ON A THEOREM OF SPITZER AND STONE AND RANDOM WALKS WITH ABSORBING BARRIERS

BY

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1. Introduction

Consider a sequence $X_1, X_2, \cdots$ of independent, identically distributed random variables, taking integer values only. We assume that every integer is a possible value (compare [5]), i.e., if

$$S_n = \sum_{i=1}^{n} X_i, \tag{1.1}$$

then there exist integers u and v such that[1]

$$P\{S_u = +1\} > 0 \quad \text{and} \quad P\{S_v = -1\} > 0. \tag{1.2}$$

Let I be any finite set of integers, containing $\mu(I)$ points, and put

(1.3) $N_I(A)$ = the number of terms S_k in the infinite sequence $S_1, S_2, \cdots$, such that $S_k \in I$ and $S_i \leqq A$ for $i = 1, 2, \cdots, k$,

(1.4) $N_I(A, -B)$ = the number of terms S_k in the infinite sequence $S_1, S_2, \cdots$ such that $S_k \in I$ and $-B \leqq S_i \leqq A$ for $i = 1, 2, \cdots, k$.

In a recent research note (Theorem 6 of [11]) of Spitzer and a paper [13] of Spitzer and Stone the asymptotic distributions of $N_I(A)$ and $N_I(A, -A)$ were given for the case $\mu(I) = 1$, X_i symmetrically distributed and $EX_i^2 < \infty$. At the same time Spitzer suggested in [11] that some formulae would be valid for any finite $\mu(I)$ and even for nonsymmetrically distributed X_i with zero mean. We shall drop the condition $EX_i^2 < \infty$ but instead assume that the characteristic function $\phi(t) = Ee^{itX_1}$ is such that

$$\lim_{t \downarrow 0} (1 - \phi(t))/t^\alpha = Q \qquad \text{with} \quad \operatorname{Re} Q > 0$$

for some α with $1 \leqq \alpha \leqq 2$. (In some places $0 < \alpha < 1$ is also considered.)

The generalizations suggested by Spitzer for $N_I(A)$ will be derived, and the corresponding results for $1 \leqq \alpha < 2$ are also found. If there are two barriers, we consider mostly variables with symmetric dlstributions, i.e., for which $P\{X_i = k\} = P\{X_i = -k\}$. We do not require, however, that the barriers be symmetrically placed, i.e., we shall find the asymptotic distribution of $N_I(A, -B)$ where B not necessarily equals A.

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[1] As usual, $P\{A\}$ = probability of the event A;
$P\{A \mid B\}$ = conditional probability of A, given B;
$E\{X\}$ = expectation of the random variable X; and
$E\{X \mid B\}$ = conditional expectation of X, given B.

For $1 < \alpha \leqq 2$ the results are of the form

$$\lim_{A\to\infty} P\{N_I(A) \leqq A^{\alpha-1}C\mu(I)x\} = 1 - e^{-x},$$

and if $P\{X_i = k\} = P\{X_i = -k\}$, for $c > 0$

$$\lim_{A\to\infty} P\{N_I(A, -cA) \leqq A^{\alpha-1}D\mu(I)x\} = 1 - e^{-x},$$

where C and D are constants depending on α, Q, and c.

If $\alpha = 1$, one obtains similar formulae with $A^{\alpha-1}$ replaced by $\log A$.

The case $\alpha < 1$ does not lead to theorems of this type, since for $\alpha < 1$

$$P\{S_k \in I \text{ for infinitely many } k\} = 0.$$

It is of course possible to derive similar results if (1.2) is not fulfilled, but if instead

$$P\{S_u = +d\} > 0, \qquad P\{S_v = -d\} > 0, \qquad P\{S_n = j\} = 0$$

if $j \not\equiv 0 \pmod d$.

There is some duplication between this note and the papers [12] and [13], especially in Section 3. Since we treat slightly more general cases than Spitzer, most proofs are nevertheless reproduced in full. The behavior of $N_I(A)$ for $1 \leqq \alpha < 2$ follows from Spitzer's method just as well. The main difference lies in the methods for $N_I(A, -B)$.

