BORELSTRUCTURES FOR FUNCTION SPACES[1]

BY
ROBERT J. AUMANN

If X and Y are topological spaces, then Y^X denotes the set of all continuous mappings from X into Y. For a given topology on Y^X, we may ask whether the natural mapping $\varphi: Y^X \times X \to Y$ defined by $\varphi(f, x) = f(x)$ is continuous; if it is, then the topology on Y^X is said to be *admissible* [1]. It is always possible to find an admissible topology; for instance, the discrete topology on Y^X is always admissible. Moreover, when X is locally compact, Y^X has a unique smallest[2] admissible topology; this is the familiar "compact-open" topology. These and related questions concerning topologies for function spaces have been investigated in considerable detail by several authors [1, 4].

We are interested in the analogous situation when X and Y are Borel spaces[3] rather than topological spaces; in this case we define Y^X as the set of all Borel mappings[4] from X into Y. Unfortunately, it turns out that even for some of the simplest Borel spaces, it is impossible to define a Borel structure on Y^X so that φ is a Borel mapping; even if we impose the discrete structure on Y^X, φ will in general not be Borel. As a substitute, we may ask ourselves the following questions: For which *subsets* F of Y^X is it possible to impose a Borel structure on F so that $\varphi \mid F \times X$ will be Borel? If is is possible for a given F, what can we say about the appropriate structures? In particular, is there always a smallest such structure (corresponding to the compact-open topology)?

Let us introduce some terminology. We will write "space" instead of "Borel space", "structure" instead of "Borel structure", and φ_F instead of $\varphi \mid F \times X$. A structure R on F for which φ_F is Borel will be called *admissible*; a subset F of Y^X on which it is possible to impose an admissible structure is

Received November 14, 1960.

[1] The results proved here were announced in [2]. The author is grateful to Prof. P. R. Halmos for many helpful suggestions, and in particular for the counterexample in Section 7; also to Dr. M. Rabin for a number of helpful discussions.

[2] I.e., weakest, with fewest open sets. We remark that the local compactness condition on X may be replaced by certain other conditions on X and Y; see [1, 4].

[3] A *Borel space* is a set X together with a σ-ring of subsets of X called *Borel sets* whose union is all of X. The σ-ring of Borel sets is called the *Borel structure* of X, or simply its *structure*. The structure of the cartesian product of two Borel spaces X and Y is taken to be that generated by the *Borel rectangles*—the products of a Borel set in X and a Borel set in Y. Our definition of Borel space is slightly more general than Mackey's definition [7], in which it is demanded that the structure be a σ-field rather than a σ-ring; as it turns out, most of our theorems and examples refer to the more restricted kind of space anyway. In [5] and [2], the word "measurable" is used in the sense that "Borel" is used here.

[4] A *Borel mapping* is a mapping such that the inverse image of every Borel set is Borel. It is called "Borel function" in [7] and "measurable transformation" in [2, 5].

also called *admissible*. We will be chiefly concerned with characterizing, for given X and Y, the admissible sets F and the admissible structures on them.

We first state our theorems, then give some illustrations and applications. The following three general theorems may be established fairly easily:

THEOREM A. *A set consisting of a single Borel mapping is admissible.*

THEOREM B. *A subset of an admissible set is admissible. Indeed, if $G \subset F$, R is an admissible structure on F, and R_G is the subspace structure on G induced*[5] *by R, then R_G is admissible on G.*

THEOREM C. *The union of denumerably many admissible sets is admissible. Indeed, if $F = \bigcup_{i=1}^{\infty} F_i$ and $R_1, R_2, \cdots$ are admissible structures on $F_1, F_2, \cdots$ respectively, then the structure R on F generated by the members of all the R_i is admissible on G.*

Much more can be said if X and Y are assumed to be *separable*, i.e., to have structures with countable generating families.[6] To state our theorems in this case, we need the concept of Banach class, closely related to that of Baire class. Let X be a space, $\mathfrak{A}$ a countable generating family for its structure. For each denumerable ordinal number $\alpha \geqq 1$, we will define a family $Q_\alpha(\mathfrak{A})$ of Borel subsets of X; roughly $Q_\alpha(\mathfrak{A})$ consists of all those sets that can be constructed from $\mathfrak{A}$ by means of at most α operations, where each operation consists of forming a denumerable union and a complement. Thus the union of all the $Q_\alpha(\mathfrak{A})$ is precisely the structure of X. Now let $\mathfrak{B}$ be a countable generating family for Y, and for each denumerable ordinal $\alpha \geqq 0$, define the *Banach class*[7] $L_\alpha(\mathfrak{A}, \mathfrak{B})$ to be the family of all functions $f: X \to Y$ such that for all $B \in Q_1(\mathfrak{B})$, $f^{-1}(B) \in Q_{\alpha+1}(\mathfrak{A})$. The union of all the Banach classes is precisely Y^X. If X and Y are separable metric spaces and Y is pathwise connected, and if $\mathfrak{A}$ and $\mathfrak{B}$ are appropriately chosen, then the Banach classes coincide with the Baire classes.[8]

Let $F \subset Y^X$. If we fix α but do not specify $\mathfrak{A}$ and $\mathfrak{B}$, then of course we can not say whether or not F is included in the α^{th} Banach class $L_\alpha(\mathfrak{A}, \mathfrak{B})$. However, we will be interested not so much in the question of whether or not F is in a Banach class of a specified order, but rather of whether there exists a Banach class of *any* order which includes F. The answer to *this* question is independent of the choice of $\mathfrak{A}$ and $\mathfrak{B}$. In other words, if $\mathfrak{A}$ and $\mathfrak{A}'$ are countable generating families for the structure of X, and $\mathfrak{B}$ and $\mathfrak{B}'$ for that of Y,

---

[5] R_G consists of all intersections of G with members of R.

[6] As in [7], a generating family $\mathfrak{A}$ for the structure of a space X is a set of Borel subsets of X with the property that every σ-ring containing $\mathfrak{A}$ is the structure of X. The term "separable" is used by analogy with its use in topology; we will apply it indiscriminately to the space and to the structure.

[7] After the work that Banach [3] did in characterizing these families.

[8] In this case the structures of X and Y are taken to be those generated by the closed sets, and $\mathfrak{A}$ and $\mathfrak{B}$ are taken to consist of the open spheres with rational radius.

and if α is a denumerable ordinal such that $F \subset L_\alpha(\mathfrak{A}, \mathfrak{B})$, then there is a denumerable ordinal α' such that $F \subset L_{\alpha'}(\mathfrak{A}', \mathfrak{B}')$. In this case we will say that F is of *bounded* Banach class; this concept depends on X and Y only, not on any particular choice of countable generating families for their structures.

