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NON-LINEAR BALAYAGE AND APPLICATIONS

MURALI RAO AND JAN SOKOŁOWSKI

ABSTRACT. A theory of capacities has been extentively studied for Besov spaces [1]. However not much seems to have been done regarding non-linear potentials. We develop some of this here as consequences of the form of certain metric projections.

The non-linear potential theory is used to derive the form of tangent cones for a class of convex sets in Besov spaces. Tangent cones for obstacle problem arise when studying differentiability of metric projection. Characterising the tangent cones is the first step in these considerations. This has been done in some of the Sobolev spaces using Hilbert space methods. In this article we describe tangent cones for obstacle problems precisely, using non-linear potential theoretic ideas, for all Besov spaces $B_\alpha^{p,q}$, $1 < p < \infty, 1 < q < \infty, \alpha > 0$.

1. Introduction

Classical or *linear* potential theory has played a fundamental role in boundary value problems. A particular case of these potentials, the so called equilibrium potentials, and the resulting capacity theory are crucial in the description of *small sets*. In the *non-linear* setting, we find a very well developed theory of capacities and capacitary potentials in the literature; our main reference is [1]. However not much is said about non-linear potentials.

Our main objective in this paper is the development of a theory of non-linear potentials. We will develop some properties of these potentials that are analogues of their classical counterparts. The theory is non linear because the sum of two potentials is not necessarily a potential. Our main method of attack is the determination of the form of metric projections onto special convex sets.

In Section 2 we introduce *kernels* on $\mathbb{R}^N$ with values in $l^{q'}(L^{p'}(\mathbb{M}))$ spaces. The conditions on these kernels are general enough to include the Besov $B_\alpha^{p,q}$ spaces, $1 < p < \infty$, $1 < q < \infty$, $\alpha > 0$. The action of any $l^q(L^p(\mathbb{M}))$ function on this kernel then determines a *potential* on $\mathbb{R}^N$ and the action of any measure on $\mathbb{R}^N$ determines a *potential* on $l^{q'}(L^{p'}(\mathbb{M}))$. The main result of this section, Theorem 1, characterizes the elements of $l^{q'}(L^{p'}(\mathbb{M}))$ which are non-negative on elements of $l^q(L^p(\mathbb{M}))$ giving rise to non-negative potentials on $\mathbb{R}^N$ as potentials of non-negative Radon measures on $\mathbb{R}^N$.

We will make crucial use of Theorem 1 in Section 3 to introduce balayage and capacitary potentials in this setup. These happen to be elements of smallest norm in suitable closed closed convex sets. It is shown that these are given as nonlinear

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potentials. It might seem that Theorem 2 is a consequence of the results in [1]. However note that in [1], two versions of $\alpha - p$ capacities are given. A first version presented in pp. 19–23 is valid only for special kernels (such as Bessel Kernels). As the proof on p. 22 shows one *needs* the (distributional) inverse of such kernels and the fact that positive distributions are measures. This argument does not extend to the context of Besov Spaces. In [1], another version of $\alpha - p$ capacities is developed in Sections 2.3–2.5 precisely for this purpose—see the remarks on p. 85 and p. 104 of [1]. These remarks show that the authors consider the theory of capacities developed for Bessel kernels in pp. 22–23 inadequate to deal with Besov Spaces. A nice abstract theory of nonlinear capacities is developed. To do this the authors need a version of the min-max principle which, while very interesting, is not exactly easy. On the other hand its applicability in the context of Besov Spaces has not gone beyond a theory of capacities. In particular it is not clear how this can be used to extend balayage theory for Besov Spaces. And this is precisely what is needed to get tangent cones. Balayage has not, to our knowledge, been discussed in the nonlinear setting. We do this in Section 3.

In Section 4 we develop this nonlinear balayage further. We prove that capacity zero sets are subsets of *poles* of non-linear potentials, that the set of non-linear potentials while not necessarily convex is necessarily complete etc. Those are counterparts of results for classical potential theory. Finally, in Section 5, the results of Sections 2 and 3 and the characterization of Besov $B_\alpha^{p,q}$ spaces given in Chapter 4 of [1] allows us to precisely describe the form of the tangent cone $\mathfrak{T}_{\mathfrak{K}}(z)$ for any z in the convex set

$$\mathfrak{K} = \{f \in B_\alpha^{p,q}(\mathbb{R}^N) \mid f(x) \geq \psi(x) \quad \text{q.e.}\}, \quad 1 < p, q < \infty, \alpha > 0\,.$$

This result of the paper generalizes the previous results [3],[4],[5] obtained in the framework of the Hilbert space theory of Sobolev spaces combined with the linear potential theory to the general setting of non-linear potential theory in the Besov spaces.

2. A general result

We derive a result for $l^q(L^p)$ spaces which will be useful for applications. The same result can be proved for the L^p spaces using the same arguments.

Let $\mathbb{M}$ be a measure space with a σ-finite measure ν. Let

$$X = \mathbb{R}^N \times \mathbb{M} \times \mathbb{N}\,,$$

where $\mathbb{N}$ is the set of non-negative integers.

Fix $1 < p, q < \infty$, and let

$$\|f(\cdot, n)\|_p^p = \int_{\mathbb{M}} |f(y, n)|^p \nu(dy) = \int |f(y, n)|^p \nu(dy).$$

The space $l^q(L^p(\mathbb{M}))$ is defined as the set of $f = f(y,n)$, $y \in \mathbb{M}$, $n \in \mathbb{N}$ such that

$$\|f\|_{p,q} = \left(\Sigma_{n\in\mathbb{N}}\|f(\cdot,n)\|_p^q\right)^{\frac{1}{q}} < \infty,$$

i.e., the sequence $\|f(\cdot,n)\|_p \in l^q$. The space $l^q(L^p(\mathbb{M}))$ is a reflexive Banach space with the dual space $(l^q(L^p(\mathbb{M})))' = l^{q'}(L^{p'}(\mathbb{M}))$.