The author is indebted to Professor F. Spitzer for communicating results and methods before they appeared in print.

2. The exponential form of the limiting distributions

$X_1, X_2, \cdots$ is a sequence of independent, identically distributed random variables, such that, with

$$S_n = \textstyle\sum_{i=1}^n X_i, \tag{2.1}$$

there exist positive integers u and v for which

$$P\{S_u = +1\} > 0, \qquad P\{S_v = -1\} > 0. \tag{2.2}$$

Putting $\phi(t) = Ee^{itX_1}$, we also assume

$$\lim_{t\downarrow 0} (1 - \phi(t))/t^\alpha = Q \qquad \text{with } \operatorname{Re} Q > 0 \tag{2.3}$$

for some $\alpha > 0$. In general we take $1 \leqq \alpha \leqq 2$. Since $\phi(-t) = \overline{\phi(t)}$, it follows from Theorem 3 in [5] that all integers are recurrent values in this case, i.e., $S_k = b$ for infinitely many k with probability 1 for every integer b. Thus, if (2.2) and (2.3) are satisfied,

$$\textstyle\sum_{k=1}^\infty P\{S_k = b,\ S_i \neq b \quad \text{for} \quad 1 \leqq i < k\} = 1.$$

This implies immediately

$$\lim_{A\to\infty} \textstyle\sum_{k=1}^\infty P\{S_k = b,\ b \neq S_i \leqq A \quad \text{for} \quad 1 \leqq i < k\} = 1 \tag{2.4}$$

as well as

$$\lim_{A\to\infty, B\to\infty} \sum_{k=1}^{\infty} P\{S_k = b,\ S_i \neq b,\ -B \leqq S_i \leqq A \quad \text{for} \quad 1 \leqq i < k\} = 1. \tag{2.5}$$

Put

$$p_b(A) = \sum_{k=1}^{\infty} P\{S_k = b,\ b \neq S_i \leqq A \quad \text{for} \quad 1 \leqq i < k\} \tag{2.6}$$

and

$$p_b(A, -B) = \sum_{k=1}^{\infty} P\{S_k = b,\ S_i \neq b,\ -B \leqq S_i \leqq A \quad \text{for} \quad 1 \leqq i < k\}. \tag{2.7}$$

$p_b(A)$ is the probability to visit b, before any partial sum exceeds A. In terms of random walks, we can think of $A + 1$ as an absorbing barrier, and then $p_b(A)$ is the probability of reaching b without absorption. A similar interpretation can be given to $p_b(A, -B)$.

If I contains just the point 0, one has almost immediately from the definition (1.3)

$$P\{N_{\{0\}}(A) \geqq N\} = [p_0(A)]^N, \tag{2.8}$$

and thus by (2.4) and (2.6)

$$\lim_{A\to\infty} P\{N_{\{0\}}(A) \geqq x(1 - p_0(A))^{-1}\} = \lim_{A\to\infty} [1 - (1 - p_0(A))]^{x(1-p_0(A))^{-1}} = e^{-x}. \tag{2.9}$$

The generalization of this formula for general finite sets I is proved in the following lemma.

LEMMA 1. *Let I be any finite set of integers, containing $\mu(I)$ points. Define $N_I(A)$ and $N_I(A, -B)$ by* (1.3) *and* (1.4). *Then*

$$\lim_{A\to\infty} P\{N_I(A) \geqq \mu(I)x(1 - p_0(A))^{-1}\} = e^{-x} \tag{2.10}$$

and

$$\lim_{A\to\infty, B\to\infty} P\{N_I(A, -B) \geqq \mu(I)x(1 - p_0(A, -B))^{-1}\} = e^{-x}. \tag{2.11}$$

Proof. We shall only prove (2.10), the proof of (2.11) being practically the same. If J is any set of integers, put $M_J(n)$ = the number of terms S_k in the finite sequence $S_1, S_2, \cdots, S_n$, such that $S_k \in J$. Let I now consist of the μ integers $a_1, \cdots, a_\mu$. By Theorem 2 of [3] and its corollaries one has

$$\lim_{n\to\infty} M_{\{0\}}^{-1}(n) M_{\{a_i\}}(n) = 1 \qquad (i = 1, \cdots, \mu), \tag{2.12}$$

and therefore

$$\lim_{n\to\infty} M_{\{0\}}^{-1}(n) M_I(n) = \mu \qquad \text{with probability 1.} \tag{2.13}$$

