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A WITTEN STYLE PROOF OF MORSE INEQUALITIES FOR ORBIT SPACES

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ABSTRACT. Let G be a compact Lie group acting on a closed manifold M. Witten's method is used to prove Morse inequalities for the cohomology of the orbit space M/G.

1. Introduction

Let M be a closed manifold with a smooth action of a compact Lie group G. Let $C^\infty(M/G)$ denote the space of G-invariant smooth functions on M. Such functions can be considered as "smooth" functions on the orbit space M/G, defining a so called "differentiable structure" on M/G. If $f \in C^\infty(M/G)$, a G-orbit F is said to be a *critical orbit* of f if one of its points (and thus all of them) is a critical point of f. A critical orbit F is said to be *nondegenerate* if, for any smooth transversal Σ of F, the points in $F \cap \Sigma$ are nondegenerate critical points of $f|_\Sigma$; if this holds for some transversal of F, then it holds for all of them.

A function $f \in C^\infty(M/G)$ is called a *G-Morse function* if its critical orbits are nondegenerate. For such a function f, let $\mathrm{Crit}_G(f)$ be the set of its critical orbits, which is a finite set. The existence of nondegenerate G-Morse functions was proved by Wasserman [12]; indeed Wasserman has shown the density of the space of such functions in $C^\infty(M/G)$ with respect to the C^∞ topology.

If F is a nondegenerate critical orbit of a function $f \in C^\infty(M/G)$, then the Hessian of f defines a nondegenerate quadratic form $H_F f$ on the normal bundle N_F of F. So $H_F f$ yields a decomposition of N_F as direct sum of the subbundles $N_{F,+}$ and $N_{F,-}$, where $H_F f$ is respectively positive and negative definite. The *index* of F with respect to f is the rank m_F of $N_{F,-}$. All of these vector bundles are G-vector bundles in a canonical way.

MAIN THEOREM. *Let G be a compact Lie group acting on a closed manifold M of dimension n, and G_0 its connected component containig the identity. Let $H^*(M/G)$ denote the real cohomology of the orbit space M/G, and $\beta_j = \dim H^j(M/G)$ (the corresponding Betti numbers). For any G-Morse function f on M let*

$$\mu_j = \#\left\{F \in \mathrm{Crit}_G(f) \mid m_F = j \textit{ and } N_{F,-} \textit{ is } G\textit{-orientable and } G_0\textit{-trivial}\right\}.$$

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Then we have the inequalities

$$\begin{aligned}\beta_0 &\leq \mu_0,\\ \beta_1 - \beta_0 &\leq \mu_1 - \mu_0,\\ \beta_2 - \beta_1 + \beta_0 &\leq \mu_2 - \mu_1 + \mu_0,\end{aligned}$$

etc., and the equality

$$\sum_{j=0}^{n}(-1)^j \beta_j = \sum_{j=0}^{n}(-1)^j \mu_j.$$

The proof of this theorem is an adaptation of the method of Witten [13], especially as it is shown by Roe in [9] (see also [1]). We fix a G-invariant Riemannian metric on M, and the G-Morse function is used to modify the Laplacian on "basic forms" so that its Schwarz kernel concentrates around the critical orbits, whose cohomological contribution is thus obtained by a local study using the Koszul Slice Theorem.

In the Main Theorem, the function $\bar{f}\colon M/G \to \mathbb{R}$ defined by f can be considered as a Morse function on the orbit space M/G, so that this result establishes some kind of Morse inequalities for M/G. Nevertheless the index of $\bar{f}$ at the critical orbits, in the classical sense[1], may not be well defined—this was solved by using intersection cohomology instead of real cohomology in the more general setting of singular stratified spaces [5]. So the Main Theorem shows that, at least on some singular stratified spaces, certain Morse type inequalities hold for the real cohomology independently of having no well defined index at some critical points.

The above remarks can be shown in the following simple example. Consider the group $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ acting on $\mathbf{S}^1 \subset \mathbb{R}^2$ by symmetry with respect to the vertical axis. Since the orbits are discrete, the $\mathbb{Z}_2$-Morse functions on $\mathbf{S}^1$ are just the usual Morse functions that are $\mathbb{Z}_2$-invariant; for instance, the "high" function $f(x, y) = y$ is one of them, which moreover satisfies that $\bar{f}$ is a homeomorphism of $\mathbf{S}^1/\mathbb{Z}_2$ onto $[-1, 1]$, and a diffeomorphism of $(\mathbf{S}^1/\mathbb{Z}_2) \setminus \{F_1, F_2\}$ onto $(-1, 1)$, where F_1, F_2 are the south and north pole orbits, which are singletons. Thus F_1, F_2 are the only critical points of $\bar{f}$ in the suitable sense, where it respectively reaches the maximum and minimum. Now the index of $\bar{f}$ at F_1 is clearly zero because $\bar{f}(F_1) = -1$ and $\bar{f}$ itself induces $(\bar{f}^{-1+\epsilon}, \bar{f}^{-1-\epsilon}) \equiv ([-1, -1+\epsilon], \emptyset)$ for $0 < \epsilon < 2$. However $\bar{f}$ has no well defined index at F_2 because $\bar{f}(F_2) = 1$ and $(\bar{f}^{1+\epsilon}, \bar{f}^{1-\epsilon}) \equiv ([-1, 1], [-1, 1-\epsilon])$ for $0 < \epsilon < 2$. On the other hand we clearly have $m_{F_1} = 0$ and $m_{F_2} = 1$ because f reaches the minimum and maximum at these orbits of codimension one. But since the $\mathbb{Z}_2$-action does not preserve the orientation of the tangent space of $\mathbf{S}^1$ at the north pole, which is equal to $N_{F_2,-}$, we get $\mu_0 = 1$ and $\mu_1 = 0$ according to the definition in the Main Theorem. So our Morse inequalities are equalities in this case.

[1] Recall that, for a Morse function f separating critical points, which is defined in same reasonable way on some space, and with the usual notation $f^a = f^{-1}(-\infty, a]$ for any $a \in \mathbb{R}$, one can classically define the *index* of f at a critical point x to be equal to i_x when $H^i(f^{a+\epsilon}, f^{a-\epsilon}) \neq 0$ for small enough $\epsilon > 0$ if and only if $i = i_x$, where $f(x) = a$ and we consider relative real cohomology.

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2. The basic complex

Let $(\Omega(M), d)$ denote the de Rham complex of M. The *basic complex* of the G-action on M is the subcomplex $\Omega(M/G) \subset \Omega(M)$ of G-invariant forms $\alpha \in \Omega(M)$ such that $\iota_X\alpha = 0$ if $X \in \mathfrak{X}(M)$ is tangent to the orbits of G. It was proved by A. Verona [11] that the cohomology of the basic complex is isomorphic to the real cohomology of M/G. Thus we shall write $H^\star(M/G) = H^\star(\Omega(M/G), d_b)$, where d_b is the restriction of d. The Hilbert space of L^2 differential forms on M will be denoted by $L^2\Omega(M)$, and let $L^2\Omega(M/G)$ be the closure of $\Omega(M/G)$ in $L^2\Omega(M)$.

