ILLINOIS JOURNAL OF MATHEMATICS
Volume 44, Number 2, Summer 2000

WEIGHTED INEQUALITIES FOR HANKEL CONVOLUTION OPERATORS

JORGE J. BETANCOR AND LOURDES RODRÍGUEZ-MESA

ABSTRACT. In this paper we obtain weighted inequalities for Hankel convolution operators. Also, a weighted version of Mihlin-Hörmander theorem for Hankel multipliers is given. Some inequalities for maximal functions play an important role.

1. Introduction and preliminaries

The purpose of this paper is to derive weighted inequalities for Hankel convolution operators. As a particular case we obtain a weighted version of a Mihlin-Hörmander type theorem for Hankel multipliers that extends the results of Gosselin and Stempak [7, Corollary 1.2] and [16, Theorem 5].

Consider the measure space $(I, d\gamma)$ where $I = (0, \infty)$ and $d\gamma = \frac{x^{2\mu+1}}{2^\mu \Gamma(\mu+1)} dx$, $\mu > -1/2$. The measure γ satisfies the doubling condition, that is, there exists a positive constant $C > 0$ such that

$$\gamma(B(x, 2\epsilon)) < C\gamma(B(x, \epsilon)),$$

where $B(x, \epsilon) = \{y \in I\colon |x - y| < \epsilon\}$, $x \in I$ and $\epsilon > 0$. Let w be a nonnegative measurable function on I. By $L_{p,w}(\gamma)$, $1 \leq p < \infty$, we denote the space of measurable functions f on I such that

$$\|f\|_{p,w} = \left\{ \int_0^\infty |f(x)|^p w(x) x^{2\mu+1}\, dx \right\}^{1/p} < \infty.$$

When $w \equiv 1$, to simplify the notation, we write $L_p(\gamma)$ and $\| \ \|_p$ instead of $L_{p,w}(\gamma)$ and $\| \ \|_{p,w}$, respectively. Let L_∞ denote the space of essentially bounded functions on $(0, \infty)$.

We represent by C_0 the space of continuous and compactly supported functions on I.

As usual the Hankel transform $h_\mu f$ of $f \in L_1(\gamma)$ is defined by

$$h_\mu(f)(y) = \int_0^\infty (xy)^{-\mu} J_\mu(xy) f(x) x^{2\mu+1}\, dx, \qquad y \in I,$$

Received March 10, 1998; received in final form June 17, 1999.
1991 Mathematics Subject Classification. Primary 46F12.
The authors were partially supported by DGICYT Grant PB 97-1489 (Spain).

where J_μ represents the Bessel function of the first kind and order $\mu > -1/2$. Since h_μ is an isometry on $L_2(\gamma)$ and maps $L_1(\gamma)$ boundedly into L_∞ it follows that h_μ can be extended to a bounded operator from $L_p(\gamma)$ into $L_{p'}(\gamma)$, $1 < p \leq 2$, $p' = \frac{p}{p-1}$ [9, Theorem 3].

The convolution operation for h_μ-transformation was investigated by Cholewinski [6], Haimo [8] and Hirschman [10]. If f and g are in $L_1(\gamma)$ the convolution $f\#g$ of f and g is defined by

$$(f\#g)(x) = \int_0^\infty (\tau_x f)(y) g(y)\, d\gamma(y), \qquad x \in I,$$

where the Hankel translation $\tau_x f$ of f is

$$(\tau_x f)(y) = \int_0^\infty D_\mu(x, y, z) f(z)\, d\gamma(z), \qquad x, y \in I,$$

and

$$\begin{aligned} &D_\mu(x, y, z) \\ &= 2^{2\mu}\Gamma(\mu+1)^2 \int_0^\infty (xt)^{-\mu} J_\mu(xt)(yt)^{-\mu} J_\mu(yt)(zt)^{-\mu} J_\mu(zt) t^{2\mu+1} dt, \quad x, y, z \in I. \end{aligned}$$

The #-convolution defines a bilinear bounded mapping from $L_p(\gamma) \times L_q(\gamma)$ into $L_r(\gamma)$, provided that $1 \leq p, q, r < \infty$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$ [10, Theorem 2.b]. The Hankel translation τ_x is a contractive operator in $L_p(\gamma)$, for every $x \in I$ and $1 \leq p \leq \infty$ [16, p. 16].

Let k be a locally integrable function on I and consider the convolution operator T_k defined by $T_k f = k\#f$. The function k is usually called the convolution kernel of the operator T_k. By taking into account the fact that $(\tau_x f)(y) = (\tau_y f)(x)$, $x, y \in I$, the following result follows from [5, Theorem 2.4].

THEOREM 1.1. *Let $1 < p < \infty$. Assume the following conditions:*
(i) *There exists $C_p > 0$ such that $\|T_k f\|_p \leq C_p \|f\|_p$, $f \in L_p(\gamma)$.*
(ii) *There exist two positive constants a and b such that for every $x, y \in I$,*

$$\int_{|x-z|>b|y-x|} |(\tau_x k)(z) - (\tau_y k)(z)|\, d\gamma(z) \leq a, \tag{1}$$

holds.
Then for every $1 < q < p$ there exists $C_q > 0$ for which

$$\|T_k f\|_q \leq C_q \|f\|_q, \qquad f \in L_q(\gamma),$$

and there exists $C_1 > 0$ such that $\gamma(\{x \in I\colon |T_k f(x)| > \lambda\}) \leq \frac{C_1}{\lambda}\|f\|_1$ for each $\lambda > 0$ and $f \in L_1(\gamma)$. Moreover, C_q, $q \in [1, p)$, depends only on C_p, a and b.

Note that (1) is the Hankel version of the well-known Hörmander condition.