Define $n(\gamma)$ as the first index for which $S_n > \gamma$, i.e., $n(\gamma) = k$ if $S_k > \gamma$ while $S_i \leqq \gamma$ for $i = 1, \cdots, k - 1$. One has then for any $\varepsilon > 0$ and integer m

$$P\{\mu N_{\{0\}}(A)(1-\varepsilon) \leqq N_I(A) \leqq \mu N_{\{0\}}(A)(1+\varepsilon)\}$$
$$\geqq 1 - P\{n(A) < m\} - P\{\text{there exists an } n \geqq m \text{ such that } |\mu^{-1}M_{\{0\}}^{-1}(n)M_I(n) - 1| > \varepsilon\}.$$

Since by (2.13)

$$\lim_{m\to\infty} P\{\text{there exists an } n \geqq m \text{ such that } |\mu^{-1}M_{\{0\}}^{-1}(n)M_I(n) - 1| > \varepsilon\} = 0,$$

and for each fixed m

$$\lim_{A\to\infty} P\{n(A) < m\} = 0,$$

it follows that for each $\varepsilon > 0$

$$\lim_{A\to\infty} P\{|\mu^{-1}N_{\{0\}}(A)\cdot N_I(A) - 1| \leqq \varepsilon\} = 1.$$

This together with (2.9) implies (2.10).

From the lemma we see that one only needs to find the asymptotic behavior of $1 - p_0(A)$ and $1 - p_0(A, -B)$ in order to find the asymptotic distributions of $N_I(A)$ and $N_I(A, -B)$. From (2.9)

$$EN_{\{0\}}(A) = p_0(A)(1 - p_0(A))^{-1}, \tag{2.14}$$

and thus, by using (2.4) and (2.6)

$$\lim_{A\to\infty} (1 - p_0(A))EN_{\{0\}}(A) = 1. \tag{2.15}$$

Similarly

$$\lim_{A\to\infty} (1 - p_0(A, -B))EN_{\{0\}}(A, -B) = 1. \tag{2.16}$$

The relations (2.15) and (2.16) will be one of the main tools in the next two sections.

3. The asymptotic behavior of $1 - p_0(A)$ and $N_I(A)$

We have already interpreted in (2.14) $(1 - p_0(A))^{-1}$ as the expected number of terms S_k in the infinite sequence $S_0, S_1, S_2, \cdots$ ($S_0 = 0$) with $S_k = 0$ and $S_i \leqq A$ for $i \leqq k$. Let us put

$$S_k^+ = \max(0, S_k) \quad\text{and}\quad S_k^- = \max(0, -S_k).$$

Then

$$((1 - p_0(A))^{-1} = \textstyle\sum_{p=0}^{A} \sum_{n=0}^{\infty} P\{S_n = 0, \max_{1\leqq k\leqq n} S_k^+ = p\}. \tag{3.1}$$

In order to find the asymptotic behavior of this sum we shall derive an expression for

$$\textstyle\sum_{p=0}^{\infty} e^{-2sp} \sum_{n=0}^{\infty} P\{S_n = 0, \max_{1\leqq k\leqq n} S_k^+ = p\}$$

and apply Karamata's Tauberian theorem to this expression.

The computations in the next few lemmas could be greatly simplified by considering symmetric distributions of X_i only. The reader may find it profitable to consider that case only (i.e., $\phi(t) = \phi(-t)$, $\beta = 0$). It seemed worth while though, to do the more general case to obtain the expressions (3.6) and (3.7) for $C(\alpha, Q)$ (cf. also the remark immediately after the proof of Theorem 1).