THEOREM D. *Assume that X and Y are separable. Then a necessary and sufficient condition for a subset of Y^X to be admissible is that it be of bounded Banach class.*

THEOREM E. *If X and Y are separable, then every admissible subset of Y^X has a separable admissible structure.*

A space Z and its structure are called *regular*[9] if for all $x, y \in Z$, there is a Borel set in Z containing x but not y. It is known [7] that a space is separable and regular if and only if it is isomorphic[10] to a subspace of I, where I denotes the unit interval [0, 1] with the usual Borel structure.[11]

THEOREM F. *If X and Y are separable and regular, then every admissible subset of Y^X has a separable and regular admissible structure.*

The *natural* admissible structure on a given admissible set F is defined to be the smallest admissible structure on F, if it exists. Alternatively, it may be defined to be the intersection of all the admissible structures on F, in case this is admissible. Not every admissible set need have a natural admissible structure; the counterexample, which is due to P. R. Halmos, is given in Section 7.

If $a \in X$ and $B \subset Y$, define $F(a, B) = \{f : f \in F, f(a) \in B\}$. It is not hard to prove that if B is Borel and a is arbitrary, then every admissible structure on F must contain $F(a, B)$. A "converse" would be that the structure generated by the $F(a, B)$ is admissible, and it would follow that it is also natural. This "converse" is not in general true; the best we have been able to establish is the following:

THEOREM G. *If X and Y are separable metric spaces and F contains continuous functions only, then F has a natural admissible structure, which is generated by the set of all $F(a, B)$, where B is Borel and a is arbitrary.*

We now give some applications. A space is said to have the *discrete* structure if every subset is Borel. Let J be the space consisting of 0 and 1 only, and K the space of all positive integers, both with the discrete structure. If X is an arbitrary space, then X^J and X^K are both admissible, and possess

[9] Mackey [7] calls this a "separated" space. We do not use this term because we wish to avoid confusion with "separable".

[10] Two spaces are *isomorphic* if there is a one-one correspondence between them that sends Borel sets into Borel sets (in both directions).

[11] Mackey [7] uses the term "countably generated" for what we call "separable and regular" spaces.

natural admissible structures which make them isomorphic to $X \times X$ and $\times_{i=1}^{\infty} X_i$ respectively, where the X_i are copies of X. In particular, J^K is admissible and has a natural admissible structure which makes it isomorphic to I. These results are relatively trivial or at least easily derivable from known results.

The situation changes when we pass to exponent spaces with nondiscrete structures. For example, J^I may be considered the set of all Borel subsets of I. It is not itself admissible. The set of all open subsets of I is admissible, as is the set of all closed subsets, the set of all G_δ, etc. In general, a subset F of J^I is admissible if and only if all members of F can be constructed from the open subsets of I by taking denumerable unions and intersections at most α times, where α is an arbitrary denumerable ordinal number (which is fixed for given F, but may differ for different F). Whether or not every admissible subset of J^I has a natural admissible structure remains an open question; but if F is admissible, then we may endow it with an admissible structure in such a way that it will be isomorphic to a subset of I.

I^I is not admissible. The set of all continuous functions from I into I is admissible; more generally, a necessary and sufficient condition that a subset F of I^I be admissible is that there exist a denumerable ordinal number α such that all members of F are of Baire class α at most. The set H of all continuous functions from I into I has a natural admissible structure; it is the Borel structure of H when considered as a metric space (in the uniform convergence topology). Again, whether or not every admissible subset of I^I has a natural admissible structure remains an open question; but if F is admissible, we may endow it with an admissible structure in such a way that it will be isomorphic to a subset of I.

Section 1 is devoted to a brief summary of terminology and to proving Theorems A, B, and C. In Section 2 we give the precise definition of Banach class and justify the remarks about these classes made above. Sections 3 and 4 are devoted to a proof of Theorem D when it is assumed that X and Y are regular as well as separable; in Section 3 we also establish the inadmissibility of J^I and I^I. In Sections 5 and 6 we prove Theorems F and G respectively. Section 7 is devoted to Halmos's counterexample. Finally, in Section 8 we prove Theorem E and remove the regularity restriction on the previous proof of Theorem D.

1. Theorems A, B, and C

We first lay down a number of conventions to which we will adhere throughout Sections 1 through 8. "Countable" and "denumerable" will mean "of cardinality at most $\aleph_0$". α, β, and γ will denote denumerable ordinal numbers, also when decorated with subscripts, primes, etc.; Ω will denote the first nondenumerable ordinal number. X, Y, and Z will denote spaces. The structures of X and Y will be denoted S and T respectively. F will be a set of Borel mappings from X into Y. In unquantified statements, the universal quantifier is to be understood. The symbol ▌ will signal the end of a proof

In addition to the conventions laid down above, we will occasionally make use of conventions which will be valid only throughout a section or a part of it. The rule is that a convention stated within the statement or proof of a lemma is valid only until the proof is completed, and all other conventions are valid for the remainder of the section.

Let U and V be two σ-rings, not necessarily on the same abstract space. A function ψ from U into V is called a *homomorphism* if for all $A_1, A_2, \cdots$ in U we have

$$\psi(\bigcup_{i=1}^{\infty} A_i) = \bigcup_{i=1}^{\infty} \psi(A_i) \quad \text{and} \quad \psi(A_1 - A_2) = \psi(A_1) - \psi(A_2).$$

If ψ is also one-one and its inverse is a homomorphism, then it is called an *isomorphism.*[12]

LEMMA 1.1 *If $\psi: U \to V$ is a homomorphism onto, and if Γ generates U, then $\psi(\Gamma)$ generates V.*

The lemma is easily verified.

Proof of Theorem A. Let $f \in Y^X$, and let B be a Borel subset of Y. Then $\varphi_{\{f\}}^{-1}(B) = \{f\} \times f^{-1}(B)$. Since f and B are both Borel, so is $f^{-1}(B)$. ▌

Proof of Theorem B. The set U of all Borel subsets of $F \times X$ and the set V of all subsets of $G \times X$ of the form $C \cap (G \times X)$, where C is a Borel subset of $F \times X$, are both σ-rings. The function $\psi: U \to V$ defined by $\psi(C) = C \cap (G \times X)$ is a homomorphism onto. The set Γ of all rectangles of the form $D \times A$, where D and A are Borel subsets of F and X respectively, generates U. Hence by Lemma 1.1, $\psi(\Gamma)$ generates V. Now $\psi(\Gamma)$ is the set of all rectangles of the form $D' \times A$, where D' and A are Borel subsets of G and X respectively. In other words, $\psi(\Gamma)$ generates the structure of $G \times X$. This structure is therefore identical with V. Thus for a set to be Borel in $G \times X$, it is necessary and sufficient that it be Borel in V.

Let B be a Borel subset of Y. Then

$$\begin{aligned} \varphi_G^{-1}(B) &= \{(f, x): f \in G, x \in X, f(x) \in B\} \\ &= \{(f, x): f \in F, x \in X, f(x) \in B\} \cap \{(f, x): f \in G\} \\ &= \varphi_F^{-1}(B) \cap (G \times X). \end{aligned}$$

Hence $\varphi_G^{-1}(B) \in V$, and hence it is a Borel subset of $G \times X$. Hence the structure R_G on G makes φ_G measurable; therefore it is admissible.