Any bounded function m on I defines a Hankel multiplier operator $\mathcal{M}_m$ by $h_\mu(\mathcal{M}_m f) = m h_\mu(f)$. It is clear that the operators $\mathcal{M}_m$ depend on μ. Note that if in addition $m \in L_1(\gamma)$ and $h_\mu(m) \in L_1(\gamma)$ then by invoking [10, Theorem 2.d and Corollary 2.e] we can write $\mathcal{M}_m f = h_\mu(m) \# f$, for every $f \in L_1(\gamma)$, that is, the multiplier operator $\mathcal{M}_m$ is actually a convolution operator. In [1, Corollary 3.1] we established conditions on a function $m \in L_p(\gamma)$ that implies that $h_\mu(m) \in L_1(\gamma)$. Using Theorem 1.1 we prove the following result, a Hankel version of the Mihlin-Hörmander multiplier theorem for the Fourier transform. This theorem is a generalization of [7, Theorem 1.1].

Throughout this paper C will represent a positive constant not necessarily the same in each ocurrence.

THEOREM 1.2. *Let $1 < r \leq 2$ and $s > \frac{\mu+1}{r}$. Also, assume that $m \in C^{2s}(I)$ is a bounded function on I such that there exists $C > 0$ for which*

$$\left\{ \int_{R/2}^{R} \left| \left(\frac{1}{x} D \right)^{\alpha} m(x) \right|^{r} d\gamma(x) \right\}^{1/r} \leq C R^{2(\mu+1)/r - 2\alpha}, \quad R > 0 \text{ and } 0 \leq \alpha \leq 2s. \quad (2)$$

Then the Hankel multiplier operator $\mathcal{M}_m$ associated to m defines a bounded operator from $L_p(\gamma)$ into itself, for every $1 < p < \infty$, and it is of weak type (1,1), that is,

$$\gamma(\{x \in I \colon |\mathcal{M}_m(f)(x)| > \lambda\}) \leq C \frac{\|f\|_1}{\lambda} \qquad \text{for every } \lambda > 0,$$

with $C > 0$ independent of $\lambda > 0$ and $f \in L_1(\gamma)$.

Note that condition (2) imposed on the multiplier m in Theorem 1.2 is similar to the property that characterizes the class $M(s, \lambda)$ of Fourier multipliers in [14]. Here the operator $\frac{1}{x} D$ plays the role of the derivative in the definition of $M(s, \lambda)$ [14]. Recently, Prof. K. Stempak has pointed us that condition (2) could allow to prove the boundedness of the multiplier operator $\mathcal{M}_m$ on the spaces $L_{p,x^\alpha}(\gamma)$, by establishing the Hankel version of [14, Theorems 1.2-1.5]. This will be the objective of our next paper.

Motivated by the papers of Córdoba and Fefferman [4], Kurtz and Wheeden [13] and Kurtz [12] we introduce a class of kernels satisfying a Hörmander type condition involving L_p-norm that will allow us to deduce that if the convolution operator T_k is bounded from $L_r(\gamma)$ into $L_{p,h}(\gamma)$, for some p and r and a particular weight h, then T_k is also bounded between other weighted L_p-spaces.

Let $l \in \mathbb{N} \setminus \{0\}$. In what follows we will consider the metric ρ_l on I defined by $\rho_l(x, y) = |x^l - y^l|^{1/l}$, $x, y \in I$. We note that the following doubling condition holds: there exists $C_l > 0$ such that

$$\gamma(B_l(x, 2\epsilon)) \leq C_l \gamma(B_l(x, \epsilon)), \qquad x \in I \text{ and } \epsilon > 0. \qquad (3)$$

Hence (I, ρ_l, γ) is a space of homogeneous type in the sense of Coifman and Weiss [5].

We denote by M^l the maximal function on I associated to the measure γ and the metric ρ_l. That is, if f is a locally integrable function on I, we define

$$(M^l f)(x) = \sup_{\epsilon>0} \frac{1}{\gamma(B_l(x,\epsilon))} \int_{B_l(x,\epsilon)} |f(z)|\, d\gamma(z), \qquad x \in I,$$

where $B_l(x,\epsilon) = \{y \in I\colon\ \rho_l(x,y) < \epsilon\}$, for every $x \in I$ and $\epsilon > 0$.

Definition. Let k be a locally integrable function on I. We will say that k belongs to $K(\mu, r, q, l)$, where $\mu > -\frac{1}{2}$, $1 \le r, q < \infty$ and $l = 1, 2, \dots$ if k satisfies the following conditions:

(i) There exists a non-decreasing function S defined on $(0, 1)$ such that $\sum_{j=1}^{\infty} S(2^{-j}) < \infty$ and

$$\left\{\int_{R<|x-y_0|<2R} |(\tau_y k)(x) - (\tau_{y_0} k)(x)|^{q'}\, d\gamma(x)\right\}^{1/q'} \le R^{-2(\mu+1)l/r} S\left(\frac{\rho_l(y, y_0)}{R}\right),$$

for every $R > 0$ and every $y_0, y \in I$ such that $\rho_l(y, y_0) < \frac{R}{2}$.

(ii) There exists $C > 0$ such that

$$\|T_k f\|_{r,h_l} \le C\|f\|_q, \qquad f \in L_q(\gamma),$$

where $h_l(y) = y^{2(\mu+1)(l-1)}$, $y \in I$.

The next theorem corresponds to Lemma 3.4 and Theorem 3.5 of [12].

THEOREM 1.3. *Let $l \in \mathbb{N} \setminus \{0\}$, $1 \le q, r < \infty$, $lq \le r$ and $1 < p < \infty$. Assume that $k \in K(\mu, r, q, l)$ and v and w are nonnegative measurable functions on I satisfying the following*:

(i) *There exists $C > 0$ and $\chi > 0$ such that*

$$\frac{\int_{B_l(x,\epsilon)\cap E} w(y)\, d\gamma(y)}{\int_{B_l(x,\epsilon)} w(y)\, d\gamma(y)} \le C\left(\frac{\gamma(B_l(x,\epsilon) \cap E)}{\gamma(B_l(x,\epsilon))}\right)^{\chi},$$

for every E Lebesgue measurable set, $x \in I$ and $\epsilon > 0$ (that is, $w \in A_\infty$ respect to (I, ρ_l, μ)).