LEMMA 2. *If (2.3) is satisfied, then for $s > 0$*

$$\sum_{p=0}^{\infty} e^{-2sp} \sum_{n=0}^{\infty} P\{S_n = 0, \max_{1 \leq k \leq n} S_k^+ = p\}$$
$$= \frac{1}{2\pi} \int_{-\pi}^{+\pi} dy \exp - \left(\frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{s}{(t-y)^2 + s^2} \log [1 - \phi(t)]\, dt\right).$$

Proof. Spitzer (Theorem 6.1 in [9]) has shown that

$$\sum_{n=0}^{\infty} x^n \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} e^{-(s-iy)p} e^{-(s+iy)q} \cdot P\{\max_{1 \leq k \leq n} S_k^+ = p, \max_{1 \leq k \leq n} S_k^+ - S_n = q\}$$
$$= \exp \sum_{k=1}^{\infty} \frac{x^k}{k} [\psi_k^+(s - iy) + \psi_k^-(s + iy) - 1], \tag{3.2}$$

where

$$\psi_k^+(s) = Ee^{-sS_k^+}, \qquad \psi_k^-(s) = Ee^{-sS_k^-}.$$

Hence

$$\sum_{n=0}^{\infty} x^n \sum_{p=0}^{\infty} e^{-2sp} P\{\max_{1 \leq k \leq n} S_k^+ = p, S_n = 0\}$$
$$= \sum_{n=0}^{\infty} x^n \sum_{p=0}^{\infty} e^{-2sp} P\{\max_{1 \leq k \leq n} S_k^+ = p, \max_{1 \leq k \leq n} S_k^+ - S_n = p\}$$
$$= \frac{1}{2\pi} \int_{-\pi}^{+\pi} dy \sum_{n=0}^{\infty} x^n \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} e^{-s(p+q)} e^{iy(p-q)} \tag{3.3}$$
$$\cdot P\{\max_{1 \leq k \leq n} S_k^+, \max_{1 \leq k \leq n} S_k^+ - S_n = q\}$$
$$= \frac{1}{2\pi} \int_{-\pi}^{+\pi} dy \exp \sum_{k=1}^{\infty} \frac{x^k}{k} [\psi_k^+(s - iy) + \psi_k^-(s + iy) - 1].$$

In another place (Theorem 3 in [10]), Spitzer showed that if

$$\int_{-1}^{+1} \left| \frac{1 - \phi(t)}{t} \right| dt < \infty,$$

then for $s \geqq 0, 0 \leqq x < 1$

$$\sum_{k=1}^{\infty} \frac{x^k}{k} [\psi_k^+(s) - 1] = \exp\left(\frac{1}{2\pi} \int_0^x dz \int_{-\infty}^{+\infty} \frac{s}{t(t - is)} \frac{\phi(t) - 1}{(1 - z)(1 - z\phi(t))} dt\right).$$

By analytic continuation we are allowed to replace s by $s \pm iy$ for $s > 0$, y real. Changing $+X_i$ into $-X_i$ we obtain

$$\sum_{k=1}^{\infty} \frac{x^k}{k} [\psi_k^-(s + iy) - 1]$$
$$= \exp\left(\frac{1}{2\pi} \int_0^x dz \int_{-\infty}^{+\infty} \frac{s + iy}{t(t + y - is)} \frac{\phi(-t) - 1}{(1 - z)(1 - z\phi(-t))} dt\right)$$
$$= \exp\left(\frac{1}{2\pi} \int_0^x dz \int_{-\infty}^{+\infty} \frac{s + iy}{t(t - y + is)} \frac{\phi(t) - 1}{(1 - z)(1 - z\phi(t))} dt\right).$$

Taking into account that

$$\exp \sum_{k=1}^{\infty} \frac{x^k}{k} = \exp \int_0^x \frac{dz}{1 - z}$$

one verifies immediately

$$\exp \sum_{k=1}^{\infty} \frac{x^k}{k} [\psi_k^+(s - iy) - 1 + \psi_k^-(s + iy) - 1 + 1]$$
$$(3.4) \qquad = \exp\left(-\frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{s}{(t - y)^2 + s^2} \log [1 - x\phi(t)]\, dt\right).$$