Proof of Theorem C. If we take R to be the structure of F, then the structure on $F \times X$ is generated by all sets of the form $G \times A$, where G is a Borel subset of some F_i, and A is Borel in X. Hence every set that is Borel in $F_i \times X$ is also Borel in $F \times X$. Now let B be a Borel subset of Y. Then

[12] This is a σ-ring isomorphism, not to be confused with a space isomorphism.

$$\begin{aligned}\varphi_F^{-1}(B) &= \{(f, x) : f \in F, f(x) \in B\} \\ &= \bigcup_{i=1}^{\infty} \{(f, x) : f \in F_i, f(x) \in B\} \\ &= \bigcup_{i=1}^{\infty} \varphi_{F_i}^{-1}(B);\end{aligned}$$

but since by hypothesis each of the $\varphi_{F_i}^{-1}(B)$ is Borel in $F_i \times X$, it follows that $\varphi_F^{-1}(B)$ is Borel in $F \times X$. Hence the structure R makes φ_F a Borel mapping, and therefore it is admissible.

COROLLARY 1.2. *Every denumerable subset of Y^X is admissible.*

2. Banach classes

The definition of Banach class was sketched in the introduction. In this section we give the precise definition, and establish the less obvious properties of Banach classes. It will be assumed that X and Y are separable.

Let $\mathfrak{A}$ be an arbitrary family of Borel subsets of X. For each denumerable ordinal $\alpha \geqq 1$, we define $P_\alpha(\mathfrak{A})$ and $Q_\alpha(\mathfrak{A})$ inductively as follows: $Q_1(\mathfrak{A})$ consists of all denumerable unions of members of $\mathfrak{A}$, and $P_1(\mathfrak{A})$ consists of all complements of members of $Q_1(\mathfrak{A})$; supposing $Q_\beta(\mathfrak{A})$ and $P_\beta(\mathfrak{A})$ to have been defined for all $\beta < \alpha$, we define

$$Q_\alpha(\mathfrak{A}) = Q_1(\bigcup_{\beta<\alpha} P_\beta(\mathfrak{A})) \quad \text{and} \quad P_\alpha(\mathfrak{A}) = P_1(\bigcup_{\beta<\alpha} P_\beta(\mathfrak{A})).$$

$Q_\alpha(\mathfrak{A}) \cup P_\alpha(\mathfrak{A})$ is the set of all subsets of X which can be "reached from $\mathfrak{A}$" by performing at most α operations, where each operation consists of forming a denumerable union and a complement; if $\mathfrak{A}$ generates the structure of X, then the union (over α) of all the $Q_\alpha(\mathfrak{A})$ (or of the $P_\alpha(\mathfrak{A})$) is the set of all Borel subsets of X. If $\mathfrak{B}$ is a family of Borel subsets of Y, we may define $P_\alpha(\mathfrak{B})$ and $Q_\alpha(\mathfrak{B})$ in a similar manner.

For the remainder of this section, let $\mathfrak{A}$ and $\mathfrak{B}$ denote countable generating families for the structures S of X and T of Y respectively. For each denumerable ordinal $\alpha \geqq 0$, we define $L_\alpha(\mathfrak{A}, \mathfrak{B})$ to be the set of all functions $f: X \to Y$ such that for all $B \in Q_1(\mathfrak{B})$, we have $f^{-1}(B) \in Q_{\alpha+1}(\mathfrak{A})$.

LEMMA 2.1. $Y^X = \bigcup_{\alpha<\Omega} L_\alpha(\mathfrak{A}, \mathfrak{B})$.

This lemma follows without difficulty from the following lemma, by setting $Z = X$.

LEMMA 2.2. *A necessary and sufficient condition that a mapping $f: Z \to Y$ be Borel is that for every $B \in \mathfrak{B}$, $f^{-1}(B)$ is Borel in Z.*

Proof. Necessity is obvious. To prove sufficiency, let U be the set of all subsets B of Y such that $f^{-1}(B)$ is Borel. U includes $\mathfrak{B}$ and is a σ-ring; hence $U \supset T$. ▌

Let $F \subset Y^X$. We shall say that F is of *bounded Banach class w.r.t.* $(\mathfrak{A}, \mathfrak{B})$ if there is an α such that $F \subset L_\alpha(\mathfrak{A}, \mathfrak{B})$.

LEMMA 2.3. *The concept of bounded Banach class is independent of the choice of countable generating families. In other words, if* $\mathfrak{A}'$ *and* $\mathfrak{B}'$ *are any other countable generating families for* S *and* T *respectively, and if* F *is of bounded Banach class w.r.t.* $(\mathfrak{A}, \mathfrak{B})$, *then it is also of bounded Banach class w.r.t.* $(\mathfrak{A}', \mathfrak{B}')$.

Because of Lemma 2.3, we can speak of F being of *bounded Banach class* without referring to the generating families.

We supply only the idea of the proof; the details may be filled in by the reader. Suppose $f \in L_\alpha(\mathfrak{A}, \mathfrak{B})$. Each member of $\mathfrak{B}'$ is Borel, and so can be "reached" in denumerably many steps from $\mathfrak{B}$; since $\mathfrak{B}'$ is denumerable, there is a denumerable upper bound on the number of steps needed. It follows that there is also such an upper bound, say γ, if we start out with $B' \in Q_1(\mathfrak{B}')$; that is, B' can then be reached in at most γ steps from $\mathfrak{B}$ (independent of the choice of a particular B'). Hence $f^{-1}(B')$ can be reached in γ steps from sets of the form $f^{-1}(B)$, where $B \in \mathfrak{B}$. But all such $f^{-1}(B)$ can be reached in $\alpha + 1$ steps from $\mathfrak{A}$; so $f^{-1}(B')$ can be reached in $\alpha + 1 + \gamma$ steps from $\mathfrak{A}$. Now each member of $\mathfrak{A}$ can be reached in denumerably many steps from $\mathfrak{A}'$, and again $\mathfrak{A}$ has only denumerably many members; so there s an upper bound on the number of steps necessary. Adding this upper bound to $\alpha + 1 + \gamma$, we obtain a denumerable α' such that

$$f^{-1}(B') \in Q_{\alpha'+1}(\mathfrak{A}').$$

Since α' is independent of the choice of B' and f, it follows that $f \in L_{\alpha'}(\mathfrak{A}', \mathfrak{B}')$ and $L_\alpha(\mathfrak{A}, \mathfrak{B}) \subset L_{\alpha'}(\mathfrak{A}', \mathfrak{B}')$. ▌

We end this section with the following lemma, whose proof may be supplied by the reader:

LEMMA 2.4. *If* $\alpha + 1 < \beta$, *then* $Q_\alpha(\mathfrak{A}) \subset Q_\beta(\mathfrak{A})$.