For any $f = f(y,n)$ we write

$$\Sigma_{n\in\mathbb{N}}\int f(y,n)\nu(dy) = \int f(y,n)\nu(dy\,dn).$$

Definition 1. By a kernel T on X we mean a non-negative function $T(x,y,n)$, $x \in \mathbb{R}^N$, $y \in \mathbb{M}$, $n \in \mathbb{N}$, such that:

1. $0 \leq T$; for each fixed (y,n), $T(\cdot,y,n)$ is continuous on $\mathbb{R}^N$ with compact support. For each $x \in \mathbb{R}^N$, $T(x,y,n)$ is measurable in (y,n).
2. For each compact set $K \subset \mathbb{R}^N$,
$$\int_K T(x,y,n)dx \in l^{q'}(L^{p'}(\mathbb{M})).$$
3. There exists a non-negative $A \in l^q(L^p(\mathbb{M}))$ such that
$$\Phi(x) = \Sigma_{n\in\mathbb{N}}\int T(x,y,n)A(y,n)\nu(dy) = \int T(x,y,n)A(y,n)\nu(dy\,dn)$$
is stricly positive on $\mathbb{R}^N$.

Remark 1. For all non-negative $f \in l^q(L^p(\mathbb{M}))$, the integral

$$Tf = \int T(x,y,n)f(y,n)\nu(dy\,dn)$$

is well-defined on $\mathbb{R}^N$. Tf is a lower semicontinuous function on $\mathbb{R}^N$, from condition 1 above.

For each non-negative measure μ on $\mathbb{R}^N$ the integral

$$\check{T}\mu = \int T(x,y,n)\mu(dx)$$

is a well-defined non-negative function on $\mathbb{M} \times \mathbb{N}$.

Remark 2. Condition 2 above implies that for each $f \in l^q(L^p(\mathbb{M}))$,

$$Tf \in L^1_{\text{loc}}(\mathbb{R}^N). \tag{1}$$

Moreover, the Hölder inequality implies that T maps $l^q(L^p(\mathbb{M}))$ continuously into $L^1(\mu)$ for each $\mu \in \mathfrak{M}$ where

$$\mathfrak{M} = \{\mu \mid \mu \text{ non-negative Radon measure, } \check{T}\mu \in l^{q'}(L^{p'}(\mathbb{M}))\}.$$

Assumption 1. We assume in the rest of the paper that the kernel T is such that Tf = 0 implies f=0. We do not give conditions on T validating this. In situations of interest, namely Besov spaces, this is true: See Theorem 4.17 in [1].

2.1. *Quasi-null sets.*

Now, we need the notion of a quasi-null set or a set of capacity zero.

Definition 2. A set $\mathcal{E}$ is called a quasi-null set if $\mathcal{E} \subset \mathcal{B}$ for a Borel set $\mathcal{B}$ such that

$$\mu(\mathcal{B}) = 0 \quad \text{for all } \mu \in \mathfrak{M}.$$

A countable union of quasi-null sets is quasi-null.

A property holds quasi everywhere, written q.e., if it holds except perhaps for a quasi-null set.

If $f \in l^q(L^p(\mathbb{M}))$, it is easily seen from the Hölder inequality that the set

$$\{x \mid T|f|(x) = \infty\}$$

is quasi-null. In particular, for each $f \in l^q(L^p(\mathbb{M}))$, Tf is q.e. well defined.

Condition 2 above implies, of course, that quasi-null sets are of Lebesgue measure zero.

We record another simple consequence:

Consequence. If a sequence $\{f_n\} \subset l^q(L^p(\mathbb{M}))$ tends to $f \in l^q(L^p(\mathbb{M}))$ then for a subsequence Tf_n tends to Tf pointwise quasi-everywhere.

Indeed, choose a subsequence $\{f_{n_i}\}$ so that

$$\Sigma_i \|f_{n_i} - f_{n_{i-1}}\|_{p,q} < \infty.$$

Then, by Fubini's theorem and the Hölder inequality, for any $\mu \in \mathfrak{M}$,

$$\begin{aligned}\Sigma_i \int |Tf_{n_i} - Tf_{n_{i-1}}|\, d\mu &\le \Sigma_i \int T|f_{n_i} - f_{n_{i-1}}|\, d\mu \\ &\le \Sigma_i \|f_{n_i} - f_{n_{i-1}}\|_{p,q} \|\check{T}\mu\|_{p',q'} < \infty.\end{aligned}$$

Thus the μ-measure of the set

$$\{x \mid \Sigma_i |Tf_{n_i} - Tf_{n_{i-1}}|(x) = \infty\}$$

is zero for every $\mu \in \mathfrak{M}$.

2.2. *Main result.*
With this setup we have the following theorem.

THEOREM 1. *For $g \in l^{q'}(L^{p'}(\mathbb{M}))$ the following are equivalent:*

1. $\int g(y,n)f(y,n)\nu(dy\,dn) \geq 0$ *for all* $f \in l^q(L^p(\mathbb{M}))$ *such that* $Tf \geq 0$ *q.e. on* $\mathbb{R}^N$.
2. $g = \check{T}\mu$ *for some non-negative Radon measure* μ *on* $\mathbb{R}^N$.

Proof. We give the proof in simple steps.

Step 1. Let

$$\mathfrak{C} = \{h \in l^{q'}(L^{p'}(\mathbb{M})) \mid h = \check{T}\mu \text{ for some non-negative Radon measure } \mu \text{ on } \mathbb{R}^N\}.$$

We claim $\mathfrak{C}$ is a closed convex cone in $l^{q'}(L^{p'}(\mathbb{M}))$.

The only item not at once clear is the closedness of $\mathfrak{C}$. Let $\{h_k\} \subset \mathfrak{C}$ converge to $h \in l^{q'}(L^{p'}(\mathbb{M}))$. By choosing a subsequence if necessary we may assume $h_k(y,n)$ converges to $h(y,n)$ μ-a.e.y for all n. If $h_k = \check{T}\mu_k$ we have

$$\int \Phi(x)\mu_k(dx) = \int h_k(y,n)A(y,n)\nu(dy\,dn) \leq \|h_k\|_{p',q'}\|A\|_{p,q}\,. \tag{2}$$

The right side is bounded in k because $\{h_k\}$ converges in $l^{q'}(L^{p'}(\mathbb{M}))$.

Now, Φ, being a strictly positive lower semicontinuous function, is bounded below on compacts. From (2) one concludes that $\{\mu_k(K)\}$ is bounded for each compact K. By choosing a subsequence if necessary we may assume that there is a Radon measure μ such that

$$\lim_k \int \varphi(x)\mu_k(dx) = \int \varphi(x)\mu(dx)$$

for each continuous function φ with compact support in $\mathbb{R}^N$. Since by assumption, for each $(y,n) \in \mathbb{M}\times\mathbb{N}$, $T(x,y,n)$ is continuous in x and has compact support, and since $h_k(y,n)$ converges for ν-a.e.y and all n to $h(y,n)$, we have for ν-a.e.y and all n

$$h(y,n) = \lim_k h_k(y,n) = \lim_k \check{T}\mu_k(y,n) = \check{T}\mu(y,n)\,.$$

This shows that $h \in \mathfrak{C}$ and thus $\mathfrak{C}$ is closed.