Since $\phi(t)$ is close to 1 only when t is close to $2\pi k$, $k = 0, \pm 1, \pm 2, \cdots$, (by (2.2)), and if $t = 2\pi k + t'$, $\log (1 - x\phi(t)) = O(|\log t'|)$ $(t' \to 0)$ (by (2.3)), it follows from the Lebesgue dominated convergence theorem that for $s > 0$

$$\lim_{x \uparrow 1} \frac{1}{2\pi} \int_{-\pi}^{+\pi} dy \exp\left(-\frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{s}{(t - y)^2 + s^2} \log [1 - x\phi(t)]\, dt\right)$$
$$= \frac{1}{2\pi} \int_{-\pi}^{+\pi} dy \exp\left(-\frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{s}{(t - y)^2 + s^2} \log [1 - \phi(t)]\, dt\right),$$

which proves the lemma.

LEMMA 3. *Let (2.2) and (2.3) be satisfied. Put*

$$f(t) = \begin{cases} \log [1 - \phi(t)] - \alpha \log |t| - \log Q & \text{for } t > 0, \\ \log [1 - \phi(t)] - \alpha \log |t| - \log \bar{Q} & \text{for } t < 0, \end{cases}$$

and

$$f(0) = 0.$$

Then

$$(3.5) \qquad \lim_{s \downarrow 0} \frac{-1}{\pi} \int_{-\infty}^{+\infty} [f(t) - f(y)] \frac{s}{(t - y)^2 + s^2} dt = 0$$

uniformly in $|y| \leqq \pi$.

Proof. Let η be some small positive number. Split the integral up in three pieces: I_1 from $-\infty$ to $y - \eta$, I_2 from $y - \eta$ to $y + \eta$, and I_3 from $y + \eta$ to $+\infty$.

$$|I_1| \leqq \frac{s}{\pi} \int_{\eta}^{\infty} \frac{|f(y - t) - f(y)|}{t^2} dt.$$

By (2.3) $\lim_{t\to 0} f(t) = f(0) = 0$, while by (2.2) $|t - k\cdot 2\pi| \geqq \varepsilon$ for all k implies $|\phi(t)| \leqq 1 - C(\varepsilon) < 1$ for some function $C(\varepsilon) > 0$. In addition by (2.3)

$$|\log [1 - \phi(t)]| = O(|\log (t - k\cdot 2\pi)|) \qquad \text{as } t \to k2\pi.$$

Using these facts, one sees

$$\int_\eta^\infty \frac{|f(y - t) - f(y)|}{t^2}\, dt = O(\eta^{-2}) \qquad (\eta \to 0).$$

Hence $|I_1| = O(s\eta^{-2})$, and similarly $|I_3| = O(s\eta^{-2})$. On the other hand

$$|I_2| \leqq \sup_{|y-t|\leqq\eta} |f(t) - f(y)| \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{s\, dt}{(t - y)^2 + s^2} \leqq \sup_{|y-t|\leqq\eta} |f(t) - f(y)|.$$

By the continuity of $f(t)$ it follows that $I_2 \to 0$, as $\eta \to 0$, uniformly in s. The lemma follows by combining the estimations for I_1, I_2, and I_3.

LEMMA 4. *If* (2.2) *and* (2.3) *are satisfied with* $1 \leqq \alpha \leqq 2$ *and* $Q = |Q|e^{i\beta}$ $(|\beta| < \pi/2)$, *then, as* $s \downarrow 0$,

$$\sum_{p=0}^\infty e^{-2sp} \sum_{n=0}^\infty P\{S_n = 0, \max_{1\leqq k\leqq n} S_k^+ = p\}$$

$$\backsim \begin{cases} (\pi|Q|)^{-1}s^{1-\alpha}\displaystyle\int_0^\infty (1 + y^2)^{-\alpha/2} \cos\left(\frac{2\beta}{\pi}\int_0^y (1 + t^2)^{-1}\, dt\right) dy & \textit{for } 1 < \alpha \leqq 2, \\ (\pi|Q|)^{-1} \cdot \cos\beta \cdot \log s^{-1} & \textit{for } \alpha = 1. \end{cases}$$

