1+\alpha) \|g\|_{L^p(\nu)} + \alpha \|\varphi\|_{\mathfrak{M}(\mathbb{T})} \left\| \left\{ \sum_{m\in J} |g_m|^2 \right\}^{1/2} \right\|_{L^p(\nu)} \\
&\quad \le \gamma (2+3\alpha) \|\varphi\|_{\mathfrak{M}(\mathbb{T})} \|g\|_{L^p(\nu)}. \qquad (4.9)
\end{aligned}$$

We can apply the first inequality in (4.7) to the vector $\sum_{m=N+1}^{M} \mathcal{S}(\mathcal{P}; \varphi, F) g_m$ in place of f, and to the partition $(u_0, u_1, \ldots, u_L)$ of $[0, 2\pi]$ specified by $0 < t_N < t_{N+1} < \cdots < t_M < 2\pi$. In view of the definition of the vectors g_m, this procedure gives

$$\begin{aligned}
&\left\| \sum_{m=N+1}^{M} \mathcal{S}(\mathcal{P}; \varphi, F) g_m \right\|_{L^p(\nu)} \\
&\quad \le \beta^{-1} \left\| \left\{ \sum_{m=N+1}^{M} |\mathcal{S}(\mathcal{P}; \varphi, F) g_m|^2 \right\}^{1/2} \right\|_{L^p(\nu)} . \qquad (4.10)
\end{aligned}$$

The required inequality (4.8) follows from (4.9) and (4.10). This completes the proof of Theorem 4.7.

Remarks. In order to emphasize its ergodic operator theory framework, all our results leading up to the establishment of Theorem 4.1 have been designed so that when applied to its context, they can furnish the claimed domination of the norm for the functional calculus of $\mathfrak{M}(\mathbb{T})$ by a constant $C(p, \mathfrak{s})$ involving any uniform bound $\mathfrak{s}$ for the ergodic averages of $|T|$. This has required due attention to the nature of the constants figuring in all our computations. In this regard, the calculations used above to establish Theorem 4.7 have been based on those for the power-bounded case which appear in the proofs of Lemmas 3.1 and 3.12 in [2]. A slightly more abstract approach to the proof of Theorem 4.7 would be to express the square function inequalities (4.6) and (4.7) in terms of Rademacher averages, and then to parallel the Banach space arguments of Lemma (4.3) in [6].

Notice that (4.6) implies $\|F(0)\| \le \alpha$. Combining Theorems 4.3 and 4.7, we arrive at the following umbrella theorem for transferring the Marcinkiewicz multiplier theorem to Lebesgue subspaces in the presence of transferred versions of the vector-valued M. Riesz property (4.6) and the Littlewood-Paley property (4.7). This circle of ideas abstracts the classical derivation of the Marcinkiewicz multiplier theorem.

THEOREM 4.8. *Assume the hypotheses of Theorem 4.7. Then, for each $\varphi \in \mathfrak{M}(\mathbb{T})$, the spectral integral $\int_0^{2\pi} \varphi(e^{it})\, dF(t)$ exists. Furthermore, the mapping*

$$\varphi \to \varphi(U) \equiv \int_{[0,2\pi]}^{\oplus} \varphi(e^{it})\, dF(t)$$

is an identity-preserving algebra homomorphism of $\mathfrak{M}(\mathbb{T})$ into $\mathfrak{B}(X)$ satisfying

$$\|\varphi(U)\| \le \{3\beta^{-1}\gamma(2+3\alpha)+\alpha\}\|\varphi\|_{\mathfrak{M}(\mathbb{T})}$$

for all $\varphi \in \mathfrak{M}(\mathbb{T})$.

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