Step 2. We will show that

$$\left\{f \in l^q(L^p(\mathbb{M})) \mid \int F(y,n)f(y,n)\nu(dy\,dn) \geq 0 \text{ for all } F \in \mathfrak{C}\right\} \\ = \{g \in l^q(L^p(\mathbb{M})) \mid Tg \geq 0 \text{ q.e. on } \mathbb{R}^N\}\,. \tag{3}$$

The set on the left in (3) clearly contains that on the right. On the other hand let $f \in l^q(L^p(\mathbb{M}))$ be such that

$$\int F(y,n)f(y,n)\nu(dy\,dn) \geq 0 \quad \text{for all } F \in \mathfrak{C}.$$

Since, $|f| \in l^q(L^p(\mathbb{M}))$ and $F \in l^{q'}(L^{p'}(\mathbb{M}))$,

$$\infty > \int |f(y,n)|F(y,n)\nu(dy\,dn) = \int |f(y,n)|T(x,y,n)\mu(dx)\nu(dy\,dn)$$

Fubini's theorem permits interchange of the order of integration. Therefore, for all non-negative measures μ on $\mathbb{R}^N$ such that $\check{T}\mu \in l^{q'}(L^{p'}(\mathbb{M}))$,

$$0 \leq \int f(y,n)\check{T}\mu(y,n)\nu(dy\,dn) = \int (Tf)(x)\mu(dx). \tag{4}$$

If $\mu \in \mathfrak{M}$, for any Borel set $\mathcal{B}$, $\mu_{\mathcal{B}}$ also belongs to $\mathfrak{M}$, where $\mu_{\mathcal{B}}$ is the restriction of μ to $\mathcal{B}$. From (4) we see that the set $\{Tf < 0\}$ has μ-measure zero for every $\mu \in \mathfrak{M}$, i.e., $Tf \geq 0$ q.e. Thus the sets are identical.

Step 3. This is the last step in the proof. To this end let $g \in l^{q'}(L^{p'}(\mathbb{M}))$ and suppose

$$\int g(y,n)f(y,n)\nu(dy\,dn) \geq 0 \text{ for each } f \text{ such that } Tf \geq 0 \text{ q.e.} \tag{5}$$

We want to show that $g \in \mathfrak{C}$. If not, by the Hahn–Banach theorem, there is a function $\varphi \in l^q(L^p(\mathbb{M}))$ and $\alpha \in \mathbb{R}$ such that

$$\int F(y,n)\varphi(y,n)\nu(dy\,dn) > \alpha \text{ for all } F \in \mathfrak{C} \tag{6}$$

but

$$\int g(y,n)\varphi(y,n)\nu(dy\,dn) < \alpha.$$

Now $\mathfrak{C}$ is a cone so $\lambda F \in \mathfrak{C}$ for all $\lambda > 0$. From the first inequality in (6) with F replaced by λF, we get

$$\lambda \int F(y,n)\varphi(y,n)\nu(dy\,dn) > \alpha \text{ for all } \lambda > 0. \tag{7}$$

Dividing by λ and letting $\lambda \to \infty$ we infer that

$$\int F(y,n)\varphi(y,n)\nu(dy\,dn) \geq 0 \quad \text{for all } F \in \mathfrak{C}$$

which, by step 2, implies

$$T\varphi(x) \geq 0 \quad \text{q.e.} \quad \text{on } \mathbb{R}^N .$$

This last inequality in turn implies, by (5), that

$$\int g(y, n)\varphi(y, n)\nu(dy\, dn) \geq 0 .$$

But then, by the second inequality in (6), we must have $\alpha > 0$. However, by (7), this cannot be valid since λ can be chosen arbitrarily small.

This contradiction completes the proof. □

3. Non-linear balayage

To motivate and explain the term non-linear balayage let us recall some classical potential theory.

Let Ω be a bounded domain in $\mathbb{R}^N$ and Δ the Dirichlet–Laplacian. A lower semi-continuous function u on Ω is called superharmonic if, in the sense of distributions, $\Delta u \leq 0$ in Ω. Let G be the integral kernel of the operator $(-\Delta)^{-1}$; a famous result of F. Riesz states:

Let $u \geq 0$ be superharmonic in Ω. Then there is a unique Radon measure μ and a harmonic function h such that

$$u(x) = \int G(x, y)\mu(dy) + h(x) .$$

When $h = 0$, u is called a potential and μ its Riesz–measure. This result is basic in classical potential theory.

The balayage or sweeping process of H. Poincaré is the following.

Let μ be a measure, $p = G\mu$ its potential and K a compact subset of Ω. Let q be the lower semicontinuous regularisation of

$$\inf \{u\colon\ 0 \leq u \text{ superharmonic}, u \geq p \text{ on } K\} .$$

Then q is a potential with its Riesz–measure ν concentrated on K. The measure ν is called the balayage of μ on K. We shall also say that q is the balayage of p on K.

Now suppose that μ has finite energy:

$$\int G\mu\, d\mu < \infty .$$

Then $G\mu \in H_0^1(\Omega)$. H. Cartan proved that q is nothing but the "projection" of p on the closed convex set

$$\{u\colon\ u \in H_0^1(\Omega), u \geq p \quad \text{q.e. on } K\} .$$

In other words "balayages" are special cases of projections onto convex sets.

We hope this explanation will be usefull in motivating our nomenclature.

Keeping the notation of previous section, let $\mathbb{B}$ denote the following space of functions on $\mathbb{R}^N$:

$$\mathbb{B} = \{Tf \mid f \in l^q(L^p(\mathbb{M}))\}. \tag{8}$$

If $u \in \mathbb{B}$, $u = Tf$, we define

$$\|u\| = \|u\|_{\mathbb{B}} \equiv \|f\|_{p,q}.$$

Then $\mathbb{B}$ is a Banach space. By suitable choices of the kernel T and the space $\mathbb{M}$ we get the Besov space $B_\alpha^{p,q}$, $\alpha > 0$. We return to this later.