2.1. *The orthogonal projection onto the basic complex.* If δ denotes the coderivative induced by a fixed G-invariant Riemannian metric on M, in general $\delta(\Omega(M/G)) \not\subset \Omega(M/G)$. Thus d_b has an adjoint δ_b on $\Omega(M/G)$ if the orthogonal projection $\Pi\colon L^2\Omega(M) \to L^2\Omega(M/G)$ preserves smoothness of differential forms; we would have $\delta_b = \Pi \circ \delta$ in this case. So we need the following.

PROPOSITION 2.1. *We have* $\Pi(\Omega(M)) = \Omega(M/G)$.

This section will be devoted to proving Proposition 2.1. Observe that this result is not obvious because of the possible existence of singular strata. Two orbits have the same *normal orbit type* if they have equivalent normal slice representations [3]. The unions of the orbits with the same normal orbit type are the different strata, which are submanifolds of M. The *blowing-up* construction yields a G-manifold $\widetilde{M}$ with no singular orbits and an equivariant projection $\tau\colon \widetilde{M} \to M$. We describe the construction of $\widetilde{M}$ and τ to prove Proposition 2.1 with the following idea. Since $\widetilde{M}$ has no singular orbits, we have an orthogonal projection on C^∞ differential forms, $\widetilde{\Pi}\colon \Omega(\widetilde{M}) \to \Omega(\widetilde{M}/G)$. Then we use τ and $\widetilde{\Pi}$ to define a projection $\Omega(M) \to \Omega(M/G)$, which turns out to be orthogonal by construction, and thus equal to the restriction of Π.

2.1.1. *Blowing-up singular strata.* Let X be a stratum of M with $\dim X = s$, $N(X)$ its normal bundle, and $P\colon P(X) \to X$ the associated projective bundle. Let $\widetilde{M}_X = (M \setminus X) \cup P(X)$ and define $\tau_X\colon \widetilde{M}_X \to M$ in the following way:

$$\tau_X(v) = \begin{cases} v & \text{if } v \in M \setminus X, \\ P(v) & \text{if } v \in P(X). \end{cases}$$

Such a set $\widetilde{M}_X$ is the blowing-up of M along X, and τ_X is the blowing-up mapping. A differential structure of $\widetilde{M}_X$ is defined as follows by using local coordinates $(x^1, \dots, x^s, x^{s+1}, \dots, x^n)$ adapted to X on an open subset U of M. On U, the stratum

X is given by $x^{s+1} = \cdots = x^n = 0$. Let $(x^1, \ldots, x^s, X^{s+1}, \ldots, X^n)$ be the induced coordinates on $N(X)$, where $X^{s+1}, \ldots, X^n$ are the coordinates induced by the frame of $N(X)$ on $X \cap U$ defined by $\partial/\partial x^{s+1}, \ldots, \partial/\partial x^n$. For any $i \in \{s+1, \ldots, n\}$, let $U_i = \{x \in U \mid x^i \neq 0\}$ and P_i the part of $P(X)$ defined by the vectors of $N(X)$ on $X \cap U$ with $X^i \neq 0$. We define coordinates on $U_i \cup P_i$ by

$$\varphi_i = (y^1, \ldots, y^n) = \begin{cases} (x^1, \ldots, x^s, x^{s+1}/x^i, \ldots, x^i, \ldots, x^n/x^i) & \text{on } U_i, \\ (x^1, \ldots, x^s, X^{s+1}/X^i, \ldots, 0, \ldots, X^n/X^i) & \text{on } P_i. \end{cases}$$

In this way, a smooth structure on $\widetilde{M}_X$ is defined by an atlas of $M \setminus X$ and a covering of X by open sets U_i with coordinates as above.

This way of describing the blowing-up of a singular stratum is taken from [8]. A different description is given in [3] and [7]; it consists of attaching a boundary to the complement of the stratum by passing to "fiberwise polar coordinates" on a tubular neighborhood of the stratum.

2.1.2. *Lifting vector fields tangent to singular strata.* Let $Z \in \mathfrak{X}(M)$ be tangent to the stratum X. With respect to a chart $(x^1, \ldots, x^n)$ on U as above, we have

$$Z = \sum_{i=1}^{n} Z^i \partial/\partial x^i \quad \text{on } U.$$

Let

$$\widetilde{Z}_X = \sum_{j=1}^{n} \widetilde{Z}_i^j \partial/\partial y_i^j \text{ on } \quad U_i \cup P_i,$$

where

$$\widetilde{Z}_i^j(y_i^1, \ldots, y_i^n) = Z^j(y_i^1, \ldots, y_i^s, y_i^{s+1} y_i^i, \ldots, y_i^i, \ldots, y_i^n y_i^i)$$

if $j = 1, \ldots, s$ or $i = j$, and

$$\begin{aligned} \widetilde{Z}_i^j(y_i^1, \ldots, y_i^n) &= \frac{1}{y_i^i} Z^j(y_i^1, \ldots, y_i^s, y_i^{s+1} y_i^i, \ldots, y_i^i, \ldots, y_i^n y_i^i) \\ &\quad - \frac{y_i^j}{y_i^i} Z^i(y_i^1, \ldots, y_i^s, y_i^{s+1} y_i^i, \ldots, y_i^i, \ldots, y_i^n y_i^i) \end{aligned}$$

otherwise. Such a $\widetilde{Z}_X$ is a C^∞ vector field on $U_i \cup P_i$. In this way we construct a C^∞ vector field on $\widetilde{M}_X$, tangent to $P(X)$, which is projectable by τ_X to Z. This $\widetilde{Z}_X$ is called the blowing-up of Z along X. This construction allows lifting of G-invariant vector fields on M to G-invariant vector fields on $\widetilde{M}_X$, because the G-invariant vector fields on M are tangent to all the singular strata—this follows from the triviality of G-invariant sections of the normal bundle of any stratum. This lifting will be denoted by i: $\mathfrak{X}(M)_G \to \mathfrak{X}(\widetilde{M})_G$; thus $\tau_{X*} \circ i = \mathrm{id}$.

Observe that, if $\tau_X(\tilde{x}) = x$, the value of $\widetilde{Z}_X$ at $\tilde{x}$ is not determined by the value of Z on x. So this procedure does not define a lifting to $\widetilde{M}_X$ of vectors tangent to X.