3. Theorem D: Necessity

We assume throughout this section that X and Y are separable, and fix denumerable generating families $\mathfrak{A}$ and $\mathfrak{B}$ for their respective structures. Z will be an arbitrary Borel space. We will prove that every admissible subset of Y^X is of bounded Banach class.

The principal tool in the proof is Lemma 3.2, which says that if B is a Borel subset of the cartesian product $X \times Z$, then there is an α (depending on B) such that every Z-section[13] of B may be constructed in at most α steps from $\mathfrak{A}$. The idea of the proof is that since B is Borel, there must be an α such that B may be constructed in at most α steps from rectangles in $X \times Z$ whose X-factors are in $\mathfrak{A}$. If we copy this construction step by step, but within a given Z-section of $X \times Z$, then we obtain the desired construction of the corresponding Z-section of B.

To illustrate the general necessity proof, we use Lemma 3.2 to show that

[13] A section parallel to the X-axis.

J^I is not admissible. If it were, then $\varphi^{-1}(1)$ would be Borel in $J^I \times I$, and so by Lemma 3.2, all the J^I-sections of $\varphi^{-1}(1)$ would be constructible in at most α steps from $\mathfrak{A}$, where α is fixed. Such a section consists of the set of all $x \in I$ such that $f(x) = 1$, where f is an arbitrary but fixed characteristic function; in other words, *any* Borel set may be represented as such a section, and so would be constructible in α steps from $\mathfrak{A}$, where α is fixed. Now this is known to be impossible if, for example, we take $\mathfrak{A}$ to be a denumerable basis for the open sets of I (cf. [6], p. 207). ∎

J^I may be considered a subset of I^I; so by Theorem B, I^I is not admissible either.

We now give the formal proofs. Write $Q_\alpha(\mathfrak{A}) = Q_\alpha$, $P_\alpha(\mathfrak{A}) = P_\alpha$, $L_\alpha(\mathfrak{A}, \mathfrak{B}) = L_\alpha$. If $B \subset X \times Z$, let B^z denote the Z-section $\{x \in X: (x, z) \in B\}$ of B. For each denumerable α, let α'' denote the largest limit ordinal no larger than α, i.e., the smallest $\beta \leqq \alpha$ such that $-\beta + \alpha$ is finite.[14] Let α' be the "finite tail" of α, i.e., $\alpha' = -\alpha'' + \alpha$.

Lemma 3.1. *$\beta < \gamma$ implies $\beta + \beta' + 1 < \gamma + \gamma'$.*

Proof. If $\beta < \gamma''$, then since γ'' is a limit ordinal and β' is finite, it follows that $\beta + \beta' + 1 < \gamma'' \leqq \gamma \leqq \gamma + \gamma'$. If $\beta \geqq \gamma''$, then $\beta' < \gamma'$, and hence $2\beta' + 1 < 2\gamma'$ (β' and γ' being finite). Hence

$$\beta + \beta' + 1 = (\beta - \beta') + 2\beta' + 1$$
$$= (\gamma - \gamma') + 2\beta' + 1 < \gamma - \gamma' + 2\gamma' = \gamma + \gamma'. \quad ∎$$

Lemma 3.2. *Let B be a Borel subset of $X \times Z$. Then there is an α such that every Z-section of B is in Q_α.*

Proof. We define N_α and M_α for each α as follows: N_1 is the set of all sets of the form $A \times C$, where $A \in Q_1$ and C is a Borel subset of Z; $M_1 = \{D_1 - D_2 : D_1, D_2 \in N_1\}$. Suppose N_β and M_β have been defined for all $\beta < \alpha$; define $N_\alpha = Q_1(\bigcup_{\beta<\alpha} M_\beta)$ and $M_\alpha = \{D_1 - D_2 : D_1, D_2 \in N_\alpha\}$.

We prove by transfinite induction on γ that $D \in N_\gamma$ implies that every Z-section of D is in $Q_{\gamma+\gamma'}$. For $\gamma = 1$ this follows from Lemma 3.1. Suppose we have shown it for all $\beta < \gamma$. Then if $\beta < \gamma$ and $D \in M_\beta$, we have $D = D_1 - D_2$, where $D_1, D_2 \in N_\beta$. Hence for arbitrary $z \in Z$, we have by induction hypothesis that $D_1^z, D_2^z \in Q_{\beta+\beta'}$. Hence $X - D_1^z \in P_{\beta+\beta'}$. But D_2^z, as a member of $Q_{\beta+\beta'}$, is of the form $\bigcup_{i=1}^\infty A_i$, where

$$A_i \in \bigcup_{\alpha<\beta+\beta'} P_\alpha \subset \bigcup_{\alpha<\beta+\beta'+1} P_\alpha \qquad (\text{for } i = 1, 2, \cdots).$$

Setting $A_0 = X - D_1^z$, we obtain $(X - D_1^z) \cup D_2^z = \bigcup_{i=0}^\infty A_i$, where again $A_i \in \bigcup_{\alpha<\beta+\beta'+1} P_\alpha$ (for $i = 0, 1, 2, \cdots$). Hence $(X - D_1^z) \cup D_2^z \in Q_{\beta+\beta'+1}$, and hence

$$D^z = (D_1 - D_2)^z = D_1^z - D_2^z = X - ((X - D_1^z) \cup D_2^z) \in P_{\beta+\beta'+1}.$$

[14] $\alpha'' = 0$ if α is finite.

Now suppose $D \in N_\gamma$; then $D = \bigcup_{i=1}^{\infty} D_i$, where $D_i \in \bigcup_{\beta<\gamma} M_\beta$. Hence $D_i^z \in \bigcup_{\beta<\gamma} P_{\beta+\beta'+1} \subset \bigcup_{\alpha<\gamma+\gamma'} P_\alpha$, the last inclusion being a consequence of Lemma 3.1. Hence $D^z = \bigcup_{i=1}^{\infty} D_i^z \in Q_{\gamma+\gamma'}$. This completes the induction.

N_1 generates the Borel structure of $X \times Z$, and hence the Borel structure of $X \times Z$ is $\bigcup_{\alpha<\Omega} N_\alpha$; hence for some γ, $B \in N_\gamma$. Hence every Z-section of B is in $Q_{\gamma+\gamma'}$. ▌

To avoid confusion in the sequel, we now make the following remarks concerning notation. We are concerned with *elements* (e.g. members x of X); with *sets* (e.g. subsets A of X); and with *families* of sets (e.g. $Q_1(\mathfrak{A})$). Functions $f: X \to Y$ are defined in the first instance on elements only; but the definition can be extended to sets in the usual way, by writing $f(A) = \{f(x): x \in A\}$ = image of A under f. We will go one step further, and extend the definition of functions that were originally defined on elements to families as well. This is done in the natural way, by writing $f(\mathfrak{C}) = \{f(A): A \in \mathfrak{C}\}$ (for an arbitrary family $\mathfrak{C}$). Thus $f(x)$ is an element, $f(A)$ a set, and $f(\mathfrak{A})$ a family. Similar remarks hold for the inverse function; we will write $f^{-1}(\mathfrak{D})$ for $\{f^{-1}(B): B \in \mathfrak{D}\}$ (where $\mathfrak{D}$ is an arbitrary family). In this notation, for example, L_α can be defined as $\{f \in Y^X: f^{-1}(Q_1(\mathfrak{B})) \subset Q_{\alpha+1}\}$. In which sense f or f^{-1} is meant in a particular case will always be clear from the context.