Proof. We have shown in Lemmas 2 and 3 that

$$\sum_{p=0}^\infty e^{-2sp} \sum_{n=0}^\infty P\{S_n = 0, \max_{1\leqq k\leqq n} S_k^+ = p\}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{+\pi} dy \exp\left[o(1) - \frac{1}{\pi}\int_0^\infty \frac{s(\alpha \log |t| + \log Q + f(y))}{(t - y)^2 + s^2}\, dt \right.$$
$$\left. - \frac{1}{\pi}\int_{-\infty}^0 \frac{s(\alpha \log |t| + \log \bar{Q} + f(y))}{(t - y)^2 + s^2}\, dt\right],$$

where $o(1) \to 0$ as $s \downarrow 0$ uniformly in $|y| \leqq \pi$. Substituting $\log Q = \log |Q| + i\beta$ (note that $|\beta| < \pi/2$ since we assumed $\operatorname{Re} Q > 0$) one can write for the integral in the exponent

$$-\log |Q| - f(y) - \frac{i\beta}{\pi}\left[\int_0^\infty \frac{s\, dt}{(t - y)^2 + s^2} - \int_0^\infty \frac{s\, dt}{(t + y)^2 + s^2}\right]$$
$$- \frac{\alpha}{\pi}\int_0^\infty \log t \left[\frac{s - iy}{t^2 + (s - iy)^2} + \frac{s + iy}{t^2 + (s + iy)^2}\right] dt$$

$$= -\log |Q| - f(y) - \frac{i\beta}{\pi}\int_{-ys^{-1}}^{+ys^{-1}} (1 + t^2)^{-1}\, dt$$

$$-\frac{\alpha}{\pi}\int_0^\infty \frac{[\log(s-iy)t+\log(s+iy)t]}{1+t^2}\,dt$$

$$= -\log|Q| - f(y) - \frac{i\beta}{\pi}\int_{-ys^{-1}}^{+ys^{-1}} (1+t^2)^{-1}\,dt - \frac{\alpha}{2}\log(s^2+y^2).$$

We therefore have to determine the asymptotic behavior of

$$\frac{1}{2\pi|Q|}\int_{-\pi}^{+\pi}(s^2+y^2)^{-\alpha/2}\exp\left(o(1)-f(y)-\frac{i\beta}{\pi}\int_{-ys^{-1}}^{+ys^{-1}}(1+t^2)^{-1}\,dt\right)dy$$

$$= \frac{1}{\pi|Q|}\int_0^\pi (s^2+y^2)^{-\alpha/2}\cos\left(\frac{2\beta}{\pi}\int_0^{ys^{-1}}(1+t^2)^{-1}\,dt\right)dy$$

$$+\frac{1}{2\pi|Q|}\int_{-\pi}^{+\pi}(s^2+y^2)^{-\alpha/2}\exp\left(-\frac{2i\beta}{\pi}\int_0^{ys^{-1}}(1+t^2)^{-1}\,dt\right)$$

$$\cdot\,[\exp(o(1)-f(y))-1]\,dy.$$

Let us prove the result for $\alpha = 1$, the proof for $1 < \alpha \leqq 2$ being very similar, even simpler. We have then

$$\left|\int_{-\pi}^{+\pi}(s^2+y^2)^{-1/2}\exp\left(-\frac{2i\beta}{\pi}\int_0^{ys^{-1}}(1+t^2)^{-1}\,dt\right)[\exp(o(1)-f(y))-1]\,dy\right|$$

$$\leqq \int_{-\varepsilon}^{+\varepsilon}(s^2+y^2)^{-1/2}\,[\exp(o(1)-f(y))-1]dy + O\left(\int_\varepsilon^\pi (s^2+y^2)^{-1/2}\,dy\right).$$

Since $\lim_{y\to 0} f(y) = 0$, the integral from $-\varepsilon$ to ε can be made less than

$$\eta\int_{-\varepsilon}^{\varepsilon}(s^2+y^2)^{-1/2}\,dy = O\left(\eta\log\frac{1}{s}\right)$$

for any $\eta > 0$, by choosing ε small enough.