2.1.3. *Lifting actions.* On $\widetilde{M}_X$ we have an action of G induced by the action of G on $M \setminus X$ and the action of G on $P(X)$ — the last one is defined by the action of G on the normal bundle $N(X)$. It is easy to see that this action on $\widetilde{M}_X$ is differentiable by using the above local coordinates. With respect to this action, the projection τ_X is a G-map, yielding

$$\tau_X^\star C^\infty(M/G) \subset C^\infty\left(\widetilde{M}_X/G\right).$$

Indeed we have the following.

LEMMA 2.1. *For any $\alpha \in L^2\Omega(M)_G$ we have $\alpha \in \Omega(M)_G$ if and only if $\tau_X^\star\alpha \in \Omega(\widetilde{M}_X)_G$.*

Proof. If $\alpha \in \Omega(M)_G$, then $\tau_X^\star\alpha \in \Omega(\widetilde{M}_X)_G$ trivially. Now assume that $\alpha \in L^2\Omega^s(M)_G$ and $X_1, \dots, X_s \in \mathfrak{X}(M)_G$.

Because $\tau_{X\star} \circ i = \mathrm{id}$, we write

$$\alpha(X_1, \dots, X_s) = \alpha(\tau_{X\star}i(X_1), \dots, \tau_{X\star}i(X_s)) = (\tau_X^\star\alpha)(i(X_1), \dots, i(X_s));$$

it is differentiable because $\tau_X^\star\alpha \in \Omega^s\left(\widetilde{M}_X\right)_G$. □

2.1.4. *Vector fields on the orbit space.* Now we show that $\Omega(M/G)$ can also be described by using "vector fields" on M/G. We take some definitions from [10]. Let $\mathfrak{X}(M)_G$ be the space of G-invariant vector fields on M, let $\mathfrak{X}_G(M)_G$ be the subspace of elements of $\mathfrak{X}(M)_G$ tangent to the orbits of G, and let $\mathfrak{X}(M/G)$ be the space of derivations in $C^\infty(M/G)$ that preserve the ideals in $C^\infty(M/G)$ defined by fixing any stratum of M and taking the functions that vanish the stratum selected. It was proved by G. W. Schwarz [10] that the sequence

$$0 \longrightarrow \mathfrak{X}_G(M)_G \longrightarrow \mathfrak{X}(M)_G \longrightarrow \mathfrak{X}(M/G) \longrightarrow 0.$$

is exact. Thus $\mathfrak{X}(M)_G/\mathfrak{X}_G(M)_G \cong \mathfrak{X}(M/G)$. Now, if $\alpha \in \Omega^r(M/G)$, the G-invariance of α implies that α can be determined by its value on G-invariant vector fields; i. e., α can be described as an antisymmetric r-linear map

$$\alpha\colon \mathfrak{X}(M)_G \times \overset{(r)}{\cdots} \times \mathfrak{X}(M)_G \longrightarrow C^\infty(M/G)$$

of $C^\infty(M/G)$-modules. Moreover, since $\iota_X\alpha = 0$ for all $X \in \mathfrak{X}_G(M)_G$, the basic form α can be described as an antisymmetric r-linear map

$$\alpha\colon \mathfrak{X}(M/G) \times \overset{(r)}{\cdots} \times \mathfrak{X}(M/G) \longrightarrow C^\infty(M/G)$$

of $C^\infty(M/G)$-modules. We point out for further use that G. W. Schwarz has proved that $\mathfrak{X}(M/G)$ is finitely generated over $C^\infty(M/G)$ [10], so the above interpretation of basic forms implies that $\Omega^r(M/G)$ is finitely generated as well. Because of this relation between $\mathfrak{X}(M/G)$ and $\Omega(M/G)$, the homomorphism $i\colon \mathfrak{X}(M)_G \to \mathfrak{X}(\widetilde{M}_X)_G$ canonically induces $i^\star\colon \Omega(\widetilde{M}_X/G) \to \Omega(M/G)$ such that $i^\star \circ \tau_X^\star = \mathrm{id}$ on $\Omega(M/G)$.

2.1.5. *Construction of a metric on the blowing-up.* Consider a singular stratum X of the G-action on M and the blowing-up map $\tau_X\colon \widetilde{M}_X \to M$. Let g be a Riemannian metric on M. We are going to modify the metric g on $M \setminus X$ inside a tubular neighborhood of X in order to get a metric that may be smoothly extended on P_X.

For a small $\rho_0 > 0$, consider the tubular neighborhood V of X of radius ρ_0. On V, the distance to X defines a function ρ that is constant on the orbits of the action. Let $\varphi\colon (0, \rho_0) \to \mathbb{R}^+$ be a C^∞ function such that $\varphi(\rho) = 1$ for ρ close to ρ_0 and $\varphi(\rho) = 1/\rho$ for ρ close to 0.

We modify the metric g on $V \setminus X$ leaving it invariant along the geodesics orthogonal to X, and multiplying it by $\varphi(\rho)$ tangentially to the tube of radius ρ around X. The new metric on $M \setminus X$ extends in a differential way to a metric $\widetilde{g}_X$ on $\widetilde{M}_X$. This construction is an adaptation of a similar one in [8]. This metric has the following property.

LEMMA 2.2. *For all $x \in M$ such that $\tau_X(\tilde{x}) = x$,*

$$\tau_{X\star}\left(T_{\tilde{x}}\left(\tau_X^{-1}(Gx)\right)^{\perp}\right) \subset T_x(Gx)^{\perp}. \tag{1}$$

Proof. On $M \setminus V$, $\widetilde{g}_X \equiv g$ and τ_X is the identity, so (1) is trivially satisfied. Now, suppose that $x \in V \setminus X$; again, τ_X is the identity, so $\tau_X^{-1}(Gx) \equiv G\tilde{x}$ which is inside a tube T_ρ, of radius ρ, where τ_X is conformal. So (1) follows.

Finally, suppose that x is in the singular stratum X. We use local coordinates to check that (1) is satisfied. Let $(x^1, \ldots, x^n)$ be local coordinates on an open subset $U \subset M$ containing x, and let $(y^1, \ldots, y^n)$ be local coordinates defined on $U_i \cap P_i \subset \widetilde{M}_X$ for $i \in \{s+1, \ldots, n\}$, as it is shown in Section 2.1.1. With respect to these coordinates, τ_X has the local expression

$$(y^1, \ldots, y^s, y^{s+1}, \ldots, y^n) \longmapsto (y^1, \ldots, y^s, y^i y^{s+1}, \ldots, y^i, \ldots, y^i y^n),$$

and thus

$$\tau_{X\star} \equiv \begin{pmatrix} \mathrm{id}_{s\times s} & 0 \\ 0 & A \end{pmatrix} \tag{2}$$

at $\tilde{x}$, where A is the $(n-s) \times (n-s)$ matrix with all elements zero except its ith column which is given by

$$(y^{s+1}, \ldots, y^{i-1}, 1, y^{i+1}, \ldots, y^n).$$

Now, by definition of P_X with respect to the chart φ_i, we have $T_{\tilde{x}}(P_X)^{\perp} = \langle \partial/\partial y^i \rangle$ and by (2) we have

$$\tau_{X\star}\left(T_{\tilde{x}}(P_X)^{\perp}\right) = \tau_{X\star}\left(\langle \partial/\partial y^i \rangle\right)$$

$$
\begin{aligned}
&= \left\langle \partial/\partial x^i + \sum_{s+j\neq i} y^{s+j}\partial/\partial x^{s+j} \right\rangle \\
&\subset \langle \partial/\partial x^{s+1}, \dots, \partial/\partial x^n \rangle \\
&= T_x(X)^\perp.
\end{aligned}
$$