Note that if $f: X \to Y$ is a Borel mapping, then $f^{-1}: T \to S$ is a homomorphism (see §1). Hence

$$f^{-1}(Q_1(\mathfrak{B})) \subset Q_1(f^{-1}(\mathfrak{B})). \tag{3.3}$$

LEMMA 3.4. *If F is an admissible subset of Y^X, then there is a γ such that $F \subset L_\gamma$.*

Proof. Impose an admissible structure U on F. Let $\mathfrak{B} = \{B_1, B_2, \cdots\}$. Then since B_j is Borel, $\varphi_F^{-1}(B_j)$ must be a Borel subset of $F \times X$. Hence by Lemma 3.2, there are α_j such that all F-sections of $\varphi_F^{-1}(B_j)$ are in Q_{α_j}. An F-section of $\varphi_F^{-1}(B_j)$ has the form $\{x: x \in X, f(x) \in B_j\}$, where $f \in F$; i.e., it has the form $f^{-1}(B_j)$. It follows that for all $f \in F$, $f^{-1}(B_j) \in Q_{\alpha_j}$. Let $\alpha = \sup \alpha_j + 1$; then by Lemma 2.4, $f^{-1}(B_j) \in Q_\alpha$ for all j, i.e., $f^{-1}(\mathfrak{B}) \subset Q_\alpha$. Hence

$$f^{-1}(Q_1(\mathfrak{B})) \subset Q_1(f^{-1}(\mathfrak{B})) \subset Q_1(Q_\alpha) \subset Q_1(P_{\alpha+1}) \subset Q_{\alpha+2},$$

where the first inclusion follows from (3.3) and the two last from the definitions of P_α and Q_α. Comparing the first and last members of this chain of inclusions, we deduce $f \in L_{\alpha+1}$. ▌

4. Theorem D: Sufficiency in the regular case

In this section we will assume that X and Y are regular as well as separable, and will prove that every subset of Y^X which is of bounded Banach class is admissible. We will retain the notation of the previous section, but choose $\mathfrak{A}$ and $\mathfrak{B}$ in a particular way, which we describe in the next paragraph (by Lemma 2.3, no loss of generality is involved). It is of course sufficient to prove that every L_γ is admissible.

Since X and Y are separable and regular, we may assume without loss of generality that they are subspaces of the unit interval I. Now as well as being considered a Borel space with the usual structure, I may also be considered a topological space with the usual topology.[15] This topology induces relative subset topologies on X and on Y, which we will call the natural topologies on X and on Y. We choose $\mathfrak{A}$ and $\mathfrak{B}$ to be denumerable bases for the natural topologies of X and Y respectively; then $\mathfrak{A}$ and $\mathfrak{B}$ also generate the Borel structures of X and Y. Note that the choice of particular denumerable bases for the natural topologies of X and Y does not affect the values of the Q_α and the P_α. In the sequel, references to topological concepts like continuity and the related Baire functions are to be understood as referring to the natural topologies. Note that Q_1 is the set of open sets of X, $Q_1(\mathfrak{B})$ is the set of open sets of Y, and L_0 is the set of continuous functions from X into Y.

LEMMA 4.1. *If $Y = I$ and $\gamma \geqq 1$, then for every $f \in L_\gamma$, there is a sequence $f_1, f_2, \cdots$ of members of $\bigcup_{\beta<\gamma} L_\beta$ such that*[16] $f = \lim_{i\to\infty} f_i$.

Proof. It is sufficient to prove that when $Y = I$, the L_γ coincide with the Baire classes of order γ. This is known to be the case (cf. [3], p. 284, also [6], p. 294). ▌

If $Y = I$, then with every $f \in Y^X$, we may associate an infinite sequence $f_1, f_2, \cdots$ of members of Y^X as follows: By Lemma 2.1, for every $f \in Y^X$ there is a unique β such that $f \in L_\beta - \bigcup_{\alpha<\beta} L_\alpha$; we construct the f_i in such a way that $f = \lim_{i\to\infty} f_i$, and $f_i \in \bigcup_{\alpha<\beta} L_\alpha$. When $\beta = 0$, define $f_i = f$ for all i. The construction is possible by Lemma 4.1; however, it is not unique. Throughout the remainder of this section, we will consider the f_i as fixed; in other words, with each $f \in Y^X$ we associate a unique sequence $f_1, f_2, \cdots$.

Let $\lambda = \{\lambda_1, \cdots, \lambda_k\}$ be a finite sequence of positive integers. We will denote the sequence $\{\lambda_1, \cdots, \lambda_k, n\}$ by (λ, n). We will also use the notation f_λ instead of $f_{\lambda_1\cdots\lambda_k}$; thus $f_{\lambda n} = f_{(\lambda, n)}$. The empty sequence of integers will be denoted by $\varnothing$; we define $f_\varnothing = f$.

Let $\mathfrak{A} = \{A_1, A_2, \cdots\}$ and $\mathfrak{B} = \{B_1, B_2, \cdots\}$. If i and j are positive integers, define

$$E_\gamma(\lambda, i, j) = \{f : f_\lambda(A_i) \subset B_j, f_\lambda \in L_0, f \in L_\gamma\}.$$

Define R_γ to be the σ-ring generated by all sets of the form $E_\gamma(\lambda, i, j)$, where λ ranges over all finite sequences of positive integers, and i and j range over all positive integers.

LEMMA 4.2. *If $Y = I$, then R_γ is an admissible structure on L_γ.*

Proof. Impose the structure R_γ on L_γ; we wish to prove that it is ad-

[15] That generated by the open intervals.
[16] Pointwise.

missible. Define

$$C_\gamma(j, \lambda, \beta) = \{(f, x): f_\lambda(x) \in B_j, f \in L_\gamma, f_\lambda \in L_\beta\}.$$

We will prove

(4.3) *For each j, each*[17] λ, *and each $\beta \leqq \gamma$, $C_\gamma(j, \lambda, \beta)$ is Borel in $L_\gamma \times X$.*

The proof is by induction on β. If $\beta = 0$, we have

$$\begin{aligned} C_\gamma(j, \lambda, 0) &= \{(f, x): f_\lambda(x) \in B_j, f \in L_\gamma, f_\lambda \text{ is continuous}\} \\ &= \{(f, x): (\exists i)(x \in A_i, f_\lambda(A_i) \subset B_j, f \in L_\gamma, f_\lambda \in L_0)\} \\ &= \textstyle\bigcup_{i=1}^{\infty}\{(f, x): f_\lambda(A_i) \subset B_j, f \in L_\gamma, f_\lambda \in L_0, x \in A_i\} \\ &= \textstyle\bigcup_{i=1}^{\infty} E_\gamma(\lambda, i, j) \times A_i \end{aligned}$$

