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THE CONLEY INDEX AND NON-EXISTENCE OF MINIMAL HOMEOMORPHISMS

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ABSTRACT. We give a brief proof of the theorem of P. Le Calvez and J.-C. Yoccoz on the non-existence of a minimal homeomorphism of the finitely punctured plane. The proof here is based on the Conley index.

A problem posed some years ago by S. Ulam and included in the well-known *Scottish Book* ([5], problem 115) asks if there exists a homeomorphism of $\mathbb{R}^n$ or of $\mathbb{R}^n$ with a single point omitted which has every complete orbit dense. Such a homeomorphism is called *minimal* because the smallest non-empty closed invariant subset is the entire space.

Various partial results can be found referenced in [5], but in full generality the problem remains unresolved. In the case of all of $\mathbb{R}^2$ it is an easy consequence of the Brouwer plane translation theorem (see [2] for example) that no minimal homeomorphism can exist. But the case of the punctured plane proved substantially more elusive. This case was completely resolved, however, in a recent important paper of P. Le Calvez and J.-C. Yoccoz [4]. Le Calvez and Yoccoz prove, in fact, that the plane with any finite number of punctures does not admit a minimal homeomorphism.

The techniques in their paper involve an impressive analysis of the dynamics in the neighborhood of a fixed point of a local homeomorphism of a two-manifold. They use this to contradict the possibility of a minimal homeomorphism of the finitely punctured plane by compactifying the plane (adding the missing puncture points and a point at infinity) to transfer the problem to S^2. The local analysis then allows them to show that the existence of a minimal homeomorphism on the punctured plane would contradict the Lefschetz fixed point theorem.

In this paper we give an alternate proof of the this result on the non-existence of minimal homeomorphisms which is based on the use of the Conley index (described below). Le Calvez and Yoccoz have independently re-proved their result using ideas similar to those presented here. While the Conley index provides a much shorter path to this result it does not provide the deep local analysis of the dynamics in the neighborhood of a fixed point which can be found in [4].

1. Introduction and definitions

We begin with a brief review of the basic definitions and properties of the Conley index.

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Definition 1.1. Suppose Λ is a compact invariant set of homeomorphism $f\colon U \to f(U)$ where U is an open subset of a manifold. We say that Λ is *isolated* and N is an *isolating neighborhood* if $N \subset U$ is compact and contains Λ in its interior and $\Lambda = \bigcap_{n \in \mathbb{Z}} f^n(N)$. An isolating neighborhood N is an *isolating block* provided $N \cap f(N) \cap f^{-1}(N) \subset \mathrm{Int} N$.

Definition 1.2. A *regular index pair* for an isolated invariant set Λ with isolating neighborhood N is a compact pair (X, A) with $A \subset X$ such that

1. $f(X) \cap N \subset X$ and $f(A) \cap N \subset A$,
2. $X \setminus A \subset f^{-1}(N)$,
3. $\Lambda \subset \mathrm{Int}(X \setminus A)$, and
4. $\mathrm{cl}(X \setminus A) \cap \mathrm{cl}(f(A) \setminus X) = \emptyset$.

In the situation which we consider it will be the case that $X = N$ and N is an isolating block, not just an isolating neighborhood. Two results which we will need are the following theorems which we quote in the form given in [6]. The first is a result due to R. Easton [1]. It is also given as Proposition 4.8 of [6].

PROPOSITION 1.3. *If Λ is an isolated invariant set then every neighborhood of Λ contains an isolating block N for Λ.*

The following result can be found as Proposition 4.7 of [6].

PROPOSITION 1.4. *If N is an isolating block for Λ and $L = N \setminus f^{-1}(\mathrm{Int} N)$ then (N, L) is a regular index pair for Λ with isolating neighborhood N.*

LEMMA 1.5. *Suppose Λ is a compact connected isolated invariant set of a homeomorphism $f\colon U \to f(U)$ where U is an open subset of $\mathbb{R}^2$ and suppose that there is no isolating neighborhood V of Λ such that either $f(V) \subset V$ or $V \subset f(V)$. Then Λ has a regular index pair (N, L) such that $H_k(N, L; \mathbb{Q})$ is finite dimensional if $k = 1$, and vanishes if $k \neq 1$.*

Proof. Let N_0 be an isolating block for the isolated invariant set Λ. We can alter N_0 to be a compact manifold with boundary. This is done by constructing a smooth non-negative real-valued function g on U which vanishes precisely on N_0 and letting $N = g^{-1}([0, \epsilon])$ where ϵ is a small regular value of g. If N is sufficiently close to N_0 then N is an isolating block for Λ. Since Λ is connected, it is contained in a single component of N. Replacing N by this component (which is also an isolating block) we may assume N is connected.