Again, by (2) we also have $\tau_{X\star} = 0$ on

$$\langle \partial/\partial y^{s+1}, \dots, \partial/\partial y^{i-1}, \partial/\partial y^{i+1}, \dots, \partial/\partial y^n \rangle,$$

and $\tau_{X\star} = \mathrm{id}$ on

$$\langle \partial/\partial y^1, \dots, \partial/\partial y^s \rangle.$$

It follows that

$$\tau_{X\star}\left(T_{\tilde{x}}(\tau_X^{-1}(Gx))^\perp \cap T_{\tilde{x}}(P_X)\right) \subset \left(T_x(Gx)^\perp \cap T_x(X)\right),$$

and (1) is also satisfied in this case. □

LEMMA 2.3. *For all $x \in M$ and $\tilde{x} \in \widetilde{M}$ with $\tau_X(\tilde{x}) = x$, the tangent bundle of the orbit $G\tilde{x}$ contains the horizontal subbundle of the fiber bundle $\tau_X\colon \tau_X^{-1}(Gx) \to Gx$.*

Proof. We denote by $H, V \subset T\left(\tau_X^{-1}(Gx)\right)$ the horizontal and vertical subbundles of the fiber bundle $\tau_X\colon \tau_X^{-1}(Gx) \to Gx$.

If $x \in M \setminus X$, then $\tau_X \equiv \mathrm{id}$ and $V_{\tilde{x}} = \{0\}$, and the result follows trivially in this case.

If $x \in X$ we consider the local chart φ_i, and we get the matrix representation of $\tau_{X\star}$ given in (2). So $V_{\tilde{x}}$ is the space generated by

$$\partial/\partial y^{s+1}, \dots, \partial/\partial y^{i-1}, \partial/\partial y^{i+1}, \dots, \partial/\partial y^n$$

at $\tilde{x}$, and $H_{\tilde{x}}$ must be inside the subspace generated by $\partial/\partial y^1, \dots, \partial/\partial y^s$ at $\tilde{x}$ because $\partial/\partial y^i$ is orthogonal to $\tau_X^{-1}(Gx)$ by the definition of the metric. Moreover, by construction of blowing-up, we have

$$T\left(\tau_X^{-1}(Gx)\right) \cap \langle \partial/\partial y^1, \dots, \partial/\partial y^s \rangle \subset T(G\tilde{x})$$

at $\tilde{x}$. Hence $H_{\tilde{x}} \subset T_{\tilde{x}}(G\tilde{x})$. □

2.1.6. *The global blowing-up.* We repeat the above blowing-up process until we get a G-manifold with regular orbits, obtaining a tower of blowing-up projections whose composition is the projection $\tau\colon \widetilde{M} \to M$. There is also a lifting of the G-action to $\widetilde{M}$, and an extension of G-invariant vector fields, $i\colon \mathfrak{X}(M)_G \to \mathfrak{X}(\widetilde{M})_G$, that induces a homomorphism $i^\star\colon \Omega(\widetilde{M}/G) \to \Omega(M/G)$ such that $i^\star \circ \tau^\star = \mathrm{id}$ on $\Omega(M/G)$.

From Lemmas 2.1, 2.2 and 2.3 we get the following.

COROLLARY 2.1. *For any* $\alpha \in L^2\Omega(M)_G$ *we have* $\alpha \in \Omega(M)_G$ *if and only if* $\tau^\star\alpha \in \Omega(\widetilde{M})_G$.

COROLLARY 2.2. *There exists a* G*-invariant Riemannian metric on* $\widetilde{M}$ *such that for all* $x \in M$, *if* $\tau(\tilde{x}) = x$ *then*

$$\tau_\star\left(T_{\tilde{x}}\left(\tau^{-1}(Gx)\right)^\perp\right) \subset T_x(Gx)^\perp,$$

and $G\tilde{x}$ *contains the horizontal subbundle of the fiber bundle* $\tau\colon \tau^{-1}(Gx) \to Gx$.

2.1.7. *Smoothness is preserved by* Π. The regularity of the G-orbits on $\widetilde{M}$ means that their connected components are the leaves of a G-invariant regular foliation $\mathcal{F}$ on $\widetilde{M}$. The leaves of $\mathcal{F}$ are mapped by τ to the connected components of the G-orbits of M (which are the leaves of a singular foliation). Since $\mathcal{F}$ is a regular G-invariant foliation, we have the orthogonal decomposition

$$T\widetilde{M} = T\mathcal{F}^\perp \oplus T\mathcal{F} \tag{3}$$

of G-vector bundles, yielding

$$\bigwedge T\widetilde{M}^\star = \bigwedge T\mathcal{F}^{\perp\star} \otimes \bigwedge T\mathcal{F}^\star$$

as G-vector bundles. Observe that the space of G-invariant sections of $\bigwedge T\mathcal{F}^{\perp\star}$ is $\Omega\left(\widetilde{M}/G\right)$. Thus we get an orthogonal decomposition

$$\Omega\left(\widetilde{M}\right)_G = \Omega\left(\widetilde{M}/G\right) \oplus \left(\Omega\left(\widetilde{M}/G\right)^\perp \cap \Omega\left(\widetilde{M}\right)_G\right),$$

defining, pointwise, an orthogonal projection

$$\widetilde{\Pi}\colon \Omega\left(\widetilde{M}\right)_G \longrightarrow \Omega\left(\widetilde{M}/G\right),$$

depending only on the decomposition (3). Then define $\overline{\Pi} = i^\star\,\widetilde{\Pi}\,\tau^\star\colon \Omega(M)_G \to \Omega(M/G)$, where we consider $\tau^\star\colon \Omega(M)_G \to \Omega(\widetilde{M})_G$.