(ii) *There exists $C > 0$ such that for every $x \in I$ and $\epsilon > 0$*

$$\gamma(B(x,\epsilon))^{-lq/r} \left(\int_{B(x,\epsilon)} w\, d\gamma\right)^{q/p} \left(\int_{B(x,\epsilon)} v^{-1/(h-1)}\, d\gamma\right)^{1/h'} \le C,$$

where $1 < h < \frac{p}{q}$ and $v^{-1/(h-1)} d\gamma$ satisfies the doubling condition with respect to the usual metric on I.

(iii) *For every* $f \in \mathcal{C}_0$, $M^l(T_k f) \in L_{p,w}(\gamma)$.
Then there exists a positive constant C such that

$$\|T_k f\|_{p,w} \leq C\|f\|_{qh,v}, \qquad f \in \mathcal{C}_0.$$

Moreover, the last constant C depends on the function S and the constants C appearing in the hypothesis.

As a consequence of Theorem 1.3 we obtain the following weighted version of the Mihlin-Hörmander theorem for Hankel multipliers.

THEOREM 1.4. *Let* $1 < r \leq 2, r \leq q < \infty, d \in \mathbb{N}\setminus\{0\}$ *and* $\frac{\mu+1}{r} < d < \frac{\mu+1}{r}+1$. *Assume that* $m \in C^{2d}(I)$ *is a bounded function on* I *such that* $m \in L_1(\gamma)$, *that* $h_\mu(m) \in L_1(\gamma)$ *and that there exists* $C > 0$ *for which*

$$\left\{\int_{R/2}^{R}\left|\left(\frac{1}{x}D\right)^{\alpha} m(x)\right|^{r} d\gamma(x)\right\}^{1/r} \leq CR^{2(\mu+1)/r-2\alpha}, \qquad R > 0 \text{ and } 0 \leq \alpha \leq 2d,$$

and

$$|h_\mu(m)(x)| \leq Cx^{-\alpha}, \qquad x \in I, \tag{4}$$

for a certain $\alpha > \log_2(C_l)$, *where* C_l *is the constant appearing in* (3) *for* $l = 2(d+1)$.

Suppose also that $T_{h_\mu(m)}$ *is bounded from* $L_q(\gamma)$ *into* $L_{lq,h_l}(\gamma)$, *where* $h_l(y) = y^{2(\mu+1)(l-1)}$, $y \in I$. *Then* $\mathcal{M}_m$ *is a bounded operator from* $L_{qh,v}(\gamma)$ *into* $L_{p,w}(\gamma)$, *where* $1 < h < \frac{p}{q}$ *and* $q < p < \infty$, *provided that* v *and* w *are nonnegative measurable functions on* I *satisfying condition* (ii) *in Theorem* 1.3 *and the following one: there exists* $C > 0$ *for which*

$$\int_{B_l(x,\epsilon)} w(y)\,d\gamma(y)\left(\int_{B_l(x,\epsilon)} w^{\frac{-1}{p-1}}\,d\gamma(y)\right)^{p-1} \leq C\gamma(B_l(x,\epsilon))^p,$$

for every $x \in I$ *and* $\epsilon > 0$ (*that is,* $w \in A_p$ *with respect to* (I, ρ_l, γ)).

2. Inequalities for maximal functions

In this section we present certain inequalities for maximal functions that will be useful in the sequel.

As usual the fractional maximal function M_α, $0 \leq \alpha < 1$, associated to the measure γ and the usual metric ρ_l on I is defined for every locally integrable function f on I by

$$(M_\alpha f)(x) = \sup_{\epsilon>0}\frac{1}{\gamma(B(x,\epsilon))^{1-\alpha}}\int_{B(x,\epsilon)}|f(z)|\,d\gamma(z), \qquad x \in I,$$

where $B(x,\epsilon) = \{y \in I\colon |x-y| < \epsilon\}$, for each $x \in I$ and $\epsilon > 0$. Note that when $\alpha = 0$ the fractional maximal function reduces to the usual maximal function.

The following result follows from [18, Theorem 4].

PROPOSITION 2.1. *Let $1 < p < q < \infty$ and $0 \leq \alpha < 1$. Let v and w be nonnegative measurable functions on I. If $v^{-1/(p-1)}d\gamma$ satisfies the doubling condition with respect to the usual metric on I, then the norm inequality*

$$\left(\int_0^\infty |M_\alpha f(x)|^q w(x)\, d\gamma(x)\right)^{1/q} \leq C\left(\int_0^\infty |f(x)|^p v(x)\, d\gamma(x)\right)^{1/p},$$

holds for all locally integrable function f on I, with C independent of f, provided that

$$\gamma(B(x,\epsilon))^{\alpha-1}\left(\int_{B(x,\epsilon)} w\, d\gamma\right)^{1/q}\left(\int_{B(x,\epsilon)} v^{-1/(p-1)}\, d\gamma\right)^{1/p'} \leq C,$$

for all $x \in I$ and $\epsilon > 0$, where C does not depend on x and ϵ.

Also, for every $l \in \mathbb{N}\setminus\{0\}$, we consider the sharp maximal function $M_l^\#$, associated to the metric ρ_l already defined, given by

$$(M_l^\# f)(x) = \sup_{\epsilon>0} \frac{1}{\gamma(B_l(x,\epsilon))}\int_{B_l(x,\epsilon)} |f - f_{B_l(x,\epsilon)}|\, d\gamma, \qquad x \in I,$$

where f is locally integrable on I and $f_{B_l(x,\epsilon)}$ denotes the average of f on $B_l(x,\epsilon)$, that is

$$f_{B_l(x,\epsilon)} = \frac{1}{\gamma(B_l(x,\epsilon))}\int_{B_l(x,\epsilon)} f\, d\gamma, \qquad x \in I, \epsilon > 0.$$

From [2, Theorem 2] we can immediately deduce the following result.

PROPOSITION 2.2. *Let $1 \leq p < \infty$ and $l \in \mathbb{N}\setminus\{0\}$. Assume that w is a nonnegative measurable function on I that satisfies the condition* (*i*) *in Theorem* 1.3. *Then there exists $C > 0$ such that*

$$\|f\|_{p,w} \leq C\|M_l^\#(f)\|_{p,w}.$$

provided that f is a locally integrable function f on I and $M^l f \in L_{p,w}(\gamma)$.