According to Proposition 1.4 above, if we define $L_0 = N \setminus f^{-1}(\mathrm{Int} N)$ then (N, L_0) is a regular index pair for Λ. The hypothesis that neither $f(N) \subset N$ nor $N \subset f(N)$ implies that $L_0 \cap \partial N \neq \emptyset$ and that ∂N is not a subset of L_0. The problem is that it is possible for L_0 to have infinitely many components. We will alter L_0 to make it a nicer set.

Note that fact that (N, L_0) is a *regular* index pair implies $\mathrm{cl}(N \setminus L_0)$ is disjoint from $\mathrm{cl}(f(L_0) \setminus N) = f(L_0) \cap (\mathrm{Int}N)^c$, where superscript c indicates complement. It follows that $f(L_0)$ is disjoint from $\mathrm{cl}(N \setminus L_0)$ because $f(L_0) = f(N) \setminus \mathrm{Int}N$ so $f(L_0) \subset (\mathrm{Int}N)^c$.

From this we can conclude that any sufficiently small neighborhood L of L_0 has $f(L)$ disjoint from $\mathrm{cl}(N \setminus L_0)$. We again choose a smooth non-negative real-valued function g on U which this time vanishes precisely on L_0 and let $L_1 = g^{-1}([0, \epsilon])$ where ϵ is a small regular value of g. Then L_1 is a compact manifold with boundary which is a small neighborhood of L_0 in U. Perturbing L_1 slightly we can assume that ∂L_1 transversely intersects ∂N. We retain the property that L_1 is a sufficiently small neighborhood of L_0 that $f(L_1)$ is disjoint from $cl(N \setminus L_0)$. Setting $L = N \cap L_1$ it follows that $f(L)$ is disjoint from $\mathrm{cl}(N \setminus L_0)$ and hence from $\mathrm{cl}(N \setminus L)$.

From this one checks readily that (N, L) is a regular index pair for Λ. Since L can be chosen as an arbitrarily small neighborhood of L_0 in N we can arrange that $L \cap \partial N \neq \emptyset$ and that ∂N is not a subset of L. This implies that $H_0(N, L; \mathbb{Q}) = 0$ and $H_2(N, L; \mathbb{Q}) = 0$. All $H_k(N, L; \mathbb{Q}) = 0$ for $k > 2$ because N is a subset of the plane.

Finally by its construction it is possible to triangulate N with L a finite subcomplex. It follows that $H_1(N, L; \mathbb{Q})$ is finite dimensional. $\square$

Note that both the index pairs (N, L_0) and (N, L) in the proof above had the additional nice property that $f(L)$ is disjoint from $\mathrm{cl}(N \setminus L)$ (and that $f(L_0)$ is disjoint from $\mathrm{cl}(N \setminus L_0)$). This turns out to be a useful property.

PROPOSITION 1.6. *Suppose (N, L) is a regular index pair for Λ with isolating neighborhood N and suppose that $f(L)$ is disjoint from $cl(N \setminus L)$. Let N_L denote the quotient space obtained by collapsing L to a point. Then f induces a continuous map $\hat{f}$: $N_L \to N_L$. Moreover, if $[L]$ denotes the distinguished point in N_L to which L has been collapsed then $[L]$ has a neighborhood in N_L which is mapped by $\hat{f}$ to the point $[L]$.*

Proof. Note that we can identify $N_L \setminus \{[L]\}$ with $N \setminus L$. Recall that property 2 of the definition of regular index pair implies that $f(\mathrm{cl}(N \setminus L)) \subset N$. For $x \in \mathrm{cl}(N \setminus L)$ let $\hat{f}(x) = p(f(x))$ where p: $N \to N_L$ is the quotient map and define $\hat{f}([L]) = [L]$.