Claim. If $\alpha \in \Omega(M)_G$ and $x \in M$, then $(\overline{\Pi}\alpha)(x) \in \bigwedge T_x(Gx)^{\perp\star}$ is the component of $\alpha(x)$ with respect to the decomposition

$$\bigwedge T_xM^\star = \bigwedge T_x(Gx)^{\perp\star} \oplus \left(\overset{+}{\bigwedge} T_x(Gx)^{\perp\star} \otimes \bigwedge T_x(Gx)^\star\right). \tag{4}$$

To prove this claim, let β be the (possibly non-continuous) differential form on M, defined at each $x \in M$ as the component of $\alpha(x)$ in $\bigwedge T_x(Gx)^{\perp\star}$ by (4). Such a β is measurable because it is smooth in the dense open set of regular orbits whose complement is a finite union of manifolds of lower dimension and thus of null measure. Furthermore, $\beta \in L^2\Omega(M/G)$ because $|\beta(x)| \leq |\alpha(x)|$ for every $x \in M$.

Now fix $x \in M$ and $\tilde{x} \in \tau^{-1}(x)$. We write

$$T_{\tilde{x}}\widetilde{M} = A \oplus B \oplus C,$$

where

$$\begin{aligned} A &= T_{\tilde{x}}(G\tilde{x}), \\ B &= T_{\tilde{x}}\left(\tau^{-1}(Gx)\right) \cap T_{\tilde{x}}(G\tilde{x})^{\perp}, \\ C &= T_{\tilde{x}}\left(\tau^{-1}(Gx)\right)^{\perp}. \end{aligned}$$

If H and V are the horizontal and vertical subbundles respectively of the fiber bundle $\tau\colon \tau^{-1}(Gx) \to Gx$, we can write

$$A \oplus B = T_{\tilde{x}}\left(\tau^{-1}(Gx)\right) = H_{\tilde{x}} \oplus V_{\tilde{x}}.$$

We have $(\widetilde{\Pi}\,\tau^\star\alpha)(\tilde{x}) \in \bigwedge B^\star \oplus \bigwedge C^\star$. But by Corollary 2.2 we know that $H \subset A$ and $B \subset V$, so we can consider $(\widetilde{\Pi}\,\tau^\star\alpha)(\tilde{x}) \in \bigwedge C^\star$, and $(\tau^\star\beta)(\tilde{x}) \in \bigwedge C^\star$ again by Corollary 2.2. Hence, $(\widetilde{\Pi}\,\tau^\star\alpha)(\tilde{x}) = (\tau^\star\beta)(\tilde{x})$.

Since $x \in M$ is arbitrary, we get $\tau^\star\beta = \widetilde{\Pi}\,\tau^\star\alpha \in \Omega(\widetilde{M}/G)$, yielding $\beta \in \Omega(M/G)$ by Corollary 2.1, and we have

$$\beta = i^\star\,\tau^\star\beta = i^\star\,\widetilde{\Pi}\,\tau^\star\alpha = \overline{\Pi}\alpha;$$

the claim follows.

Now $\overline{\Pi}$ is easily seen to be an orthogonal projection, which is thus the restriction of $\Pi\colon L^2\Omega(M)_G \to L^2\Omega(M/G)$. On the other hand we have the orthogonal projection $\Omega(M) \to \Omega(M)_G$ defined by

$$\alpha \longmapsto \frac{1}{\mathrm{Vol(G)}}\int_G A_g^\star\alpha\, dg,$$

where $A\colon G \times M \to M$ denotes the given action, and we consider any biinvariant metric on G. Then the composition

$$\Omega(M) \longrightarrow \Omega(M)_G \xrightarrow{\overline{\Pi}} \Omega(M/G)$$

is an orthogonal projection too, and thus equal to the restriction of Π. This finishes the proof of Proposition 2.1.

2.2. *The basic Dirac operator.* As we said, the operator δ_b on $\Omega(M/G)$, defined as the composition

$$\Omega(M/G) \xrightarrow{\delta} \Omega(M) \xrightarrow{\Pi} \Omega(M/G),$$

is the adjoint of d_b by Proposition 2.1. We define the basic Dirac operator on $\Omega(M/G)$ by

$$D_b = d_b + \delta_b = \Pi \circ D|_{\Omega(M/G)},$$

where $D = d + \delta$ is the Dirac operator on $\Omega(M)$.

LEMMA 2.4. *D_b is essentially self-adjoint in $L^2\Omega(M/G)$.*

Proof. Since D is essentially self-adjoint in $L^2\Omega(M)$, $\Pi D \Pi$ is also essentially self-adjoint in $L^2\Omega(M)$ by Lemma XII.1.6 (c) of [4]. But

$$\Pi D \Pi = \begin{pmatrix} D_b & 0 \\ 0 & 0 \end{pmatrix}$$

with respect to the decomposition

$$L^2\Omega(M) = L^2\Omega(M/G) \oplus L^2\Omega(M/G)^{\perp},$$

and the result follows. □

LEMMA 2.5. *The following properties hold:*

(i) $[\delta, \Pi] = K\delta - \delta K$, *where* $K = (-1)^{nr+r}[\star, \Pi]\star$ *on* $\Omega^r(M)$.
(ii) *If* $\alpha \in \Omega(M/G)$ *and* $\beta \in \Omega(M)$, *then* $\Pi(\alpha \wedge \beta) = \alpha \wedge \Pi(\beta)$ *and* $K(\alpha \wedge \beta) = \alpha \wedge K(\beta)$.
(iii) *If* $\alpha \in \Omega(M/G)$ *and* $\beta \in \Omega(M)$, *then* $\Pi(\alpha \vee \beta) = \alpha \vee \Pi(\beta)$ *and* $K(\alpha \vee \beta) = \alpha \vee K(\beta)$, *where* $\alpha \vee = (\alpha \wedge)^*$.

Proof. For any $\alpha \in \Omega^r(M)$ we have

$$\begin{aligned}
[\delta, \Pi]\,\alpha &= (-1)^{nr+n+1}[\star d \star, \Pi]\alpha \\
&= (-1)^{nr+n+1}(\star d \star \Pi - \Pi \star d \star + \star \Pi d \star - \star \Pi d \star)\,\alpha \\
&= (-1)^{nr+n+1}([\star, \Pi] d \star + \star d [\star, \Pi])\alpha \\
&= (-1)^{nr+r}([\star, \Pi] \star \delta + \delta \star [\star, \Pi])\alpha \\
&= (K\delta - \delta K)\alpha,
\end{aligned}$$

where

$$K\alpha = (-1)^{nr+r}[\star, \Pi] \star \alpha = -(-1)^{nr+r} \star [\star, \Pi]\alpha$$

because $\star^2\alpha = (-1)^{nr+r}\alpha$. This proves (i).

Let $\gamma \in L^2\Omega(M/G)$. We have

$$\langle \alpha \wedge \Pi(\beta), \gamma \rangle = \langle \Pi(\beta), \alpha \vee \gamma \rangle = \langle \beta, \alpha \vee \gamma \rangle = \langle \alpha \wedge \beta, \gamma \rangle = \langle \Pi(\alpha \wedge \beta), \gamma \rangle$$

because $\alpha \vee \gamma \in \Omega(M/G)$, so the first part of (ii) follows.