(the second equality follows from the openness of B_j). The last expression is a denumerable union of Borel rectangles in $L_\gamma \times X$, and is therefore a Borel set. Now suppose that (4.3) has been proved for all $\alpha < \beta$. For each j, let $B_{k(j,1)}, B_{k(j,2)}, \cdots$ be a sequence of basic open sets, each of whose closures is contained in B_j and for which $\bigcup_{i=1}^{\infty} B_{k(j,i)} = B_j$. Suppose b_1, $b_2, \cdots$ is a convergent sequence of points in Y. If $\lim_{n\to\infty} b_n \in B_j$, then there must be an i such that $\lim_{n\to\infty} b_n \in B_{k(j,i)}$, i.e.,

$$(\exists i)(\exists N)(\text{for all } n \geqq N)(b_n \in B_{k(j,i)}). \tag{4.4}$$

Conversely, if (4.4) holds, then $\lim_{n\to\infty} b_n$ must be in the closure of $B_{k(j,i)}$, and hence in B_j. Therefore

$$\begin{aligned} C_\gamma(j, \lambda, \beta) &= \{(f, x): f_\lambda(x) \in B_j, f \in L_\gamma, f_\lambda \in L_\beta\} \\ &= \{(f, x): \lim_{n\to\infty} f_{\lambda n}(x) \in B_j, f \in L_\gamma, f_\lambda \in L_\beta\} \\ &= \{(f, x): (\exists i)(\exists N)(\text{for all } n \geqq N)(f_{\lambda n}(x) \in B_{k(j,i)}, f \in L_\gamma, f_\lambda \in L_\beta)\} \\ &= \textstyle\bigcup_{i=1}^{\infty}\bigcup_{N=1}^{\infty}\{(f, x): (\text{for all } n \geqq N)(f_{\lambda n}(x) \in B_{k(j,i)}, f \in L_\gamma), f_\lambda \in L_\beta\} \\ &= \textstyle\bigcup_{i=1}^{\infty}\bigcup_{N=1}^{\infty}\{(f, x): (\text{for all } n \geqq N)(f_{\lambda n}(x) \in B_{k(j,i)}, f \in L_\gamma), \text{ and} \\ &\qquad\qquad (\text{for all } n \geqq N)(\exists \alpha)(\alpha < \beta, f_{\lambda n} \in L_\alpha)\} \\ &= \textstyle\bigcup_{i=1}^{\infty}\bigcup_{N=1}^{\infty}\{(f, x): (\text{for all } n \geqq N)(\exists \alpha) \\ &\qquad\qquad (\alpha < \beta, f_{\lambda n}(x) \in B_{k(j,i)}, f \in L_\gamma, f_{\lambda n} \in L_\alpha)\} \\ &= \textstyle\bigcup_{i=1}^{\infty}\bigcup_{N=1}^{\infty}\bigcap_{n=N}^{\infty}\bigcup_{\alpha<\beta}\{(f, x): f_{\lambda n}(x) \in B_{k(j,i)}, f \in L_\gamma, f_{\lambda n} \in L_\alpha\} \\ &= \textstyle\bigcup_{i=1}^{\infty}\bigcup_{N=1}^{\infty}\bigcap_{n=N}^{\infty}\bigcup_{\alpha<\beta} C_\gamma(k(j, i), (\lambda, n), \alpha). \end{aligned}$$

The set $C(k(j, i), (\lambda, n), \alpha)$ is a Borel subset of $L_\gamma \times X$ by induction hypothesis, and $\bigcup_{\alpha<\beta}$ is a denumerable union; hence the last expression obtained is Borel in $L_\gamma \times X$. This completes the inductive proof of (4.3).

[17] Including the empty one.

We now have

$$\begin{aligned}\varphi_{L_\gamma}^{-1}(B_j) &= \{(f, x) : \varphi_{L_\gamma}(f, x) \in B_j , f \in L_\gamma\} \\ &= \{(f, x) : f(x) \in B_j , f \in L_\gamma\} \\ &= \{(f, x) : f_\varnothing(x) \in B_j , f \in L_\gamma , f_\varnothing \in L_\gamma\} \\ &= C_\gamma(j, \varnothing, \gamma).\end{aligned}$$

Hence by (4.3) and Lemma 2.2, φ_{L_γ} is Borel. ▌

Lemma 4.2 establishes the admissibility of L_γ when $Y = I$. In the general case, when Y is merely a subset of I, we may consider Y^X as a subset of I^X. Let $\mathfrak{B}'$ be a basis for the open sets of I, such that $\mathfrak{B}$ consists precisely of the intersections of Y with members of $\mathfrak{B}'$. Set $L_\gamma^I = L_\gamma(\mathfrak{A}, \mathfrak{B}')$; L_γ^I is admissible by Lemma 4.2. Now it may be established that $L_\gamma = L_\gamma^I \cap Y^X$; hence L_γ is also admissible. (Note that the admissibility of a set F is a property of F alone, and not of the function space in which F happens to be imbedded. Here $L_\gamma \subset Y^X \subset I^X$, and we have established the admissibility of L_γ in I^X; but this is no different from its admissibility in the intermediate space Y^X.)

We have established

LEMMA 4.5. *L_γ is admissible.*

5. Theorem F

We retain the conventions of Sections 3 and 4; in particular it is assumed that X and Y are separable and regular. We wish to prove that every admissible subset of Y^X has a separable and regular admissible structure. For this it suffices to show that L_γ has such a structure.

As in the previous section, we first assume $Y = I$.

LEMMA 5.1. *If $Y = I$, then the structure R_γ on L_γ is separable and regular.*

Proof. Separability is immediate. To prove regularity let f and g be distinct members of L_γ. There must be an i_1 such that $f_{i_1} \neq g_{i_1}$, for otherwise $f = \lim_i f_i = \lim_i g_i = g$. Suppose we have defined $i_1, \cdots, i_k$ so that $f_{i_1 \cdots i_k} \neq g_{i_1 \cdots i_k}$; then we can define i_{k+1} so that $f_{i_1 \cdots i_{k+1}} \neq g_{i_1 \cdots i_{k+1}}$. We thus obtain an infinite sequence $\{i_1, i_2, \cdots\}$. For each k, let α_k and β_k be such that

$$f_{i_1 \cdots i_k} \in L_{\alpha_k} - \bigcup_{\alpha < \alpha_k} L_\alpha \quad \text{and} \quad g_{i_1 \cdots i_k} \in L_{\beta_k} - \bigcup_{\beta < \beta_k} L_\beta .$$