We must check that $\hat{f}$ is continuous. If $x \in N_L$ is not equal to $[L]$ then there is a neighborhood U of x with $U \subset (N \setminus L)$. On U the map $\hat{f}$ is equal to the composition of p and f and hence is continuous.

Thus we need only check continuity at $[L]$. The condition that $f(L)$ is disjoint from $\mathrm{cl}(N \setminus L)$ implies there is a small neighborhood U of L in N such that $f(U)$ is disjoint from $\mathrm{cl}(N \setminus L)$. Thus if $y \in U \setminus L$ then $f(y) \in L$. This implies there is a small neighborhood $V = p(U)$ of $[L]$ in N_L such that $\hat{f}(V) = \{[L]\}$. Let $\{x_i\}$ be a sequence in N_L converging to $[L]$. Then this sequence is eventually in V so $\{\hat{f}(x_i)\}$ is eventually the constant sequence with every term equal to $[L]$. Hence $\hat{f}$ is continuous at $[L]$. □

The space N_L and the map $\hat{f}$ and even their homotopy types depend on the choice of index pair (N, L). However there is a close relationship between $\hat{f}\colon N_L \to N_L$ and $\hat{f}\colon M_P \to M_P$ if (N, L) and (M, P) are two regular index pairs for the same isolated invariant set Λ. This relationship has been investigated by several authors (see [6], [8]. and [9]). The best formulation for our purposes is one observed by D. Richeson in his thesis [7], based on results of [9].

Definition 1.7. Two maps $f\colon X \to X$ and $g\colon Y \to Y$ are *shift equivalent* provided there are maps $r\colon X \to Y$ and $s\colon Y \to X$ and $n > 0$ such that $r \circ f = g \circ r$, $f \circ s = s \circ g$, $s \circ r = f^n$, and $r \circ s = g^n$.

Shift equivalence is a natural and dynamically significant equivalence relation for maps. Note that if f and g are homeomorphisms (i.e., invertible) then they are shift equivalent if and only if they are topologically conjugate. If they are shift equivalent a conjugacy is given by $h = r \circ f^{-n} = g^{-n} \circ r$ and $h^{-1} = s$.

The following result of D. Richeson based on work of Szymczak can be found in [7].

PROPOSITION 1.8. *If (N, L) and (M, P) are regular index pairs for the isolated invariant set Λ then the maps $\hat{f}\colon N_L \to N_L$ and $\hat{f}\colon M_P \to M_P$ are shift equivalent.*

The *homotopy Conley index* for an isolated invariant set Λ is then defined to be the shift equivalence class of $\hat{f}\colon N_L \to N_L$ in the homotopy category, but we will not make use of this.

2. The Lefschetz theorem

In this section we point out the relationship between the Lefschetz fixed point theorem and the Conley index.

PROPOSITION 2.1. *If (N, L) is a regular index pair for the isolated invariant set Λ then for any k the trace of $\hat{f}_{*k}\colon H_k(N_L, [L], \mathbb{Q}) \to H_k(N_L, [L], \mathbb{Q})$ is independent of (N, L).*

Proof. Let the $r\colon N_L \to M_P$ and $s\colon M_P \to N_L$ define a shift equivalence on the induced maps on the pointed spaces coming from regular index pairs (N, L) and (M, P) respectively. Then $r_*\colon H_*(N_L, [L]; \mathbb{Q}) \to H_*(M_P, [P]; \mathbb{Q})$ and $s_*\colon H_*(M_P, [P]; \mathbb{Q}) \to H_*(N_L, [L]; \mathbb{Q})$ defines a shift equivalence of induced linear maps $\hat{f}_{*k}$.

If the linear maps R and S between vector space endomorphisms A and B define a shift equivalence then for every $n > 0$,

$$R(\ker(A - \lambda I)^n) \subset \ker(B - \lambda I)^n,$$

and

$$S(\ker(B - \lambda I)^n) \subset \ker(B - \lambda I)^n.$$

Also if $\lambda \neq 0$ the maps $R \circ S$ and $S \circ R$ are isomorphisms on these generalized eigenspaces. It follows that if $\lambda \neq 0$ then λ has the same multiplicity as an eigenvalue of A and of B. From this it follows that for any k the trace of $\hat{f}_{*k}$: $H_k(N_L, [L], \mathbb{Q}) \to H_k(N_L, [L], \mathbb{Q})$ is independent of (N, L). □

PROPOSITION 2.2. *If (N, L) is a regular index pair for the isolated invariant set Λ and $I(\Lambda, f)$ denotes the Lefschetz index of Λ then*

$$I(\Lambda, f) = \sum_{k=0}^{n} (-1)^k tr \hat{f}_{*k}$$

where $\hat{f}_{*k}$: $H_k(N_L, [L], \mathbb{Q}) \to H_k(N_L, [L], \mathbb{Q})$.