Let $\gamma \in L^2\Omega(M/G)$. We have

$$\langle \alpha \vee \Pi(\beta), \gamma\rangle = \langle \Pi(\beta), \alpha \wedge \gamma\rangle = \langle \beta, \alpha \wedge \gamma\rangle = \langle \alpha \vee \beta, \gamma\rangle = \langle \Pi(\alpha \vee \beta), \gamma\rangle$$

because $\alpha \wedge \gamma \in \Omega(M/G)$, yielding the first part of (iii).

The second parts of (ii) and (iii) follow from the equality $K = (-1)^{nr+r}[\star, \Pi]\star$ on $\Omega^r(M)$. □

LEMMA 2.6. *Any $C^\infty(M/G)$-linear operator T on $\Omega(M/G)$ defines a bounded operator on $L^2\Omega(M/G)$.*

Proof. Since $\Omega(M/G)$ is finitely generated as a $C^\infty(M/G)$-module, let $\{\alpha_1, \dots, \alpha_k\}$ be a finite set of generators. Write any $\alpha \in \Omega(M/G)$ as $\alpha = \sum_i f_i\alpha_i$ with $f_i \in C^\infty(M/G)$. Then

$$\|T\alpha\| = \left\|T\sum_i f_i\alpha_i\right\| \le \sum_i \|f_iT\alpha_i\| \le \sum_i \|f_i\|\,\|T\alpha_i\| \le \text{constant}\,\|\alpha\|.$$

So T defines a bounded operator on $L^2\Omega(M/G)$. □

LEMMA 2.7. *The operator $D - \Pi D$ is bounded on $L^2\Omega(M/G)$.*

Proof. We have

$$D - \Pi D = D\Pi - \Pi D\Pi = (\mathrm{id} - \Pi)D\Pi = [D, \Pi]\Pi.$$

Thus if $[D, \Pi]$ is bounded on $L^2\Omega(M/G)$, so is $D - \Pi D$. This holds by Lemma 2.4 because $[D, \Pi] = [\delta, \Pi]$ is $C^\infty(M/G)$-linear on $\Omega(M/G)$, which in turn follows from Lemma 2.5 since, for all $f \in C^\infty(M/G)$ and $\alpha \in \Omega(M/G)$, we have

$$\begin{aligned}
[\delta, \Pi]f\alpha &= (K\delta - \delta K)(f\alpha) \\
&= K\delta(f\alpha) - \delta K(f\alpha) \\
&= K(f\delta\alpha - df \vee \alpha) - \delta(fK(\alpha)) \\
&= fK\delta\alpha - df \vee K(\alpha) - f\delta K(\alpha) + df \vee K(\alpha) \\
&= f(K\delta - \delta K)\alpha \\
&= f[\delta, \Pi]\alpha.
\end{aligned}$$

□

Using Lemmas 2.4 and 2.7, we finally get the following result.

THEOREM 2.1. *With the above notation, there is a complete orthonormal system $\{\phi_i, i = 1, 2, \dots\} \subset \Omega(M/G)$ of the Hilbert space $L^2\Omega(M/G)$ given by the eigenforms of Δ_b with eigenvalues λ_i, $i = 1, 2, \dots$, satisfying the inequalities $0 \le \lambda_1 \le \lambda_2 \le \cdots$, with $\lambda_i \uparrow \infty$ if $\dim \Omega(M/G) = \infty$. In particular we have the Hodge type decomposition*

$$\Omega(M/G) = \mathrm{Ker}\ \Delta_b \oplus \mathrm{Im}\ d_b \oplus \mathrm{Im}\ \delta_b.$$

Proof. Let

$$\begin{aligned}\|\alpha\|_{D,k} &= \left\|(\mathrm{id}+D)^k\alpha\right\|_0, \quad \alpha\in\Omega(M),\\ \|\alpha\|_{D_b,k} &= \left\|(\mathrm{id}+D_b)^k\alpha\right\|_0, \quad \alpha\in\Omega(M/G).\end{aligned}$$

Let $W^k\Omega(M/G)$ be the closure of $\Omega(M/G)$ in the kth Sobolev completion $W^k\Omega(M)$ of $\Omega(M)$. We know that, on $\Omega(M/G)$,

$$D_b = \Pi D = \Pi D\Pi = D\Pi - (D\Pi - \Pi D\Pi) = D - (D - \Pi D).$$

So the difference $D_b - D$ is a bounded operator on $L^2\Omega(M/G)$ by Lemma 2.7. Hence the norms $\| \ \|_{D,k}$ and $\| \ \|_{D_b,k}$ are equivalent on $\Omega(M/G)$, and thus $W^k\Omega(M/G)$ is the $\| \ \|_{D_b,k}$-completion of $\Omega(M/G)$. Moreover, the compactness of the inclusion $W^k\Omega(M)\hookrightarrow W^{k-1}\Omega(M)$ implies the compactness of the inclusion $W^k\Omega(M/G)\hookrightarrow W^{k-1}\Omega(M/G)$. On the other hand, $\bigcap_k W^k\Omega(M/G)\subset\bigcap_k W^k\Omega(M)=\Omega(M)$, yielding $\bigcap_k W^k\Omega(M/G) = L^2\Omega(M/G)\cap\Omega(M)=\Omega(M/G)$. Combining these facts with Lemma 2.4 and [2, Proposition 2.44], we get the stated result. □

3. Differential forms on a neighborhood of an orbit

For $x\in M$, let $F = Gx$ be the orbit of x. Let V be a tubular neighborhood of the 0-section of $TF^\perp$, with radius λ. By the Koszul Slice Theorem [10] we know that $V\equiv G\times_K B$, where $K = G_x$ is the isotropy group and B is the ball of radius λ in $\mathbb{R}^\nu$ centered at 0 ($n=\dim M$, $r=\dim TF$, $\nu = n-r$). We can take λ small enough so that the exponential map of M is a diffeomorphism of V onto some open G-invariant subset $U\subset M$. So the composition of the canonical identity $V\equiv G\times_K B$ and $\exp_M$ defines an equivariant diffeomorphism ϕ: $G\times_K B\to U$. Thus ϕ^* defines an isomorphism $\Omega(U)_G\cong\Omega(G\times_K B)_G$ of graded differential algebras. We also have the following isomorphisms of graded differential algebras:

$$\begin{aligned}
&(5) & \Omega(G\times_K\mathbb{R}^\nu)_G &\cong \Omega(G\times\mathbb{R}^\nu)_{G,K,\iota_{\mathfrak{K}}=0}\\
&(6) & &\cong (\Omega(G)_G\otimes\Omega(\mathbb{R}^\nu))_{K,\iota_{\mathfrak{K}}=0}\\
&(7) & &\cong \left(\Omega(G)_{G,\iota_{\mathfrak{K}}=0}\otimes\bigwedge\mathfrak{k}^*\otimes\Omega(\mathbb{R}^\nu)\right)_{K,\iota_{\mathfrak{K}}=0}\\
&(8) & &\cong (\Omega(G)_{G,\iota_{\mathfrak{K}}=0}\otimes\Omega(\mathbb{R}^\nu))_K\,.
\end{aligned}$$