In the next proposition we prove a relation between the sharp and fractional maximal functions that will be very useful in the sequel.

PROPOSITION 2.3. *Let $l = 1, 2, \ldots$ and let $1 \leq r, q < \infty$ be such that $lq < r$. Assume that $k \in K(\mu, r, q, l)$. Then there exists a constant $C > 0$ depending on μ, r, q and l such that for every $f \in L_q(\gamma)$ we have*

$$M_l^\#(T_k f)(y) \leq C\left\{M_\eta(|f|^q)(y)\right\}^{1/q}, \qquad y \in I,$$

where $\eta = 1 - \frac{lq}{r}$.

Proof. Suppose that $f \in L_q(\gamma)$ and $k \in K(\mu, r, q, l)$. Let $y_0 \in I$ and $\epsilon > 0$. We define the functions f_j, $j \in \mathbb{N}$, as follows:

$$f_0(y) = f(y)\chi_{\{y\in I:\ |y-y_0|<2\epsilon\}}(y), \qquad y \in I,$$

and

$$f_j(y) = f(y)\chi_{\{y\in I:\ 2^j\epsilon\le|y-y_0|<2^{j+1}\epsilon\}}, \qquad y \in I \text{ and } j \in \mathbb{N}\setminus\{0\}.$$

It is clear that $f = \sum_{j=0}^{\infty} f_j$ on I, and that $T_k f = \sum_{j=0}^{\infty} k\#f_j$.

Since T_k is a bounded operator from $L_q(\gamma)$ into $L_{r,h_l}(\gamma)$, where $h_l(y) = y^{2(\mu+1)(l-1)}$, $y \in I$, a straightforward manipulation allows us to write

$$\begin{aligned}
&\frac{1}{\gamma(B_l(y_0,\epsilon))}\int_{B_l(y_0,\epsilon)} |(k\#f_0)(y)|\,d\gamma(y)\\
&\le \frac{1}{\gamma(B_l(y_0,\epsilon))}\left\{\int_{B_l(y_0,\epsilon)} y^{-2(\mu+1)r'\frac{l-1}{r}}\,d\gamma(y)\right\}^{1/r'}\left\{\int_0^\infty \left|y^{2(\mu+1)\frac{l-1}{r}}(k\#f_0)(y)\right|^r d\gamma(y)\right\}^{1/r}\\
&\le C\left\{\frac{1}{\gamma(B(y_0,2\epsilon))^{1-\eta}}\int_{B(y_0,2\epsilon)} |f(y)|^q\,d\gamma(y)\right\}^{1/q},
\end{aligned}$$

where $\eta = 1 - \frac{lq}{r}$.

Moreover for every $j \in \mathbb{N}\setminus\{0\}$ we have

$$\begin{aligned}
(k\#f_j)(y) &= (k\#f_j)(y_0) + \int_0^\infty [(\tau_y k)(z) - (\tau_{y_0}k)(z)]f_j(z)\,d\gamma(z)\\
&= c_j + \epsilon_j, \qquad y \in I.
\end{aligned}$$

Note that c_j, $j \in \mathbb{N}\setminus\{0\}$, does not depend on $y \in I$. Then, for every $j \in \mathbb{N}\setminus\{0\}$,

$$\begin{aligned}
|\epsilon_j| &\le \int_{2^j\epsilon\le|z-y_0|<2^{j+1}\epsilon} |(\tau_y k)(z) - (\tau_{y_0}k)(z)|\,|f(z)|\,d\gamma(z)\\
&\le \left\{\int_{2^j\epsilon\le|z-y_0|<2^{j+1}\epsilon} |(\tau_y k)(z)-(\tau_{y_0}k)(z)|^{q'}\,d\gamma(z)\right\}^{1/q'}\left\{\int_{B(y_0,2^{j+1}\epsilon)} |f(z)|^q\,d\gamma(z)\right\}^{1/q}\\
&\le CS\left(\frac{\rho_l(y,y_0)}{2^j\epsilon}\right)\left\{\frac{1}{\gamma(B(y_0,2^{j+1}\epsilon))^{1-\eta}}\int_{B(y_0,2^{j+1}\epsilon)} |f(z)|^q\,d\gamma(z)\right\}^{1/q},
\end{aligned}$$

when $\rho_l(y, y_0) < 2^{j-1}\epsilon$. Here, as above, $\eta = 1 - \frac{lq}{r}$.

In particular, if $\rho_l(y, y_0) < \epsilon$ we have

$$|\epsilon_j| \le CS\left(\frac{1}{2^j}\right)[M_\eta(|f|^q)(y_0)]^{1/q}, \qquad j \in \mathbb{N}\setminus\{0\}.$$

Hence we conclude that

$$\begin{aligned}\frac{1}{\gamma(B_l(y_0,\epsilon))}&\int_{B_l(y_0,\epsilon)}|(k\#f)(y)-\sum_{j=1}^{\infty}c_j|\,d\gamma(y)\\ &\le \frac{1}{\gamma(B_l(y_0,\epsilon))}\int_{B_l(y_0,\epsilon)}|(k\#f_0)(y)|\,d\gamma(y)\\ &\quad+\sum_{j=1}^{\infty}\frac{1}{\gamma(B_l(y_0,\epsilon))}\int_{B_l(y_0,\epsilon)}|(k\#f_j)(y)-c_j|\,d\gamma(y)\\ &\le C\left(\sum_{j=1}^{\infty}S\left(\frac{1}{2^j}\right)+1\right)\left(M_\eta(|f|^q)(y_0)\right)^{1/q}.\end{aligned}$$

Then it follows that

$$M_l^{\#}(T_kf)(y_0)\le C(M_\eta(|f|^q)(y_0))^{1/q}.$$

Thus the proof of the proposition is finished. □

3. Proofs of theorems

In this section we prove Theorems 1.2, 1.3 and 1.4. We recall as mentioned in Section 1, that Theorem 1.1 is an immediate consequence of [5, Theorem 2.4].