Remark 3. Condition 2 on the kernel T guarantees that

$$\mathbb{B} \subset L^1_{\text{loc}}(\mathbb{R}^N).$$

We have seen that if a sequence $\{u_k\}$ in $\mathbb{B}$ converges to $u \in \mathbb{B}$ then a subsequence of $u_k(x)$ converges to $u(x)$ for quasi every x.

Remark 4. Using the strict convexity of L^p-spaces it is not difficult to see that every closed convex set in $\mathbb{B}$ has a unique element of smallest norm.

Theorem 1 permits the introduction of *balayage* into $\mathbb{B}$:

Let h be any measurable function on $\mathbb{R}^N$ and let

$$\mathfrak{C}_h = \{u \in \mathbb{B} \mid u \geq h \text{ q.e.}\}. \tag{9}$$

Assume $\mathfrak{C}_h$ is not empty. From the above remarks we infer that $\mathfrak{C}_h$ is a closed convex subset of $\mathbb{B}$ and there is a unique element of smallest norm in $\mathfrak{C}_h$.

This element of smallest norm we call balayage of h and denote it by $\mathcal{R}h$.

Let us look at this a bit more closely. Let

$$\mathcal{R}h = T\varphi, \quad \varphi \in l^q(L^p(\mathbb{M})).$$

Then $T\varphi \geq h$ q.e. and for any $t > 0$ and any $f \in l^q(L^p(\mathbb{M}))$ such that $Tf \geq 0$ q.e.,

$$T(\varphi + tf) \geq h \quad \text{q.e.}$$

In the other words $\varphi + tf \in \mathfrak{C}_h$. By the definition of φ,

$$\|\varphi\|_{p,q} \leq \|\varphi + tf\|_{p,q}, \ t > 0, \quad Tf \geq 0 \text{ q.e.} \tag{10}$$

Written in full (10) is the same as

$$\Sigma_n \|\varphi(\cdot, n)\|_p^q \leq \Sigma_n \|\varphi(\cdot, n) + tf(\cdot, n)\|_p^q \tag{11}$$

for all $t > 0$ and all f such that $Tf \geq 0$ q.e. The derivative relative to t of the n-th term on the right side is

$$q\|\varphi(\cdot,n)+tf(\cdot,n)\|_p^{q-p}\int|\varphi(y,n)+tf(y,n)|^{p-2}(\varphi(y,n)+tf(y,n))f(y,n)\nu(dy), \tag{12}$$

whose absolute value, by the Hölder inequality, is dominated by

$$\begin{aligned} q\|\varphi(\cdot,n)+tf(\cdot,n)\|_p^{q-p}\|f(\cdot,n)\|_p\|\varphi(\cdot,n)+tf(\cdot,n)\|_p^{\frac{p}{p'}} \\ = q\|\varphi(\cdot,n)+tf(\cdot,n)\|_p^{q-1}\|f(\cdot,n)\|_p. \end{aligned}$$

Using this estimate and the Hölder inequality we see that the series of derivatives of the terms on right side of (11) is uniformly convergent on compacts. Therefore, term by term differentiation of the right side in (11) is permissible and (11) says the derivative at $t = 0$ is non-negative. From (12),

$$q\Sigma_n\|\varphi(\cdot,n)\|_p^{q-p}\int|\varphi(y,n)|^{p-2}\varphi(y,n)f(y,n)\nu(dy)\geq 0 \quad \text{if } Tf\geq 0 \text{ q.e.} \tag{13}$$

The function

$$a = a(y,n) = \|\varphi(\cdot,n)\|_p^{q-p}|\varphi(y,n)|^{p-2}\varphi(y,n) \tag{14}$$

belongs to $l^{q'}(L^{p'}(\mathbb{M}))$ as can be verified using the Hölder inequality.

(13) and Theorem 1 imply

$$a = \check{T}\mu \tag{15}$$

for some non-negative Radon measure μ on $\mathbb{R}^N$. Using (14) and (15) we get

$$\varphi(y,n) = \|\check{T}\mu(\cdot,n)\|_{p'}^{q'-p'}(\check{T}\mu(y,n))^{p'-1}. \tag{16}$$

This we state as:

THEOREM 2. *Let h be any measurable function and suppose the closed convex set $\mathfrak{C}_h$ is not empty. There is a unique element $T\varphi$ of smallest norm in $\mathfrak{C}_h$ where φ is given by* (16) *for some non-negative Radon measure μ on $\mathbb{R}^N$.*

Remark 5. We have not used any special properties of $\mathbb{R}^N$ or the Lebesgue measure. Therefore, in Theorems 1 and 2, $\mathbb{R}^N$ can be replaced by a locally compact second countable space provided with a σ-finite measure satisfying condition 2 of Definition 1.

In particular, if K is a compact subset of $\mathbb{R}^N$ which is not quasi-null, there is a measure $\eta \in \mathfrak{M}$ such that $\eta(K) > 0$. Replace $\mathbb{R}^N$ and the Lebesgue measure by K and the restriction of η to K to get the following, stronger version of Theorem 2:

THEOREM 3. *Let K be a non-quasi-null compact subset of $\mathbb{R}^N$, and let h be measurable and suppose the set*

$$C_h = \{u \in \mathbb{B} \mid u \geq h \quad \text{q.e. on } K\}$$

is not empty. Then, there is a measure μ on K such that the unique element of the smallest norm in C_h is given by $T\varphi$ where

$$\varphi(y, n) = \|\check{T}\mu(\cdot, n)\|_{p'}^{q'-p'} (\check{T}\mu(y, n))^{p'-1} . \tag{17}$$

Specialising to $h = 1_K$ in Theorem 3, we conclude (by condition 3 of Definition 1) that the set C_h is not empty.

COROLLARY 4. *To each compact set K there corresponds a measure μ on K such that $T\varphi$, where φ given by* (17) *satisfies*

$$T\varphi \geq 1, \quad \text{q.e. } on\ K$$

and $\|\varphi\|_{p,q}$ is minimum.

This unique element of $\mathbb{B}$ is called the *capacitary potential* of K and the *capacity* of K is defined to be

$$C(K) = \|\varphi\|_{p,q}^q . \tag{18}$$

Using (17) we get

$$C(K) = \|\varphi\|_{p,q}^q = \|\check{T}\mu\|_{p',q'}^{q'} = \int T\varphi(x) d\mu(x) . \tag{19}$$

For more information on capacities we refer to [1].