The integral from ε to π is $O(\log \varepsilon^{-1})$ uniformly in $s > 0$. Hence

$$\lim_{s\downarrow 0}(\log s^{-1})^{-1}\int_{-\pi}^{+\pi}(s^2+y^2)^{-1/2}\exp\left(-\frac{2i\beta}{\pi}\int_0^{ys^{-1}}(1+t^2)^{-1}\,dt\right)$$

$$\cdot\,[\exp(o(1)-f(y))-1]\,dy = 0.$$

Since

$$\cos\left(\frac{2\beta}{\pi}\int_0^{ys^{-1}}(1+t^2)^{-1}\,dt\right) - \cos\beta \to 0 \quad\text{as}\quad s\downarrow 0,$$

one also has

$$\lim_{s\downarrow 0}(\log s^{-1})^{-1}\int_0^\pi (s^2+y^2)^{-1/2}\left[\cos\left(\frac{2\beta}{\pi}\int_0^{ys^{-1}}(1+t^2)^{-1}\,dt\right)-\cos\beta\right]dy = 0.$$

Finally

$$\cos\beta\cdot\int_0^\pi (s^2+y^2)^{-1/2}\,dy = \cos\beta\int_0^{\pi s^{-1}}(1+y^2)^{-1/2}\,dy \backsim \cos\beta\cdot\log s^{-1}$$

as $s \downarrow 0$, from which the required formula follows.

If $Q = |Q|e^{i\beta}$ with $|\beta| < \pi/2$ and if $1 < \alpha \leqq 2$, we put

$$C(\alpha, Q) \tag{3.6}$$
$$= (\pi |Q| 2^{1-\alpha}\Gamma(\alpha))^{-1} \int_0^\infty (1 + y^2)^{-\alpha/2} \cos\left(\frac{2\beta}{\pi} \int_0^y (1 + t^2)^{-1} dt\right) dy,$$

and for the same Q with $\alpha = 1$ we put

$$C(1, Q) = (\pi |Q|)^{-1} \cos \beta. \tag{3.7}$$

We then have the following

THEOREM 1. *If* (2.2) *and* (2.3) *are satisfied with* $1 \leqq \alpha \leqq 2$, *then*

$$\lim_{A\to\infty} A^{1-\alpha}(1 - p_0(A))^{-1} = C(\alpha, Q) \quad \text{if} \quad 1 < \alpha \leqq 2 \tag{3.8}$$

and

$$\lim_{A\to\infty} (\log A)^{-1}(1 - p_0(A))^{-1} = C(1, Q) \quad \text{if} \quad \alpha = 1. \tag{3.9}$$

Consequently, if $1 < \alpha \leqq 2$,

$$\lim_{A\to\infty} P\{N_I(A) \leqq A^{\alpha-1}C(\alpha, Q)\mu(I)x\} = 1 - e^{-x}, \tag{3.10}$$

and if $\alpha = 1$,

$$\lim_{A\to\infty} P\{N_I(A) \leqq \log A \cdot C(1, Q)\mu(I)x\} = 1 - e^{-x}. \tag{3.11}$$

Proof. (3.8) and (3.9) follow immediately from (3.1) and Lemma 4 by applying Karamata's Tauberian theorem [6]. That (3.8) and (3.9) imply (3.10) and (3.11) respectively has already been proved in Lemma 1.

Note that changing $+X$ into $-X$ only changes the sign of β. Hence $N_I(+\infty, -A)$ = the number of terms S_k in the infinite sequence $S_1, S_2, \cdots$ such that $S_k \in I$ and $S_i \geqq -A$ for $i = 1, 2, \cdots k$ has the same asymptotic behavior as $N_I(A)$ even though we did not require X_i to be symmetrically distributed.

4. Asymptotic behavior of $1 - p_0(A, -B)$ and $N_I(A, -B)$

We shall derive the asymptotic behavior of $N_I(A, -B)$ directly from that of $N_I(A)$ without any such explicit expression as given by Lemma 2. Except for the case $\alpha = 2$ we shall assume in this section that the X_i have a symmetric distribution, i.e.,

$$P\{X_i = k\} = P\{X_i = -k\}. \tag{4.1}$$

Define $n(\gamma)$ as in the proof of Lemma 1 by

$$n(\gamma) = k \quad \text{if} \quad S_k > \gamma \quad \text{while} \quad S_i \leqq \gamma \quad \text{for} \quad i = 1, \cdots k - 1. \tag{4.2}$$

Put now

$$Z = S_{n(0)}. \tag{4.3}$$

By definition Z is the first positive term in the sequence $S_1, S_2 \cdots$.