$\{\alpha_1, \alpha_2, \cdots\}$ and $\{\beta_1, \beta_2, \cdots\}$ are both strictly decreasing sequences of ordinal numbers, and therefore they both terminate; that is, there are j and k such that $\alpha_i = 0$ for all $i \geqq j$ and $\beta_i = 0$ for all $i \geqq k$. Let $m = \max(j, k)$, and let $\lambda = \{i_1, \cdots, i_m\}$. Then $g_\lambda \in L_0$, $f_\lambda \in L_0$, and $g_\lambda \neq f_\lambda$; in other words, both f_λ and g_λ are continuous, and there is an $x \in X$ such that $g_\lambda(x) \neq f_\lambda(x)$. Hence there are disjoint basic open sets B_j and B_k such that $f_\lambda(x) \in B_j$ and

$g_\lambda(x) \in B_k$. From the continuity of f and g it now follows that there is an $A_i \in \mathfrak{A}$ such that $x \in A_i$, $f_\lambda(A_i) \subset B_j$, and $g_\lambda(A_i) \subset B_k$. Hence

$$f \in E_\gamma(\lambda, i, j), \quad \text{and} \quad g \in E_\gamma(\lambda, i, k);$$

but since B_j and B_k are disjoint, so are $E_\gamma(\lambda, i, j)$ and $E_\gamma(\lambda, i, k)$. ∎

We now drop the assumption $Y = I$ and assume only $Y \subset I$. We proceed as in the previous section, and construct R'_γ from $\mathfrak{B}'$ in the same way that R_γ was constructed from $\mathfrak{B}$ in the case $Y = I$. We have just seen that R'_γ is a separable and regular admissible structure for L^I_γ. Since

$$L_\gamma = L^I_\gamma \cap Y^X \subset L^I,$$

we may apply Theorem B, and deduce that $(R'_\gamma)_{L_\gamma}$ is a separable and regular admissible structure on L_γ. This completes the proof of Theorem F.

6. Theorem G

Let $\mathfrak{A}$ and $\mathfrak{B}$ be as in footnote 8; set $\mathfrak{A} = \{A_1, A_2, \cdots\}$, $\mathfrak{B} = \{B_1, B_2, \cdots\}$. For each j, let $B_{k(j,1)}, B_{k(j,2)}, \cdots$ be a sequence of members of $\mathfrak{B}$ each of whose closures is contained in B_j and for which $\bigcup_{m=1}^{\infty} B_{k(j,m)} = B_j$. For each i, let $\{a_{i1}, a_{i2}, \cdots\}$ be a denumerable dense set in A_i. Suppose

$$f: X \to Y$$

is continuous. If $f(x) \in B_j$, then there is an m for which $f(x) \in B_{k(j,m)}$. By the continuity of f, there is an i such that $x \in A_i$ and $f(A_i) \subset B_{k(j,m)}$; hence

$$(\exists m)(\exists i)(\forall n)(x \in A_i, f(a_{in}) \in B_{k(j,m)}). \tag{6.1}$$

Conversely, assume (6.1); since the a_{in} are dense in A_i and f is continuous, it follows that $f(A_i)$ is included in the closure of $B_{k(j,m)}$, which in turn is a subset of B_j. Hence $f(x) \in f(A_i) \subset B_j$. Thus we have shown that for continuous f, $f(x) \in B_j$ is equivalent to (6.1). Hence if F contains continuous functions only, then

$$\begin{aligned} \varphi_F^{-1}(B_j) &= \{(f, x): f(x) \in B_j, f \in F\} \\ &= \{(f, x): (\exists m)(\exists i)(\forall n)(x \in A_i, f(a_{in}) \in B_{k(j,m)}, f \in F)\} \\ &= \textstyle\bigcup_{m=1}^{\infty} \bigcup_{i=1}^{\infty} \bigcap_{n=1}^{\infty} F(a_{in}, B_{k(j,m)}) \times A_i . \end{aligned}$$

If we impose on F the structure R generated by all the $F(a, B)$, then the last expression obtained is Borel in $F \times X$. Hence by Lemma 2.2, R is admissible. But by the remarks preceding the statement of Theorem G in the introduction, all admissible structures on F must contain R; hence R is a natural admissible structure. This completes the proof of Theorem G.

The statement that the natural structure on the set of all continuous members of I^I is the same as that induced by the uniform convergence topology is a consequence of Theorem G.

7. An admissible set without a natural admissible structure[18]

LEMMA 7.1. *Under the continuum hypothesis, there exists a one-one mapping θ of I onto itself such that $\theta = \theta^{-1}$ and such that if A is an uncountable Borel subset of I with an uncountable complement, then $\theta(A)$ is not Borel.*

Proof. Let $\{A_\alpha : \alpha < \Omega\}$ be a well-ordering of the Borel sets referred to above. Begin the definition of θ by interchanging some point of A_0 with some point of $I - A_0$. At stage α the mapping θ is already defined at a countable set. For each $\beta \leqq \alpha$ find an unused point in A_α, and let θ interchange it with an unused point in $I - A_\beta$. When the induction is finished, let θ be the identity at all points not yet used. Then $\theta(A_\alpha) \neq A_\beta$ whenever $\beta \leqq \alpha$, and hence $\theta(A_\alpha)$ is never measurable in I. ∎

Let[19] U be the set of all Borel subsets of I, $V = \theta(U)$, W the unit interval $[0, 1]$, S the structure on W generated by $U \cup V$, and D the *diagonal* of $W \times W$, i.e., the set $\{(x, x) : x \in W\}$.

LEMMA 7.2. *The σ-ring $(U \cap V) \times S$ on $W \times W$ does not contain D.*

Proof. Set $T = U \cap V$. By Lemma 7.1, for each set A in T, either A or $W - A$ is countable; let $q(A)$ be the countable one. If $C \in T \times S$, then every "horizontal" section C^x of C belongs to T. Write C^* for the union of all the sets $q(C^x)$, and let M be the class of all those sets C in $T \times S$ for which C^* is countable. M is a σ-ring included in $T \times S$ and containing all rectangles in $T \times S$; hence $M = T \times S$. Since M does not contain the diagonal, the proof of the lemma is complete.

LEMMA 7.3. *Both $U \times S$ and $V \times S$ contain D.*

Proof. $D \in U \times S$ follows from $D \in U \times U$. Next, $\theta(S)$ is a σ-ring and includes $\theta(U)$ and $\theta(V)$, i.e., V and U; hence $\theta(S) \supset S$. Define $\theta \times \theta : W \times W \to W \times W$ by $(\theta \times \theta)(x, y) = (\theta(x), \theta(y))$. Then

$$(\theta \times \theta)(U \times S) = \theta(U) \times \theta(S) \subset V \times S.$$

But since θ is onto, $(\theta \times \theta)(D) = D$. Hence from $D \in U \times S$ we obtain $D = (\theta \times \theta)(D) \in (\theta \times \theta)(U \times S) \subset V \times S$. ∎

Let X be the space whose underlying abstract space is W and whose structure is S. For each $t \in W$, define $f_t \in J^X$ by $f_t(s) = 0$ for $t \neq s$, $f_t(t) = 1$. Define $F \subset J^X$ by $F = \{f_t : t \in W\}$. Let $\lambda : W \to F$ be the natural map, defined by $\lambda(t) = f_t$, and let e be the identity on W. Define $D^* \subset F \times X$ by $D^* = (\lambda \times e)(D)$. Since $\varphi_F^{-1}(J) = \{D^*, F \times X - D^*\}$, it follows that a necessary and sufficient condition for a structure R on F to be admissible is that $R \times S$ contains D^*. Hence from Lemma 7.3 it follows that $\lambda(U)$ and $\lambda(V)$ are admissible, whereas it follows from Lemma 7.2 that $\lambda(U) \cap \lambda(V)$ (which is the same as $\lambda(U \cap V)$) is not admissible. Hence the intersection

[18] I am indebted to Prof. P. R. Halmos, who supplied the substance of this section.