In (7), K-invariance and $\iota_K = 0$ are considered with respect to the action of K defined by $a\cdot(z,b,v) = \left(za^{-1}, ba^{-1}, av\right)$. Isomorphism (5) is defined by the canonical projection $G\times\mathbb{R}^\nu\to G\times_K\mathbb{R}^\nu$. Isomorphism (6) is canonical because G only acts on the first factor G. Isomorphism (7) is induced by

$$\Omega(G)_{\iota_{\mathfrak{K}}=0}\otimes\bigwedge\mathfrak{k}^*\cong\Omega(G),\quad \alpha\otimes\gamma\longmapsto\alpha\wedge\omega_\wedge\gamma,$$

where $\omega_\wedge\colon \bigwedge \mathfrak{k}^\star \to \Omega(G)$ is the canonical extension of the algebraic connection $\omega\colon \mathfrak{k}^\star \to \Omega^1(G)$ [6]. Isomorphism (8) is defined by the diffeomorphism η on $G \times K \times \mathbb{R}^\nu$ given by $\eta(z, a, v) = \big(za^{-1}, a, av\big)$.

From (5)–(7) we get

$$\Omega(U)_G \cong (\Omega(G)_{G,\iota_{\mathrm{K}}=0} \otimes \Omega(\mathrm{B}))_{\mathrm{K}}. \tag{9}$$

In particular,

$$C^\infty(U/G) \cong C^\infty(B/K). \tag{10}$$

4. Proof of the Main Theorem

4.1. *Concentration around the critical orbits.* For any G-Morse function f and any $s \in \mathbb{R}$, define

$$d_{b,s} = e^{-sf} d_b e^{sf} = d_b + s\,df\wedge, \qquad \delta_{b,s} = e^{sf}\delta_b e^{-sf} = \delta_b - s\,df\vee, \tag{11}$$

$$D_{b,s} = d_{b,s} + \delta_{b,s} = D_b + s(df \wedge - df\vee) = D_b + sH, \qquad \Delta_{b,s} = D_{b,s}^2. \tag{12}$$

Suppose that ϕ is a positive even Schwarz function on $\mathbb{R}$ with $\phi(0) = 1$. Then $\phi(D_{b,s})$ is of trace class, and let

$$\mu_j^s = Tr\left(\phi(D_{b,s})|_{L^2\Omega^j(M/G)}\right).$$

The following result follows with the same arguments as in [9].

PROPOSITION 4.1. *With the above notation we get the inequalities*

$$\begin{aligned} \beta_0 &\le \mu_0^s, \\ \beta_1 - \beta_0 &\le \mu_1^s - \mu_0^s, \\ \beta_2 - \beta_1 + \beta_0 &\le \mu_2^s - \mu_1^s + \mu_0^s, \end{aligned}$$

etc., and the equality

$$\sum_{j=0}^n (-1)^j \beta_j = \sum_{j=0}^n (-1)^j \mu_j^s.$$

Again with the same arguments as in [9] we get the next result.

LEMMA 4.1. *With the above notation we get:*

(i) *H^2 is the endomorphism given by multiplication by $|df|^2$.*
(ii) *$HD_{b,s} + D_{b,s}H$ is an endomorphism of order zero.*

Now consider the Fourier transform $\widehat{\phi}$ of ϕ, which has compact support contained in some interval $[-\rho, \rho]$ for some large enough $\rho > 0$. The next result follows from Lemma 4.1 as in Roe [9].

LEMMA 4.2. *On the product of M/G and the complement of a 2ρ-neighborhood of the union of the critical orbits of f, the Schwarz kernel of $\phi(D_{b,s})$ tends uniformly to zero as $s \to \infty$.*

Even though ρ is fixed, by dilating the metric transversally to the critical orbits, the 2ρ-neighborhood of the critical orbits can be made small. So, as in [9, Chapter 12], Lemma 4.2 will be used to obtain the trace of $\phi(D_{b,s})$ as the sum of the contributions from the critical orbits.

For any fixed critical orbit F, we define the operator ψ_F on $\Omega(M/G)$ of multiplication by a non negative G-function on M, equal to 1 in a G-invariant 2ρ-neighborhood of F, and supported in a G-invariant 3ρ-neighborhood. From Lemma 4.2 we get

$$\mu_j^s - \sum_{F \in \mathrm{Crit}_G(f)} Tr\left(\psi_F \phi(D_{b,s})|_{L^2\Omega^j(M/G)}\right) \longrightarrow 0 \quad \text{as} \quad s \uparrow \infty.$$

4.2. *Local computation.* The description of the forms given in Section 3 is useful to simplify the calculus of the trace of the restriction of $\psi_F\phi(D_{b,s})$ to $L^2\Omega^j(M/G)$. We can simplify the problem by going from the G-manifold M to the K-space $\mathbb{R}^\nu$. To begin with, we decompose Π as follows. From the isomorphisms (5) and (6) over $\Omega(U/G)$ we get $\Omega(U/G) \cong \Omega(B/K)$. Via this isomorphism and (9), the projection $\Pi\colon \Omega(U)_G \to \Omega(U/G)$ corresponds to the composition

$$(\Omega(G)_{G,\iota_K=0} \otimes \Omega(B))_K \longrightarrow \Omega(B)_K \xrightarrow{\Pi'} \Omega(B/K),$$

where the first arrow is the canonical projection, and Π' is the projection defined for B like Π for M. So

$$\mathrm{Tr}\left(\psi_F\phi(D_{b,s})|_{L^2\Omega^j(M/G)}\right) = \mathrm{Tr}\left(\psi_F\phi(D'_{b,s})|_{L^2\Omega^j(\mathbb{R}^\nu/K)}\right),$$

with the obvious definitions of ψ_F, ϕ, $D'_{b,s}$, etc. in the new context of $\mathbb{R}^\nu$ and K.

For any fixed $F \in \mathrm{Crit}_G(f)$, let $f' \in C^\infty(B/K)$ be the function that corresponds to $f|_U$ by (10). The origin 0 is a nondegenerate critical point of f'. So taking Morse coordinates (x_j) on some K-invariant open neighborhood of 0, we get the expression $f' = 1/2 \sum_j \lambda_j x_j^2$. The number of negative λ_j's is the index m_F of f' at 0. Assume that the first m_F of the λ_j's are negative, so the decomposition

$$\mathbb{R}^\nu = \mathbb{R}^\nu_- \oplus \mathbb{R}^\nu_+$$

defined by $H_{f'}$ on $\mathbb{R}^\nu$ is well adapted to the coordinates (x_j).