4. Applications to non-linear potential theory

In this section we shall define and study some properties of "non-linear potentials" in our setting. The results obtained will be analogues of those in classical potential theory. We point this out as we proceed.

As a first application we give the following extension (Theorem 5 below) of the classical equilibrium principle:

For each compact set K there is a non-negative measure μ on K such that $G\mu \geq 1$ on K and $G\mu = 1$, μ a.e. This has been extended to very general kernel G's not even symmetric [2]. By considering the kernel $\frac{G(x,y)}{u(x)u(y)}$ we easily see that for each compact K and each continuous $u > 0$ there is a measure μ such that

$$\int G(x, y)\mu(dy) \geq u(x) \text{ q.e. on } K \qquad \text{and equals } u,\ \mu \text{ a.e.}$$

Alas this trick does not seem to work in the non-linear case.

Consider $Rh = T\varphi$ and let $u = Tf \in C_h$ so that $u \geq h$ q.e. Then for all $0 < t < 1$,

$$T\varphi + t(u - T\varphi) = tu + (1-t)T\varphi \geq h.$$

i.e., $T\varphi + tT(f - \varphi) \geq h$. Then

$$\|T\varphi\|_{\mathbb{B}} \leq \|T(\varphi + t(f - \varphi))\|_{\mathbb{B}},$$

i.e.,

$$\|\varphi\|_{p,q}^q \leq \|\varphi + t(f - \varphi)\|_{p,q}^q, \quad 0 < t < 1.$$

Proceeding as in the proof of Theorem 2 but replacing f by $f - \varphi$ we get

$$q\Sigma\|\varphi(\cdot, n)\|_p^{q-p} \int \varphi(y, n)[f - \varphi](y, n)\nu(dy) \geq 0 \quad \text{if } Tf \in C_h$$

Recalling the definition of φ given in (16) and simplifying we get

$$\int u\,d\mu \geq \int V\mu\,d\mu \qquad \forall u \in C_h.$$

We have the following corollary.

Use the notation $V\mu$ for $T\varphi$, let $u \in \mathbb{B}$ and K compact. We know from Theorem 3 that there exists a measure μ on K such that $V\mu$ is the element of smallest norm in

$$C_u = \{v \in \mathbb{B}:\ v \geq u \text{ q.e. on } K\}.$$

From above we have

$$\int v\,d\mu \geq \int V\mu\,d\mu \qquad \forall v \in C_u.$$

In particular,

$$\int u\,d\mu \geq \int V\mu\,d\mu.$$

But since $V\mu$ is in C_u, it is $\geq u$ q.e. on K. This implies

$$u = V\mu \qquad \mu \text{ a.e.}$$

We thus have:

THEOREM 5. *For $u \in \mathbb{B}$ and K compact there exists μ on K such that $V\mu \geq u$ q.e. on K and $V\mu = u$ μ a.e.*

For the next application we need the following simple result.

PROPOSITION 6. *Let $f_n \to f$ a.e. and $\|f_n\|_p$ bounded. Then $f_n \to f$ weakly in L^p.*

Proof. By choosing a subsequence, assume $f_n \to v$ weakly. The indicator function of every set $\mathcal{O}$ of finite measure is in $L^{p'}$ and on such sets f_n is uniformly integrable because $\|f_n\|_p$ is bounded. So

$$\int_{\mathcal{O}} f_n \to \int_{\mathcal{O}} v \quad \text{by weak convergence in } L^p,$$

$$\int_{\mathcal{O}} f_n \to \int_{\mathcal{O}} f \quad \text{by uniform integrability}.$$

Therefore $v = f$. □

A classical result states that on bounded subsets of $H_0^1(\Omega)$, vague convergence of μ_k to μ is equivalent to weak convergence in $H_0^1(\Omega)$ of $G\mu_k$ to $G\mu$. An analogue of this result is given below.

THEOREM 7. *Let $\mu_k \to \mu$ vaguely. Suppose*

$$\check{T}\mu_k \text{ is bounded in } l^{q'}(L^{p'}).$$

Then (the potentials) $W\mu_k = T[(\check{T}\mu_k)^{p'-1}]$ tend to (the potential) $W\mu = T[(\check{T}\mu)^{p'-1}]$ q.e.

Proof. Since $T(\cdot, y, n)$ is continuous and has compact support and $\mu_k \to \mu$ vaguely,

$$\lim \check{T}\mu_k(y, n) = \check{T}\mu(y, n) \quad \forall y, n.$$

$\check{T}\mu_k$ is bounded in $l^{q'}(L^{p'})$ which in particular implies that for each n, $\check{T}\mu_k(\cdot, n)$ is bounded in $L^{p'}$. Hence $(\check{T}\mu_k(\cdot, n))^{p'-1}$ is bounded in L^p. And $(\hat{T}\mu_k(y, n))^{p'-1}$ tends to $(\check{T}\mu(y, n))^{p'-1}$.

Let now λ be a measure $\in \mathfrak{M}$ (i.e., $\check{T}\lambda \in l^{q'}(L^{p'})$). Then for each n, $\check{T}\lambda(\cdot, n) \in L^{p'}$. From Proposition 6 we see that

$$\int (\check{T}\mu_k(y, n))^{p'-1}\check{T}\lambda(y, n)d\nu(y) \to \int (\check{T}\mu(y, n))^{p'-1}\check{T}\lambda(y, n)d\nu(y),$$

i.e.,

$$\int W\mu_k(x)\lambda(dx) \to \int W\mu\lambda(dx), \ \lambda \in \mathfrak{M}.$$

Since $\liminf W\mu_k \geq W\mu$ the above immediately implies that

$$W\mu_k \to W\mu \quad \text{q.e.}$$

and for each $\lambda \in \mathfrak{M}$,

$$\int W\mu_k \, d\lambda \to \int W\mu \, d\lambda. \quad \square$$

Let us define the potential (non-linear) of a measure $\mu \in \mathfrak{M}$ by

$$V\mu = T\varphi_\mu \,, \quad \varphi_\mu(y,n) = \|\check{T}\mu(\cdot,n)\|_{p'}^{q'-p'} (\check{T}\mu(y,n))^{p'-1} \,. \tag{20}$$

With this definition we see that

$$\|\varphi_\mu(\cdot,n)\|_p^q = \|\check{T}\mu(\cdot,n)\|_{p'}^{q'} \,.$$

Thus $\mu \in \mathfrak{M} \Rightarrow V\mu \in \mathbb{B}$ and

$$\|V\mu\|_{\mathbb{B}}^q = \|\varphi_\mu\|_{p,q}^q = \|\check{T}\mu\|_{p',q'}^{q'} \,.$$

And

$$\int V\mu \, d\mu = \|\check{T}\mu\|_{p',q'}^{q'} \,.$$

In classical potential theory a well-known result of H. Cartan states that the set

$$\{G\mu \colon \int G\mu \, d\mu < \infty\}$$

is complete in $H_0^1(\Omega)$. We extend this result below to the non-linear setting.