Proof of Theorem 1.2. The proof of Theorem 1.2 follows the original one of Hörmander [11] and the one due to Gosselin and Stempak ([7, Theorem 1.1]).

Since m is a bounded function on I, from Theorem 3 in [9] it follows that the Hankel multiplier operator $\mathcal{M}_m$ associated to m is bounded from $L_2(\gamma)$ into $L_2(\gamma)$.

Let ψ be in $C^\infty(I)$ such that the support of ψ is contained in $(\frac{1}{2},2)$ and $\sum_{j=-\infty}^{\infty}\psi(2^{-j}x)=1, x\in I$. Define the functions ψ_j, m_j and k_j, $j\in\mathbb{Z}$, associated to m and ψ, by $\psi_j(x)=\psi(2^{-j}x)$, $m_j(x)=m(x)\psi_j(x)$ and $k_j(x)=h_\mu(m_j)(x)$, $x\in I$ and $j\in\mathbb{Z}$.

By virtue of Theorem 1.1, to see that $\mathcal{M}_m$ is a bounded operator from $L_p(\gamma)$ into itself, for every $1<p<2$, and $\mathcal{M}_m$ is of weak type (1,1) it is sufficient to prove that

$$\sum_{j=-\infty}^{\infty}\int_{|x-z|>2|y-x|}|(\tau_xk_j)(z)-(\tau_yk_j)(z)|\,d\gamma(z)\le C,\qquad x,y\in I,\tag{5}$$

for a certain $C>0$ that does not depend on $x, y\in I$.

In effect, assume that (5) holds. Define

$$R_n=\sum_{j=-n}^{n}k_j\quad\text{for every } n\in\mathbb{N}.$$

According to Theorem 1.1, for every $p \in [1,2]$ there exists $C_p > 0$ such that for every $n \in \mathbb{N}$,

$$\|T_{R_n} f\|_p \leq C_p \|f\|_p, \qquad f \in L_p(\gamma),\ 1 < p \leq 2, \tag{6}$$

and

$$\gamma\left[\{x \in I\colon |T_{R_n} f(x)| > \lambda\}\right] \leq C_1 \frac{\|f\|_1}{\lambda}, \qquad f \in L_1(\gamma), \lambda > 0. \tag{7}$$

Since $z^{-\mu} J_\mu(z)$ is a bounded function on I, we can write

$$\sup_{x \in I} |(\mathcal{M}_m f - T_{R_n} f)(x)| \leq C \left\| \left(m - \sum_{j=-n}^{n} m_j \right) h_\mu f \right\|_1 \qquad \text{for every } f \in C_0.$$

Moreover, since $\lim_{n\to\infty} \sum_{j=-n}^{n} m_j(x) = m(x)$, $x \in I$, and there exists a positive constant C such that $|\sum_{j=-n}^{n} m_j(x)| \leq C$, $n \in \mathbb{N}$, $x \in I$, we conclude that for each $f \in C_0$,

$$T_{R_n} f \longrightarrow \mathcal{M}_m f, \text{ as } n \to \infty, \text{ uniformly in } I.$$

Hence, from (6) and (7) it follows that

$$\|\mathcal{M}_m f\|_p \leq C_p \|f\|_p, \qquad f \in C_0, 1 < p \leq 2,$$

and

$$\gamma\left[\{x \in I\colon |\mathcal{M}_m f(x)| > \lambda\}\right] \leq C_1 \frac{\|f\|_1}{\lambda}, \qquad f \in C_0, \lambda > 0.$$

The theorem is established, in these cases, by extending $\mathcal{M}_m$ to $L_p(\gamma)$ by density.

To see that $\mathcal{M}_m$ defines a bounded operator from $L_p(\gamma)$ into itself, when $p > 2$, it is sufficient to use duality.

We now prove (5).

Let $j \in \mathbb{Z}$ and $x, y \in I$. As in [7, p. 661] we can write

$$\int_{|x-z|>2|y-x|} |(\tau_x k_j)(z) - (\tau_y k_j)(z)|\, d\gamma(z) \leq 2 \int_{|y-x|}^{\infty} |k_j(z)|\, d\gamma(z). \tag{8}$$

Let $t > 0$. Hölder's inequality leads to

$$\begin{aligned} \int_t^\infty |k_j(z)|\, d\gamma(z) &\leq C \|(2^j z)^{2s} k_j\|_{r'} \left\{ \int_t^\infty (2^j z)^{-2sr}\, d\gamma(z) \right\}^{1/r} \\ &\leq C \|(1 + (2^j z)^2)^s k_j\|_{r'} 2^{-2sj} t^{2\frac{\mu+1}{r} - 2s} \end{aligned} \tag{9}$$

provided that $s > \frac{\mu+1}{r}$.

Let Δ_μ denote the Bessel differential operator $x^{-2\mu-1}Dx^{2\mu+1}D$. According to a well-known operational rule for the Hankel transformation [19, (5) Lemma 5.4-1], and by [9, Theorem 3], it follows that

$$\begin{aligned}\|(1+(2^j z)^2)^s k_j\|_{r'} &= \|h_\mu[(1-2^j\Delta_\mu)^s m_j]\|_{r'} \\ &\leq C\|(1-2^{2j}\Delta_\mu)^s m_j\|_r \leq C\sum_{i=0}^{s}\binom{s}{i}2^{2ij}\|\Delta_\mu^i m_j\|_r. \qquad (10)\end{aligned}$$