Proof. Let $X = N_L$ and $A = \{[L]\}$ and consider the long exact sequence of the pair (X, A). Each element of this sequence has a linear endomorphism induced by $\hat{f}$. Because the sequence is exact the alternating sum of the traces of all of these endomorphisms is zero (see for example [3, p. 98]). If we define $\mathcal{L}(X, A, \hat{f})$ to be $\sum_{k=0}^{n}(-1)^k \operatorname{tr} \hat{f}_{*k}$ for $\hat{f}_{*k}$: $H_k(X, A; \mathbb{Q}) \to H_k(X, A; \mathbb{Q})$ and define $\mathcal{L}(X, \hat{f})$ and $\mathcal{L}(A, \hat{f})$ analogously then by re-ordering the terms of this long exact sequence we obtain. $\mathcal{L}(A, \hat{f}) - \mathcal{L}(X, \hat{f}) + \mathcal{L}(X, A, \hat{f}) = 0$.

Thus $\mathcal{L}(X, A, \hat{f}) = \mathcal{L}(X, \hat{f}) - \mathcal{L}(A, \hat{f}) = \mathcal{L}(X, \hat{f}) - 1$ since $\mathcal{L}(A, \hat{f}) = 1$ as A is a single point. Since all fixed points of $\hat{f}$ are either in Λ or the single point $[L]$ we also know that $\mathcal{L}(X, \hat{f}) = I(\Lambda \cup A, \hat{f}) = I(\Lambda), \hat{f}) + I(A, \hat{f})$. Hence we need only the fact that $I(A, \hat{f}) = 1$ to conclude that $I(\Lambda), \hat{f}) = \mathcal{L}(X, \hat{f}) - 1$.

To see this we observe that we can use any regular index we choose and hence by Proposition 1.6 we can assume there is a neighborhood V of $[L]$ with $\hat{f}(V) = \{[L]\}$. This implies that $I(A, \hat{f})$, the index of the fixed point $[L]$, is equal to one and completes the proof. □

We finish this section with a lemma on the traces of powers of an arbitrary real matrix.

LEMMA 2.3. *Let A be an arbitrary $n \times n$ real matrix. Then for infinitely many integers $k > 0$ the trace of A^k is non-negative.*

Proof. If A is nilpotent then $\operatorname{tr} A^k = 0$ for all $k > 0$. Otherwise there are non-zero eigenvalues λ_j, $j = 1 \dots m$, of A which we enumerate with multiplicity. Write $\lambda_j = r_j \exp(2\pi i \theta_j)$ where $r_j > 0$ and $\theta_j \in \mathbb{R}/\mathbb{Z}$. Then $\Theta = (\theta_1, \theta_2, \dots, \theta_m)$ is an element of the m-torus $\mathbb{T}^m$. Let G be the closure of the subgroup of $\mathbb{T}^m$ generated by Θ. Then the set $\{n\Theta \mid n \in \mathbb{Z}, n \neq 0\}$ is dense in G. It follows that for infinitely many values of k the element $k\Theta$ is close to the identity of G. Since the same is true of $(-k)\Theta$ we can assume that $k > 0$. Thus there are infinitely many positive values of k with $k\Theta = (k\theta_1, k\theta_2, \dots, k\theta_m)$ in a small neighborhood of $(0, 0, \dots, 0) \in \mathbb{T}^m$. In particular, for all $1 \leq j \leq m$ we can have $\exp(2k\pi i \theta_j)$ close enough to 1 that it has positive real part. Hence

$$\operatorname{tr} A^k = \sum_{j=1}^{m} r_j^k \exp(2k\pi i \theta_j) = \sum_{j=1}^{m} r_j^k (\operatorname{Re}(\exp(2k\pi i \theta_j)) > 0$$

for infinitely many positive integers k. $\square$

3. There is no minimal homeomorphism of the punctured plane

We are now in a position to give a simple proof of the result of Le Calvez and Yoccoz [4] which asserts the non-existence of a minimal homeomorphism of any finitely punctured two-sphere.