THEOREM 8. *The space* $\mathbb{P}$ (*of non-linear potentials*)

$$\mathbb{P} = \{V\mu \colon \mu \in \mathfrak{M}\}$$

is complete in $\mathbb{B}$.

Proof. Let $V\mu_k$ be a Cauchy sequence in $\mathbb{B}$. From the above, $\|V\mu_k\|_{\mathbb{B}}^q = \|\check{T}\mu_k\|_{p',q'}^{q'}$. Thus $\check{T}\mu_k$ is bounded in $l^{q'}(L^{p'})$. Then as observed in the proof of Theorem 1, $\mu_k(F)$ is bounded for each compact F. So (by choosing a subsequence if necessary) we may assume that μ_k tends to some μ (necessarily in $\mathfrak{M}$). Since $T(\cdot, y, n)$ is continuous with compact support,

$$\check{T}\mu_k(y,n) \to \check{T}\mu(y,n) \text{ for each } y, n. \tag{21}$$

Also, $V\mu_k$ is a Cauchy sequence in $\mathbb{B}$ if and only if $\varphi_{\mu_k}(\cdot,\cdot)$ is a Cauchy sequence in $l^q(L^p)$. Recall the definition of $\varphi_\mu(\cdot,\cdot)$ from (20). In particular this implies that $\varphi_{\mu_k}(\cdot, n)$ is a Cauchy sequence in L^p for each n.

Let $V\mu_k \to u$ in $\mathbb{B}$. In particular this implies that $\varphi_{\mu_k}(\cdot, n) \to g(\cdot, n)$ in L^p for each n. Now

$$\|\varphi_{\mu_k}(\cdot, n)\|_p = \|f_k\|^{\frac{p}{p'}\frac{q'}{q}},$$

with $f_k = (\check{T}\mu_k(\cdot, n))^{p'-1}$.

Further, since $\|V\mu_k\|_{\mathbb{B}} = \|\varphi_{\mu_k}\|^q_{p,q} = \|\check{T}\mu_k\|^{q'}_{p',q'}$ we see that $\check{T}\mu_k$ is bounded in $l^{q'}(L^{p'})$. From (21),

$$\check{T}\mu_k(y, n) \to \check{T}\mu(y, n) \quad \forall y, n\,.$$

In other words,

$$f_k(y, n) \to f(y, n) = (\check{T}\mu(y, n))^{p'-1}\,.$$

Now $\varphi_{\mu_k}(\cdot, n) = C_{k,n} f_k(\cdot, n)$ where

$$C_{k,n} = \|f_k(\cdot, n)\|_p^{(q'-p')\frac{p}{p'}}\,.$$

We have two cases:

If $\varphi_{\mu_k}(\cdot, n) \to 0$ then $\|\varphi_{\mu_k}(\cdot, n)\|_p \to 0$ but then $\|f_k\|_p \to 0$, i.e., $\check{T}\mu(\cdot, n) = 0$. So $g(\cdot, n) = \check{T}\mu(\cdot, n)$.

Suppose $\lim \|\varphi_{\mu_k}(\cdot, n)\|_p > 0$. Then $\lim_k C_{k,n} = C_n > 0$ since $\|\varphi_{\mu_k}(\cdot, n)\|_p = \|f_k(\cdot, n)\|^{\frac{pq'}{p'q}}$. Hence since $\varphi_{\mu_k}(\cdot, n) = C_{k,n} f_k(\cdot, n)$, $f_k(\cdot, n)$ converges in L^p to $(\check{T}\mu(\cdot, n))^{p'-1}$.

This completes the proof. □

The classical second maximum principle states that $G\mu \geq G\nu$ on support ν implies inequality everywhere. We give an analogue of this in the non-linear setting.

PROPOSITION 9. *Let $V\mu = T\varphi_\mu$ be defined as in* (20). *Then we have*

$$V\mu \geq V\nu \text{ on support of } \nu \quad \Rightarrow \quad \|V\mu\|_{\mathbb{B}} \geq \|V\nu\|_{\mathbb{B}}$$

for any measures μ, ν.

Proof. Indeed,

$$\begin{aligned}
\|V\nu\|^q_{\mathbb{B}} &= \int V\nu d\nu \leq \int V\mu d\nu \\
&= \int (T\varphi_\mu) d\nu = \int \phi_\mu \check{T}\nu \\
&\leq \|\varphi_\mu\|_{p,q} \|\check{T}\nu\|_{p'q'} \\
&= \|\check{T}\mu\|^{\frac{q'}{q}}_{p',q'} \|\check{T}\nu\|_{p',q'} = \|V\mu\|_{\mathbb{B}} \|V\nu\|^{\frac{q}{q'}}_{\mathbb{B}}
\end{aligned}$$

which implies $\|V\nu\|_{\mathbb{B}} \leq \|V\mu\|_{\mathbb{B}}$ as required. □

This leads to a characterization of non-linear potentials:

THEOREM 10. *Let $u \in \mathbb{B}$. Then u is a non-linear potential if and only if*

$$v \in \mathbb{B}, \ v \geq u \Rightarrow \|v\|_{\mathbb{B}} \geq \|u\|_{\mathbb{B}}.$$

Proof. Let $u \in \mathbb{B}$ and

$$\mathcal{C}_u = \{v \in \mathbb{B}, \ v \geq u \text{ q.e.}\}.$$

The unique element of smallest norm in $\mathcal{C}_u$ is a non-linear potential $V\mu$. If u has the smallest norm in $\mathcal{C}_u$, u must equal $V\mu$.