[19] See the remarks preceding Lemma 3.4.

of all admissible sets on F is not admissible, and therefore F has no natural admissible structure.

X is separable and regular, and may therefore be considered a subset of I; however, it seems unlikely that X is isomorphic to I itself (note that the example depends on the continuum hypothesis). The question remains open as to whether the admissible subsets of such sets as J^I or I^I have natural admissible structures.

8. Dropping the regularity assumption: Theorems D and E

Our principal tool in this section is the idea of structure-preserving mapping. Suppose X and X_* are spaces with structures S and S_* respectively. A mapping $\pi: X \to X_*$ is called *structure-preserving* if the mapping it induces[19] on S is an isomorphism onto S_*. Given spaces X and Y which need not be regular, our procedure will be to construct spaces X_* and Y_* which *are* regular, and are connected to X and Y by means of structure-preserving (onto) mappings $\pi_X: X \to X_*$ and $\pi_Y: Y \to Y_*$. We will then deduce the desired theorems for X and Y from the corresponding theorems for X_* and Y_*.

Any space X may be divided into equivalence classes by means of the following relation: $x \equiv y$ if and only if every Borel set that contains x also contains y. X is regular if and only if each of these equivalence classes contains exactly one point. In the general case, X_* is defined to be the space of the equivalence classes in X, with the identification structure; and π_X is the identification mapping. That X_* is regular and that π_X is structure-preserving is not difficult to verify. Separability carries over from X to X_*.

Let F be a set of Borel mappings from X into Y. Any member f of F induces a unique Borel mapping $f_*: X_* \to Y_*$ for which the diagram

(8.1)
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi_X"'] & Y \arrow[d, "\pi_Y"] \\ X_* \arrow[r, "f_*"'] & Y_* \end{tikzcd}$$

is commutative. Let us denote by F_* the set of all f_* induced in this way by members f of F, and by π_F the function from F to F_* defined by

$$\pi_F(f) = f_*.$$

The crucial step is the proof of

LEMMA 8.2. *F is admissible if and only if F_* is admissible.*

Proof. The "if" half is the easier one. Let R_* be an admissible structure on F_*. R_* induces a unique structure R on F for which π_F is structure-preserving. If we impose R_* on F_* and R on F, then in the commutative diagram

$$\begin{tikzcd}
F \times X \arrow[r, "\varphi_F"] \arrow[d, "\pi_F \times \pi_X"'] & Y \arrow[d, "\pi_Y"] \\
F_* \times X_* \arrow[r, "\varphi_{F_*}"'] & Y_*
\end{tikzcd}$$

$\pi_F \times \pi_X$ will be structure-preserving. The fact that φ_F is a Borel mapping then follows from the corresponding fact for φ_{F_*}, and from the fact that the vertical mappings are structure-preserving.

The "only if" half is slightly trickier, because if we start out with an admissible structure R on F, then there may be no structure R_* on F_* for which π_F is structure-preserving. We get around this difficulty by defining a subset G of F so that $\pi_F(G) = F_*$ and $\pi_F \mid G$ (which is the same as π_G) is one-one. The structure R_G is admissible on G by Theorem B, and induces a unique structure R_* on F_* for which π_G is structure-preserving. The remainder of the proof follows as before from the commutativity of the diagram

$$\begin{tikzcd}
G \times X \arrow[r, "\varphi_G"] \arrow[d, "\pi_G \times \pi_X"'] & Y \arrow[d, "\pi_X"] \\
F_* \times X_* \arrow[r, "\varphi_{F_*}"'] & Y_*
\end{tikzcd} \quad . \quad ▮$$

To prove Theorem E, suppose F to be an admissible subset of Y^X, where X and Y are separable but need not be regular. Then F_* is an admissible subset of $Y_*^{X_*}$, and hence by Theorem F has a separable and regular admissible structure, which we will call R_*. But then the structure R induced[20] by R_* on F will be separable. ▮

It remains only to establish sufficiency in Theorem D. Fix denumerable generating families $\mathfrak{A}$ and $\mathfrak{B}$ for the structures of X and Y respectively, and set $\mathfrak{A}_* = \pi_X(\mathfrak{A})$, $\mathfrak{B}_* = \pi_X(\mathfrak{B})$; $\mathfrak{A}_*$ and $\mathfrak{B}_*$ are denumerable generating families for the separable and regular spaces X_* and Y_*. From the commutativity of diagram (8.1) and the fact that π_X and π_Y are structure-preserving, it follows that

$$L_\gamma(\mathfrak{A}, \mathfrak{B})_* = L_\gamma(\mathfrak{A}_*, \mathfrak{B}_*). \tag{8.3}$$

Since X_* and Y_* are separable and regular, we may apply Theorem D (cf. Section 4), and deduce that the right side of (8.3) is admissible. But then

[20] See the proof of Lemma 8.2.

the left side is admissible, and therefore by Lemma 8.2, $L_\gamma(\mathfrak{A}, \mathfrak{B})$ is also admissible.

REFERENCES

1. R. ARENS AND J. DUGUNDJI, *Topologies for function spaces*, Pacific J. Math., vol. 1 (1951), pp. 5–31.
2. R. J. AUMANN, *Spaces of measurable transformations*, Bull. Amer. Math. Soc., vol. 66 (1960), pp. 301–304.
3. S. BANACH, *Über analytisch darstellbare Operationen in abstrakten Räumen*, Fund. Math., vol. 17 (1931), pp. 283–295.
4. R. H. FOX, *On topologies for function spaces*, Bull. Amer. Math. Soc., vol. 51 (1945), pp. 429–432.
5. P. R. HALMOS, *Measure theory*, Princeton, Van Nostrand, 1950.
6. F. HAUSDORFF, *Set theory*, New York, Chelsea, 1957.
7. G. W. MACKEY, *Borel structures in groups and their duals*, Trans. Amer. Math. Soc., vol. 85 (1957), pp. 134–165.

THE HEBREW UNIVERSITY
JERUSALEM, ISRAEL
PRINCETON UNIVERSITY
PRINCETON, NEW JERSEY