The proof is based on the following proposition.

PROPOSITION 3.1. *Suppose p is a fixed point of a local homeomorphism f of the plane and $\{p\}$ is an isolated invariant set and suppose that there is no isolating neighborhood V of p such that either $f(V) \subset V$ or $V \subset f(V)$. Then there are infinitely many values of $n > 0$ such that $I(p, f^n) \leq 0$. Moreover if $\Lambda = \{p_i\}_{i=1}^k$ is a finite set of fixed points each with the properties of p then there are infinitely many values of $n > 0$ such that*

$$I(\Lambda, f^n) = \sum_{i=1}^{k} I(p_i, f^n) \leq 0.$$

Le Calvez and Yoccoz actually prove a stronger result, namely that for a p as above there are positive integers r and q such that $I(p, f^n) = 1 - rq$ whenever n is a multiple of q. However, the result above is substantially easier and sufficient for our purposes.

Proof. We choose an index pair (N, L) for the isolated invariant set $\{p\}$ with the properties described in Lemma 1.5 and construct $\hat{f}\colon N_L \to N_L$. By Proposition 2.2,

$$I(p, f^n) = \sum_{k=0}^{2} (-1)^k \operatorname{tr} \hat{f}^n_{*k}$$

where $\hat{f}^n_{*k}\colon H_k(N_L, [L], \mathbb{Q}) \to H_k(N_L, [L], \mathbb{Q})$. But by Lemma 1.5 if $k \neq 1$, then $H_k(N_L, [L], \mathbb{Q}) = 0$. Hence $I(p, f^n) = -\operatorname{tr} \hat{f}^n_{*1} = -\operatorname{tr} A^n$ where A is a matrix for $\hat{f}_{*1}$. By Lemma 2.3, $\operatorname{tr} A^n \geq 0$ for infinitely many $n > 0$ so $I(p, f^n) \leq 0$ for these n.

To obtain the result for a finite set of fixed points $\{p_i\}_{i=1}^k$ let A_i be a matrix for the map on the one dimensional homology for p_i and consider the matrix $A = \bigoplus_{i=1}^k A_i$. Then

$$I(\Lambda, f^n) = \sum_{i=1}^{k} I(p_i, f^n) = \sum_{i=1}^{k} -\operatorname{tr} A_i^n = -\operatorname{tr} A^n.$$

Again Lemma 2.3 implies $\operatorname{tr} A^n \geq 0$ for infinitely many $n > 0$ so the result follows. □

THEOREM 3.2. (Le Calvez and Yoccoz [4]). *If M is the finitely punctured sphere $S^2 \setminus \{p_1, \dots, p_n\}$ then there is no homeomorphism $f\colon M \to M$ with each complete (forward and backward) orbit dense.*

Proof. We assume such an f exists and show this leads to a contradiction. The homeomorphism f extends continuously to a homeomorphism of S^2 which permutes the points of the set $\{p_1, \dots, p_k\}$. Let $g = f^n\colon S^2 \to S^2$ where n is chosen so that g preserves orientation and $g(p_i) = p_i$ for $1 \leq i \leq k$. The fact that each orbit of f on M is dense implies that each of the points p_i is an isolated invariant set for g.

If V is an isolating neighborhood of p_i for g then it cannot be the case that $g(V) \subset V$. To see this, note that V's being an isolating neighborhood implies $\operatorname{Int}(B \setminus g(B))$ is non-empty but any orbit of f can have only finitely many points in $\operatorname{Int}(B \setminus g(B))$. A similar argument shows that $V \subset g(V)$ is impossible.

Thus we can apply Proposition 3.1 and conclude

$$\sum_{i=1}^{k} I(p_i, g^j) \leq 0$$

for infinitely many values of $j > 0$. There are no other periodic points of g since such a periodic point of g would be on a finite orbit of f. Hence the sum of the indices of all fixed points of g^j is less than or equal to 0. But, by the Lefschetz index theorem, this sum is also the Euler characteristic of S^2 which is 2. This contradicts the assumption that there exists a homeomorphism $f\colon M \to M$ with all orbits dense. □

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