Conversely let $u = V\mu$ and let $V\nu$ have smallest norm in $\mathcal{C}_u$. Then $\|V\nu\| \leq \|V\mu\|$. But $V\nu \geq V\mu$ so from Proposition 9, $\|V\nu\| \geq \|V\mu\|$. By uniqueness, $V\nu = V\mu$. □

Now more generally, let $V\mu$ be a non-linear potential. Then

$$u \geq V\mu \text{ on support } \mu \Rightarrow \|u\| \geq \|V\mu\|.$$

Indeed, let $C = \{v \in \mathbb{B}\colon v \geq V\mu \text{ on support } \mu\}$ and $V\nu$ be the element of smallest norm in C. Then, By Proposition 9,

$$V\nu \geq V\mu \text{ on support } \mu \Rightarrow \|V\nu\| \geq \|V\mu\|.$$

But $V\mu \in C \Rightarrow V\nu = V\mu$.

Before the capacity theory came about, sets of capacity zero were also known as Polar sets:

A set A is polar (classical potential theory) if A is a subset of the "Poles" of a superharmonic function

$$A \subset s^{-1}(\infty), \quad s \text{ superharmonic.}$$

We generalize this below.

THEOREM 11. *A compact set K has capacity zero or quasi-null if and only if it is polar, i.e., there is a measure ν such that $V\nu \equiv \infty$ on K.*

Further, for all μ on K, $V\mu = \infty \quad \mu$ a.e.

Proof. Only one direction needs to be proved because the set $(u = \infty)$ has capacity zero for each $u \in \mathbb{B}$. Let K have capacity zero. Let $u_n \in \mathbb{B}$, $u_n \geq 1$ on K and $\|u_n\|_{\mathbb{B}} \leq 2^{-n}$. Then $\sum u_n = u \in \mathbb{B}$ and $u = \infty$ on K and $\|u\|_{\mathbb{B}} \leq 1$.

Let μ be any measure on K. Then $\infty = \int u \, d\mu \leq \|u\|_{\mathbb{B}} \ \| \ \|\check{T}\mu\|_{p',q'}$ implies that for all measures μ on K, $\int V\mu \, d\mu = \infty$.

Let $A_\gamma = \{x \in K,\ V\mu \leq \gamma\}$. Then $\int V\mu_\gamma\, d\mu_\gamma = \infty$ (as above). But $\int V\mu_\gamma\, d\mu_\gamma \leq \int V\mu\ d\mu_\gamma \leq \gamma\mu_\gamma(1) < \infty$ where $\mu_\gamma = \mu|_{A_\gamma}$. This implies $\mu(A_\gamma) = 0$.

To show that K is indeed polar let $V\nu$ be the element of smallest norm in

$$\mathcal{C}_u = \{v\colon\ v \in \mathbb{B}, v \geq u\}.$$

Then $V\nu \geq u$ q.e. and $\|V\nu\| \leq \|u\|$. $\square$

5. Tangent cones in Besov spaces

Much of the considerations below are valid in the more general setup of the previous sections.

In this section we will denote by $\mathbb{B}$ the Besov space $B_\alpha^{p,q}(\mathbb{R}^N)$, where $\alpha > 0$, $1 < p < \infty$, $1 < q < \infty$. Our reference for Besov spaces is [1], Chapter 4. $\mathbb{B}$ is a Banach space. The following characterisation of $\mathbb{B}$ will be used (see Theorem 4.4.1, page 105 of [1]).

Fix a non-negative C^∞ function η on $\mathbb{R}^N$ with support in the unit ball $B(0,1)$. We will assume

$$\int \eta(x)dx = 1.$$

Let

$$\eta_n(x) = 2^{nN}\eta(2^n x)$$

for $n \geq 0$ so that $\eta_0 \equiv \eta$.

Then a function $u \in \mathbb{B}$ iff there is a function sequence

$$f = \{f_n\} \in l^q(L^p(\mathbb{R}^N))$$

such that

$$u = \Sigma_0^\infty 2^{-n\alpha}\eta_n * f_n. \tag{22}$$

We may take the $\mathbb{B}$-norm of u to be $\|f\|_{p,q}$. All choices of η give equivalent norms.

Remark 6. u defined by (22) is in L^p. To see this let $g \in L^{p'}$. Denoting the $(L^p, L^{p'})$ pairing by $\langle\cdot,\cdot\rangle$, we have

$$\begin{aligned}\langle|u|,|g|\rangle &\leq \Sigma_0^\infty 2^{-n\alpha}\langle\eta_n * |f_n|, |g|\rangle \leq \Sigma_0^\infty 2^{-n\alpha}\|f_n\|_p\|\check{\eta}_n * |g|\|_{p'}\\ &\leq \left(\Sigma_0^\infty\|f_n\|_p^q\right)^{\frac{1}{q}}\left(\Sigma_0^\infty 2^{-n\alpha q'}\|g\|_{p'}^{q'}\right)^{\frac{1}{q'}} < \infty.\end{aligned}$$

We have used the fact that

$$\|\check{\eta}_n * |g|\|_{p'} \leq \|g\|_{p'}$$

because $\int \check{\eta}_n(x)dx = 1$, here $\check{\eta}_n(x) = \eta_n(-x)$.

For $(x, y, n) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{N}$, let

$$T(x, y, n) = 2^{-n\alpha}\eta_n(x - y), \quad n \geq 0,$$
$$\nu(dy) = dy \quad \text{Lebesgue measure}.$$

We can verify that conditions 1, 2, 3 of Definition 1 are satisfied.

Condition 1 is immediately seen to be satisfied.

If $f \in L^p(\mathbb{R}^N)$,

$$\left(\left|\int T(x, y, n) f(x) dx\right|\right)^p \leq 2^{-n\alpha p} \int \eta_n(x - y)|f(x)|^p dx$$

so that

$$\int dy \left(\left|\int T(x, y, n) f(x) dx\right|^p\right) \leq 2^{-n\alpha p}\|f\|_p^p$$

because $\int \eta_n(x - y) dy = 1$ for every n. Condition 2 follows.

Similarly, condition 3 is easy to verify.

Thus the Besov space $B_\alpha^{p,q}$ fall under our setup. The notation such as q.e. will be as before. For ease of reference let us emphasize that elements of $\mathbb{B}$ are of the form Tf for f in $l^q(L^p(\mathbb{R}^N))$ with norm $\|f\|_{p,q}$.