Moreover, for every $i \in \mathbb{N}$, $\Delta_\mu^i f = \sum_{k=0}^{i} a_{k,i}x^{2k}(\frac{1}{x}D)^{k+i}f$, where $a_{k,i}$, $k = 0, \ldots, i$, denote suitable real numbers. Hence, by (2) one has

$$\begin{aligned}\|\Delta_\mu^i m_j\|_r &\leq C\sum_{k=0}^{i}|a_{k,i}|\left\{\int_{2^{j-1}}^{2^{j+1}}\left|x^{2k}\left(\frac{1}{x}D\right)^{k+i} m_j(x)\right|^r d\gamma(x)\right\}^{1/r} \\ &\leq C\sum_{k=0}^{i}\sum_{\alpha=0}^{k+i}\left\{\int_{2^{j-1}}^{2^{j+1}}\left|x^{2k}\left(\frac{1}{x}D\right)^{\alpha}\psi_j(x)\left(\frac{1}{x}D\right)^{k+i-\alpha} m(x)\right|^r d\gamma(x)\right\}^{1/r} \\ &\leq C\sum_{k=0}^{i}\sum_{\alpha=0}^{k+i}\left\{\int_{2^{j-1}}^{2^{j+1}}\left|\left(\frac{1}{x}D\right)^{k+i-\alpha} m(x)\right|^r d\gamma(x)\right\}^{1/r} 2^{2j(k-\alpha)} \\ &\leq C2^{2j(-i+\frac{\mu+1}{r})}, \qquad i = 0, \ldots, s. \qquad (11)\end{aligned}$$

By combining (9), (10) and (11) we can conclude that

$$\int_t^\infty |k_j(z)|\,d\gamma(z) \leq C(2^j t)^{2(\frac{\mu+1}{r}-s)}, \qquad (12)$$

where C does not depend on t and j.

Hence, from (8) and (12) it follows that

$$\int_{|x-z|>2|y-x|}|(\tau_x k_j)(z)-(\tau_y k_j)(z)|\,d\gamma(z) \leq C(2^j|y-x|)^{2(\frac{\mu+1}{r}-s)}. \qquad (13)$$

Also, according to Bernstein's inequality (for the Hankel transform) [7, Corollary 2.2], it follows that

$$\int_{|x-z|>2|y-x|}|(\tau_x k_j)(z)-(\tau_y k_j)(z)|\,d\gamma(z) \leq C\|\tau_x k_j - \tau_y k_j\|_1 \leq C2^{j+1}|y-x|\;\|k_j\|_1.$$

Hölder's inequality allows us to write

$$\begin{aligned} \|k_j\|_1 &\le \|(1+(2^j z)^2)^{-s}\|_r \|(1+(2^j z)^2)^s k_j\|_{r'} \\ &\le C 2^{-2j(\mu+1)/r} \|(1+(2^j z)^2)^s k_j\|_{r'} \end{aligned}$$

because $s > \frac{\mu+1}{r}$.

By invoking (10) and (11) again we conclude that $\|k_j\|_1 \le C$, where C does not depend on j, and then

$$\int_{|x-z|>2|y-x|} |(\tau_x k_j)(z) - (\tau_y k_j)(z)|\, d\gamma(z) \le C 2^j |y-x|. \tag{14}$$

Now from (13) and (14) it follows that

$$\begin{aligned} &\sum_{j=-\infty}^{\infty} \int_{|x-z|>2|y-x|} |(\tau_x k_j)(z) - (\tau_y k_j)(z)|\, d\gamma(z) \\ &= \left(\sum_{\{j\in\mathbb{Z}:\ 2^j|y-x|\ge 1\}} + \sum_{\{j\in\mathbb{Z}:\ 2^j|y-x|<1\}} \right) \int_{|x-z|>2|y-x|} |(\tau_x k_j)(z) - (\tau_y k_j)(z)|\, d\gamma(z) \\ &\le C \left(\sum_{\{j\in\mathbb{Z}:\ 2^j|y-x|\ge 1\}} (2^j|y-x|)^{2(\frac{\mu+1}{r-s})} + \sum_{\{j\in\mathbb{Z}:\ 2^j|y-x|<1\}} 2^j|y-x| \right) \le C. \end{aligned}$$

Thus (5) is established. □

Proof of Theorem 1.3. To prove Theorem 1.3 it is sufficient to use Propositions 2.1, 2.2 and 2.3. □

Proof of Theorem 1.4. First, note that $h_\mu(m) \in L_\infty$, since $m \in L_1(\gamma)$. Hence by (4), since $C_l \ge 2^{2(\mu+1)}$, we conclude that $h_\mu(m) \in L_1(\gamma)$ and then $T_{h_\mu(m)} = \mathcal{M}_m$.

Let $f \in C_0$ and let $a > 0$ be such that $f(x) = 0$, $x > a$. We now see that $T_{h_\mu(m)} f \in L_{p,w}(\gamma)$. It is clear that

$$\|T_{h_\mu(m)} f\|_{p,w}^p = \left(\int_0^{2a} + \int_{2a}^{\infty} \right) |T_{h_\mu(m)} f(x)|^p w(x)\, d\gamma(x) = I + J.$$

By Hölder's inequality, it follows, from [10, Theorem 2.b] that for every $r > 1$,

$$\begin{aligned} |I| &\le \left\{ \int_0^{2a} |T_{h_\mu(m)} f(x)|^{pr'}\, d\gamma(x) \right\}^{\frac{1}{r'}} \left\{ \int_0^{2a} w(x)^r\, d\gamma(x) \right\}^{\frac{1}{r}} \\ &\le \|f\|_{pr'}^p \left\{ \int_0^{2a} w(x)^r\, d\gamma(x) \right\}^{\frac{1}{r}}. \end{aligned}$$

Hence, by choosing r suitably [3, Theorem 1] we conclude that $|I| < \infty$. To estimate J we start noting that according to (4) and [10, (2)],

$$\begin{aligned}|(\tau_x h_\mu(m))(y)| &\leq \int_{|x-y|}^{x+y} D_\mu(x,y,z)|h_\mu(m)(z)|\,d\gamma(z)\\ &\leq C|x-y|^{-\alpha}, \qquad x,y \in I.\end{aligned}$$

Then

$$|(T_{h_\mu(m)}f)(x)| \leq C\int_0^a \frac{|f(y)|}{|x-y|^\alpha}d\gamma(y) \leq Cx^{-\alpha}, \qquad x > 2a.$$

Hence, we can write

$$\begin{aligned}|J| &\leq \int_{2a}^\infty |(T_{h_\mu(m)}f)(x)|^p w(x)\,d\gamma(x)\\ &\leq C\int_{2a}^\infty x^{-\alpha p} w(x)\,d\gamma(x).\end{aligned}$$

Now, by using [3, Lemma 4] and by proceeding as in the proof of [17, Proposition 4.5(iv)], it follows that $|J| < \infty$.