Now, from [9, Chapter 12], we know that the eigenforms of Δ'_s on $\Omega(\mathbb{R}^\nu)$ have the expression

$$\psi_{p,q,I,s} = \prod_{j=1}^{n} h_{p_j}(x_j) \exp\left(-\frac{s}{2}\sum_j |\lambda_j| x_j^2\right) d'x_{i_1} \wedge \cdots \wedge d'x_{i_k},$$

with $p = (p_1, \dots, p_\nu)$, $p_i \in \mathbb{N}$, $q = (q_1, \dots, q_\nu)$, $q_i \in \{\pm 1\}$, $I = (i_1, \dots, i_k)$, and where the h_{p_j}'s are the Hermite polynomials, up to a normalization. The corresponding eigenvalues are

$$\lambda_{p,q,I,s} = s \sum_j (|\lambda_j|(1 + 2p_j) + \lambda_j q_j).$$

From this it follows that the absolute value of the eigenvalues of D'_s on $\Omega(\mathbb{R}^\nu)$ are also of order s as $s \uparrow \infty$, except the eigenvalue zero that corresponds to the election of $p_0 = (0, \overset{(\nu)}{\dots}, 0)$, $q_0 = (1, \overset{(m_F)}{\dots}, 1, -1, \overset{(\nu - m_F)}{\dots}, -1)$ in the above expression. The set $\{\psi_{p,q,I,s}\}$ is a complete orthonormal system of $L^2\Omega(\mathbb{R}^\nu)$, and thus $\{\Pi' \psi_{p,q,I,s}\}$ generates $L^2\Omega(\mathbb{R}^\nu / K)$ as Hilbert space.

Let $\psi_{0,s}$ denote the eigenform $\psi_{p_0, q_0, I_0, s}$, with

$$p_0 = (0, \overset{(\nu)}{\dots}, 0), \; q_0 = (1, \overset{(\mathrm{m_F})}{\dots}, 1, -1, \overset{(\nu - \mathrm{m_F})}{\dots}, -1), \; I_0 = (1, \dots, \mathrm{m_F}),$$

that corresponds to the eigenvalue zero of Δ'_s, and also of D'_s. We distinguish two cases.

Case 1. Suppose $\psi_{0,s} \in \Omega(\mathbb{R}^\nu / K)$.

This property means that $\Pi' \psi_{0,s} = \psi_{0,s}$; so we get

$$D'_{b,s} \Pi' \psi_{0,s} = D'_{b,s} \psi_{0,s} = \Pi' D'_s \psi_{0,s} = 0,$$

because $D'_s \psi_{0,s} = 0$. Moreover, we have the following.

LEMMA 4.3. *If* $\psi_{0,s} \in \Omega(\mathbb{R}^\nu / K)$, *and* $0 \neq \psi \in L^2\Omega(\mathbb{R}^\nu / K)$, *with* $\psi \perp \psi_{0,s}$, *then*

$$\langle D'_{b,s} \psi, \psi \rangle \in O(s) \quad \text{as } s \uparrow \infty.$$

Proof. For

$$0 \neq \psi \in L^2\Omega(\mathbb{R}^\nu / K), \quad \psi \perp \psi_{0,s},$$

we have

$$\begin{aligned} \langle D'_{b,s} \psi, \psi \rangle &= \langle D'_s \psi, \psi \rangle + \langle (D'_b - D') \psi, \psi \rangle \\ &\geq \min\{\lambda \neq 0 \mid \lambda \text{ is an eigenvalue of } D'_s \text{ on } \Omega(\mathbb{R}^\nu)\} \, \|\psi\| \\ &\quad + \text{constant}, \end{aligned}$$

which is of order s as $s \uparrow \infty$, because $D'_b - D'$ is bounded in $L^2\Omega(\mathbb{R}^\nu / K)$ by Lemma 2.5. □

Case 2. Suppose $\psi_{0,s} \notin \Omega(\mathbb{R}^\nu / K)$.

LEMMA 4.4. *In this case we have* $\Pi'\psi_{0,s} = 0$; *i.e.*, $\psi_{0,s} \in \Omega(\mathbb{R}^\nu/K)^\perp$.

Proof. $\omega = dx_1 \wedge \cdots \wedge dx_m$ is the volume form of the K-invariant space $\mathbb{R}^\nu_-$. If ω is not invariant we have

$$\rho_K(\omega) = \frac{1}{\mathrm{Vol(K)}} \int_K k^\star \omega \, dk = 0$$

because $k^\star\omega = \pm\omega$ for all $k \in K$. Then $\Pi'\psi_{0,s} = 0$ because $\psi_{0,s}$ is the product of a K-invariant function and ω.

Suppose ω is invariant. Then ω is not horizontal because $\psi_{0,s} \notin \Omega(\mathbb{R}^\nu/K)$. So the regular K-orbits in $\mathbb{R}^\nu_-$ have positive dimension. Now, since ω is a volume form on $\mathbb{R}^\nu_-$, it has top tangential degree along the regular G-orbits. So $\Pi'\omega = 0$ on the union of regular G-orbits, which is a dense open set, yielding $\Pi'\omega = 0$ on the whole $\mathbb{R}^\nu_-$, and thus on the whole $\mathbb{R}^\nu$. □

As in the above case we get the following.

LEMMA 4.5. *If* $\psi_{0,s} \notin \Omega(\mathbb{R}^\nu/K)$, *and* $\psi \neq 0$ *is in* $L^2\Omega(\mathbb{R}^\nu/K)$, *then*

$$\langle D'_{b,s}\psi, \psi\rangle \in O(s) \quad \textit{as } s \uparrow \infty.$$

Proof. This follows by arguing as in the above case, since in this situation we have $\psi \perp \psi_{0,s}$. □

Now we can finish the proof of the Main Theorem. From Lemma 4.3 and Lemma 4.5 we get

$$\begin{aligned}\lim_{s\to\infty} Tr\left(\psi_F\phi\left(D'_{b,s}\right)|_{L^2\Omega^j(\mathbb{R}^\nu/K)}\right) &= \lim_{s\to\infty}\sum_{p,q,I}\langle\psi_F\phi\left(D'_{b,s}\right)\Pi'\psi_{p,q,I,s}, \Pi'\psi_{p,q,I,s}\rangle \\ &= \begin{cases}1 & \text{if } \psi_{0,s} \in \Omega^j(\mathbb{R}^\nu/K), \\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Now simply observe that

$$\psi_{0,s} = \text{function } d'x_1 \wedge \cdots \wedge d'x_{m_F} \in \Omega^j(\mathbb{R}^\nu/K)$$

which means that $m_F = j$, and $N_{F,-}$ is G-orientable and G_0-trivial. So

$$\mu_j = \lim_{s\to\infty} \mu_j^s,$$

and the Main Theorem follows from Proposition 4.1.

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