5.1. *Tangent cone.*

Let $\psi \in \mathbb{B}$ and $\mathfrak{K}$ denote the closed convex set

$$\mathfrak{K} = \{f \in \mathbb{B} \mid f(x) \geq \psi(x) \quad \text{q.e.}\}. \tag{23}$$

Given $z \in \mathfrak{K}$, the *tangent cone* $\mathfrak{T}_{\mathfrak{K}}(z)$ is the closure of the set

$$\mathfrak{C}_{\mathfrak{K}}(z) = \{\varphi \in \mathbb{B} \mid \exists t > 0 \text{ such that } z + t\varphi \in \mathfrak{K}\}. \tag{24}$$

Both $\mathfrak{C}_{\mathfrak{K}}(z)$ and $\mathfrak{T}_{\mathfrak{K}}(z)$ are convex cones and contain all non-negative elements of $\mathbb{B}$. Put

$$\Xi = \{x \mid z(x) = \psi(x)\}. \tag{25}$$

Clearly every $v \in \mathfrak{T}_{\mathfrak{K}}(z)$ is non-negative q.e. on Ξ.

Since $\mathfrak{T}_{\mathfrak{K}}(z)$ is a closed convex cone, for each $V \in \mathbb{B}$, $V - \mathfrak{T}_{\mathfrak{K}}(z)$ is a closed convex set and contains a unique element of smallest norm. This element u_0 is the "projection" of V on the tangent cone:

$$\|V - u_0\| \leq \|V - u\|, \quad u \in \mathfrak{T}_{\mathfrak{K}}(z). \tag{26}$$

As observed before, each non-negative element of $\mathbb{B}$ belongs to $\mathfrak{T}_{\mathfrak{K}}(z)$. Suppose that $Tf \geq 0$ q.e. Since $\mathfrak{T}_{\mathfrak{K}}(z)$ is a cone, $u_0 + tTf \in \mathfrak{T}_{\mathfrak{K}}(z)$ for all $t > 0$: From (26),

$$\|V - u_0\| \leq \|V - u_0 - tTf\|, \quad t > 0, \quad Tf \geq 0 \quad \text{q.e.}$$

Arguing as in the last section we have:

THEOREM 12. *There is a Radon measure μ_0 such that*

$$V - u_0 = -T\varphi_0 , \tag{27}$$

where

$$\varphi_0 = \varphi_0(y, n) = \|\check{T}\mu_0(\cdot, n)\|_{p'}^{q'-p'} \left(\check{T}\mu_0(y, n)\right)^{p'-1} . \tag{28}$$

More can be said about the measure μ_0:

THEOREM 13. *Let μ_0 be as in Theorem 12. Then:*

(1) $\int u\, d\mu_0 \geq 0 , \quad \forall u \in \mathfrak{T}_{\mathfrak{K}}(z)$.
(2) *μ_0 is concentrated on Ξ.*
(3) $\int u_0\, d\mu_0 = 0$, *i.e., μ_0 is concentrated on $\{u_0 = 0\}$.*
(4) *If $\mu_0 \neq 0$,*

$$\int V\, d\mu_0 = -\int T\varphi_0(x)\mu_0(dx) = -\|\check{T}\mu_0\|_{p',q'}^{q'} < 0 . \tag{29}$$

Proof. (1)Note that $\mathfrak{T}_{\mathfrak{K}}(z)$ is a cone, so $u_0 + tu \in \mathfrak{T}_{\mathfrak{K}}(z)$ for each $t > 0$ and $u \in \mathfrak{T}_{\mathfrak{K}}(z)$. Hence

$$\|V - u_0\| \leq \|V - u_0 - tu\| .$$

Write $V - u_0 = -T\varphi$ and follow the proof leading to inequality (13) of the last section to get (1).

(2) It is known that $\mathcal{D} = \mathcal{D}(\mathbb{R}^N)$ is a multiplier for $\mathbb{B}$: $u \in \mathbb{B}, \varphi \in \mathcal{D}$ implies $\varphi u \in \mathbb{B}$. See [7], page 140.

Let $\varphi \in \mathcal{D}$; then $\varphi(z - \psi) \in \mathbb{B}$. Hence if $t = \|\varphi\|_\infty^{-1}$,

$$z - \psi + t\varphi(z - \psi) = (1 + t\varphi)(z - \psi) \geq 0 .$$

It follows that

$$\varphi(z - \psi) \in \mathfrak{T}_{\mathfrak{K}}(z), \quad \varphi \in \mathcal{D} .$$

From property 1, already established,

$$\int \varphi(z - \psi) d\mu_0 \geq 0, \quad \varphi \in \mathcal{D} .$$

This can only happen if $\int \varphi(z - \psi) d\mu_0 = 0$ i.e., μ_0 is concentrated on Ξ.

(3) $u_0 \in \mathfrak{T}_{\mathfrak{K}}(z)$, hence $tu_0 \in \mathfrak{T}_{\mathfrak{K}}(z)$ for all $t > 0$. Therefore,

$$\|V - u_0\| \leq \|V - u_0 + tu_0\| \quad \text{if } t \leq 1 .$$

Writing $V - u_0 = -T\varphi_0$ and following the proof leading to the inequality (13) of the last section (as in (1) above) we get

$$\int u_0 \, d\mu_0 \le 0 .$$

But since u_0 is in $\mathfrak{T}_{\mathfrak{K}}(z)$, (1) and the last inequality give (3). Since $u_0 \ge 0$ q.e. on Ξ (all elements of $\mathfrak{T}_{\mathfrak{K}}(z)$ satisfy this) we see that μ_0 is concentrated on the set $\{u_0 = 0\}$.

(4) Integrate both sides of (27) relative to μ_0 and use (3) to get (29). □

The following corollary characterizes the tangent cone $\mathfrak{T}_{\mathfrak{K}}(z)$:

COROLLARY 14. *$V \in \mathfrak{T}_{\mathfrak{K}}(z)$ if and only if*

$$V \ge 0 \quad \text{q.e. on} \quad \Xi .$$

Proof. Immediate from (29). □

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Murali Rao, Department of Mathematics, University of Florida, Gainesville, FL 32611
rao@math.ufl.edu

Jan Sokołowski, Institut Elie Cartan, Laboratoire de Mathématiques, Université Henri Poincaré Nancy I, B.P. 239, 54506 Vandoeuvre lès Nancy Cedex, France

Systems Research Institute of the Polish Academy of Sciences, ul. Newelska 6, 01-447 Warszawa, Poland
sokolows@iecn.u-nancy.fr