Thus we conclude that $T_{h_\mu(m)}f \in L_{p,w}(\gamma)$. By invoking [3, Theorem 3] it follows that

$$M^l(T_{h_\mu(m)}f) \in L_{p,w}(\gamma).$$

By virtue of Theorem 1.3 to establish Theorem 1.4 it is enough to prove that

$$\begin{aligned}&\left\{\sum_{j=-\infty}^{\infty}\int_{R<|x-y_0|<2R}|(\tau_y k_j)(x)-(\tau_{y_0}k_j)(x)|^{q'}\,d\gamma(x)\right\}^{1/q'}\\ &\qquad\leq CS\left(\frac{\rho_l(y,y_0)}{R}\right)R^{-2(\mu+1)/q},\end{aligned}$$

when $\rho_l(y,y_0) < \frac{R}{2}$ and $R > 0$, and for some $l \in \mathbb{N}\setminus\{0\}$, and some non-decreasing function S defined on $(0,1)$ such that $\sum_{j=1}^\infty S(2^{-j}) < \infty$. Here ψ, k_j, ψ_j and m_j, $j \in \mathbb{Z}$, are as in the proof of Theorem 1.2.

Let $R > 0$, $y, y_0 \in I$ and $j \in \mathbb{Z}$. We have

$$\begin{aligned}&\left\{\int_{R<|x-y_0|<2R}|(\tau_y k_j)(x)-(\tau_{y_0}k_j)(x)|^{q'}\,d\gamma(x)\right\}^{1/q'}\\ &\leq \left\{\int_{R<|x-y_0|<2R}|(\tau_y k_j)(x)|^{q'}\,d\gamma(x)\right\}^{1/q'} + \left\{\int_{R<|x-y_0|<2R}|(\tau_{y_0}k_j)(x)|^{q'}\,d\gamma(x)\right\}^{1/q'}.\end{aligned}$$

In the sequel we consider $l = 2(d+1)$. If $\rho_l(y, y_0) < \frac{R}{2}$, Jensen's inequality leads to

$$\begin{aligned}\int_{R<|x-y_0|<2R} &|(\tau_y k_j)(x)|^{q'}\,d\gamma(x)\\ &\le \int_{R<|x-y_0|<2R}\int_{|x-y|}^{x+y} D_\mu(x,y,z)|k_j(z)|^{q'}\,d\gamma(z)d\gamma(x)\\ &\le \int_{R/2}^{\infty}|k_j(z)|^{q'}\int_0^\infty D_\mu(x,y,z)\,d\gamma(x)\,d\gamma(z)\\ &= \int_{R/2}^{\infty}|k_j(z)|^{q'}\,d\gamma(z).\end{aligned}$$

Also, we can see that

$$\int_{R<|x-y_0|<2R}|(\tau_{y_0}k_j)(x)|^{q'}\,d\gamma(x) \le \int_{R/2}^{\infty}|k_j(z)|^{q'}\,d\gamma(z).$$

Hence, one has

$$\int_{R<|x-y_0|<2R}|(\tau_y k_j)(x)-(\tau_{y_0}k_j)(x)|^{q'}\,d\gamma(x) \le 2\int_{R/2}^{\infty}|k_j(z)|^{q'}\,d\gamma(z)$$

provided that $\rho_l(y, y_0) < \frac{R}{2}$.

By [9, Theorem 3], the operational rule [19, (5) Lemma 5.4-1] and Hölder's inequality, we get

$$\begin{aligned}\left\{\int_{R/2}^{\infty}|k_j(z)|^{q'}\,d\gamma(z)\right\}^{1/q'} &= \left\{\int_{R/2}^{\infty}|k_j(z)z^{2d}|^{q'}z^{-2dq'}d\gamma(z)\right\}^{1/q'}\\ &\le CR^{-2[d-(\mu+1)(\frac{1}{r}-\frac{1}{q})]}\left\{\int_0^\infty|h_\mu(\Delta_\mu^d m_j)(z)|^{r'}\,d\gamma(z)\right\}^{1/r'}\\ &\le CR^{-2[d-(\mu+1)(\frac{1}{r}-\frac{1}{q})]}\left\{\int_0^\infty|\Delta_\mu^d m_j(x)|^{r}\,d\gamma(x)\right\}^{1/r}\end{aligned}$$

when $q \ge r$, $1 < r \le 2$ and $d \in \mathbb{N}$, $d > (\mu+1)(\frac{1}{r}-\frac{1}{q})$.

By proceeding as in the proof of Theorem 1.2 it follows that

$$\left\{\int_0^\infty|\Delta_\mu^d m_j(x)|^{r}\,d\gamma(x)\right\}^{1/r} \le C2^{2j(\frac{\mu+1}{r}-d)}.$$

Hence, if $\rho_l(y, y_0) < \frac{R}{2}$, then

$$\left\{\int_{R<|x-y_0|<2R} |(\tau_y k_j)(x) - (\tau_{y_0} k_j)(x)|^{q'} \, d\gamma(x)\right\}^{1/q'}$$
$$\le 2\left\{\int_{R/2}^{\infty} |k_j(z)|^{q'} \, d\gamma(z)\right\}^{1/q'}$$
$$\le CR^{-2[d-(\mu+1)(\frac{1}{r}-\frac{1}{q})]} 2^{2j(\frac{\mu+1}{r}-d)} \tag{15}$$

provided that $q \ge r$, $1 < r \le 2$ and $d \in \mathbb{N}$ with $d > (\mu+1)(\frac{1}{r} - \frac{1}{q})$.

On the other hand, by invoking [9, Theorem 3] and [19, Lemma 5.4-1] again, we have

$$\left\{\int_{R<|x-y_0|<2R} |(\tau_y k_j)(x) - (\tau_{y_0} k_j)(x)|^{q'} \, d\gamma(x)\right\}^{1/q'}$$
$$\le CR^{-2[d-(\mu+1)(\frac{1}{r}-\frac{1}{q})]} \left\{\int_0^{\infty} |\Delta_\mu^d [m_j(x)((xy)^{-\mu} J_\mu(xy) - (xy_0)^{-\mu} J_\mu(xy_0))]|^r \, d\gamma(x)\right\}^{1/r},$$

with $q \ge r$, $1 < r \le 2$ and $d \in \mathbb{N}$, $d > (\mu+1)(\frac{1}{r} - \frac{1}{q})$.

Now, by taking into account the fact that $(\frac{1}{z}D)[z^{-\mu}J_\mu(z)] = -z^{-\mu-1}J_{\mu+1}(z)$, $z \in I$, that the function $z^{-\mu}J_\mu(z)$ is bounded on I and that $\Delta_\mu^i f(x) = \sum_{k=0}^{i} a_{k,i} x^{2k} (\frac{1}{x}D)^{k+i} f(x)$, where $i \in \mathbb{N}$ and $a_{k,i}$, $k = 0, \dots, i$, denotes suitable real numbers, we can conclude that

$$\left\{\int_{R<|x-y_0|<2R} |(\tau_y k_j)(x) - (\tau_{y_0} k_j)(x)|^{q'} \, d\gamma(x)\right\}^{1/q'}$$
$$\le CR^{-2(d-(\mu+1)(\frac{1}{r}-\frac{1}{q}))} \cdot \sum_{d \le \alpha+\beta \le 2d} \left(\int_0^{\infty} \left| x^{2(\alpha+\beta-d)} \left(\frac{1}{x}D\right)^{\alpha} (m_j(x)) \left(\frac{1}{x}D\right)^{\beta} [(xy)^{-\mu}J_\mu(xy) - (xy_0)^{-\mu}J_\mu(xy_0)]\right|^r d\gamma(t)\right\}^{1/r}$$
$$\le CR^{-2[d-(\mu+1)(\frac{1}{r}-\frac{1}{q})]} \sum_{1 \le \beta \le 2(d+1)} |y^{2\beta} - y_0^{2\beta}| 2^{2j(\beta-d+\frac{\mu+1}{r})}$$
$$\le CR^{-2[d-(\mu+1)(\frac{1}{r}-\frac{1}{q})]} \sum_{1 \le \beta \le l} \rho_l(y, y_0)^{2\beta} 2^{2j(\beta-d+\frac{\mu+1}{r})}$$

where $q \ge r$, $1 < r \le 2$ and $d \in \mathbb{N}$, $d > (\mu+1)(\frac{1}{r} - \frac{1}{q})$.

Hence, if $2^j\rho_l(y,y_0)\leq 1$ then

$$\left\{\int_{R<|x-y_0|<2R}|(\tau_y k_j)(x)-(\tau_{y_0}k_j)(x)|^{q'}\,d\gamma(x)\right\}^{1/q'} \leq C\rho_l(y,y_0)^2 2^{2j(1-d+\frac{\mu+1}{r})}R^{-2[d-(\mu+1)(\frac{1}{r}-\frac{1}{q})]} \tag{16}$$

with $q\geq r$, $1<r\leq 2$ and $d\in\mathbb{N}$, $d>(\mu+1)(\frac{1}{r}-\frac{1}{q})$.

By combining (15) and (16) we can obtain

$$\begin{aligned}\sum_{j=-\infty}^{\infty}&\left\{\int_{R<|x-y_0|<2R}|(\tau_y k_j)(x)-(\tau_{y_0}k_j)(x)|^{q'}\,d\gamma(x)\right\}^{1/q'}\\ &\leq\left(\sum_{\{j\in\mathbb{Z}:\ 2^j\rho_l(y,y_0)\geq 1\}}+\sum_{\{j\in\mathbb{Z}:\ 2^j\rho_l(y,y_0)<1\}}\right)\\ &\quad\times\int_{R<|x-y_0|<2R}|(\tau_y k_j)(x)-(\tau_{y_0}k_j)(x)|^{q'}\,d\gamma(x)\Bigg\}^{1/q'}\\ &\leq CR^{2[(\mu+1)(\frac{1}{r}-\frac{1}{q})-d]}\rho_l(y,y_0)^{2(d-\frac{\mu+1}{r})},\end{aligned} \tag{17}$$

when $\rho_l(y,y_0)<\frac{R}{2}$, $q\geq r$, $1<r\leq 2$ and $\frac{\mu+1}{r}<d<\frac{\mu+1}{r}+1$.

Hence by defining $S(\epsilon)=\epsilon^{2(d-\frac{\mu+1}{r})}$, $\epsilon\in(0,1)$, (17) can be rewritten as

$$\sum_{j=-\infty}^{\infty}\left\{\int_{R<|x-y_0|<2R}|(\tau_y k_j)(x)-(\tau_{y_0}k_j)(x)|^{q'}\,d\gamma(x)\right\}^{1/q'} \leq CS\left(\frac{\rho_l(y,y_0)}{R}\right)R^{-2(\mu+1)/q}, \tag{18}$$

for $\rho_l(y,y_0)<\dfrac{R}{2}$.

By taking into account the fact that $k=\sum_{j=-\infty}^{\infty}k_j$, from (18) we deduce that

$$\left\{\int_{R<|x-y_0|<2R}|(\tau_y k)(x)-(\tau_{y_0}k)(x)|^{q'}\,d\gamma(x)\right\}^{1/q'}\leq CS\left(\frac{\rho_l(y,y_0)}{R}\right)R^{-2(\mu+1)/q},$$

for $\rho_l(y,y_0)<\frac{R}{2}$.

Then, since T_k is bounded from $L_q(\gamma)$ into $L_{lq,h_l}(\gamma)$, $k\in K(\mu,lq,q,l)$, and the proof of Theorem 1.4 can be finished by using Theorem 1.3. □

Acknowledgement. We are indebted to Professor K. Stempak for comments concerning an earlier draft of this paper and for turning our attention to the paper of B. Muckenhoupt, R. Wheeden and W-S. Young [14].

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Departamento de Análisis Matemático, Universidad de La Laguna, 38271 - La Laguna, Tenerife, Islas Canarias, España
jbetanco@ull.es
lrguez@ull.es