LEMMA 5. *If* (2.3) *is satisfied for some* $0 < \alpha \leqq 2$, *and if* (4.1) *is satisfied, then for* $s \geqq 0$

$$Ee^{-sZ} = 1 - \exp\left(\frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{s}{s^2 + t^2} \log\left[1 - \phi(t)\right] dt\right) \cdot \exp \sum_{k=1}^{\infty} \frac{P\{S_k = 0\}}{2k}. \tag{4.4}$$

Proof. It is shown in Corollary 3.2 of [12] that for $s \geqq 0$

$$Ee^{-sZ} = 1 - \exp\left(-\sum_{k=1}^{\infty} \frac{1}{k}\int_{0+}^{\infty} e^{-st}\, d_t P\{S_k \leqq t\}\right).$$

But by (4.1)

$$\int_{0+}^{\infty} e^{-st}\, d_t P\{S_k \leqq t\} = \psi_k^+(s) - \tfrac{1}{2} - \tfrac{1}{2}P\{S_k = 0\}$$

where, as before,

$$\psi_k^+(s) = E\exp(-sS_k^+).$$

Putting $d = \exp(P\{S_k = 0\}/2k)$ ($d < \infty$ by Theorem 1 in [4]), one has

$$Ee^{-sZ} = 1 - d\exp\left(-\textstyle\sum_{k=1}^{\infty} (\psi_k^+(s) - \tfrac{1}{2})\right).$$

The lemma follows now by letting t tend to 1 in equation (3.2) and Lemma 2 of [10]. The distribution of Z was first found in [1].

The analogue of the next lemma for stable processes was proved by Ray in [8].

LEMMA 6.[2] *If* (2.3) *and* (4.1) *are satisfied, then, if* $\alpha = 2$,

$$\lim_{A\to\infty} P\{S_{n(A)} - A \leqq x\} = F_2(x), \tag{4.5}$$

where $F_2(x)$ *is a proper distribution function with* $\lim_{x\to\infty} F_2(x) = 1$. *If* $0 < \alpha < 2$, *then*

$$\lim_{A\to\infty} P\{S_{n(A)} - A \leqq Ax\} = F_\alpha(x), \tag{4.6}$$

where

$$F_\alpha(x) = \frac{\sin(\pi\alpha/2)}{\pi}\int_0^x t^{-\alpha/2}(1 + t)^{-1}\, dt.$$

Proof. Consider a sequence $Z_1, Z_2, \cdots$ of independent random variables, each having the same distribution as Z, put

$$U_n = \textstyle\sum_{i=1}^{n} Z_i,$$

and let $q(A)$ be the first index for which $U_n > A$, i. e., $q(A) = k$ if $U_k > A$ while $U_i \leqq A$ for $i = 1, \cdots, k - 1$. Just as $S_{n(0)} - 0 = Z$, it is easily seen that $S_{n(A)} - A$ has the same distribution as $U_{q(A)} - A$. Hence we can find

[2] This lemma does not depend on (2.2). A change of scale does change $F_2(x)$ but not $F_\alpha(x)$ for $\alpha < 2$. A similar remark holds for Lemmas 8 and 9.

the distribution of $S_{n(A)} - A$ from renewal theory if we know the distribution of Z. But using Lemma 5 and the estimates of Lemma 3 for $y = 0$ one obtains

$$\lim_{s \downarrow 0} s^{-\alpha/2}(1 - e^{-sZ}) = dQ^{-1/2}.$$

(Recall that Q in (2.3) is real when (4.1) is satisfied.) Hence for $\alpha = 2$, one has (compare Theorem 3.4 in [12])

$$EZ = dQ^{1/2} < \infty,$$

from which (4.5) follows by well known results in renewal theory (cf. for instance Theorem (3.1) and the identity (5.1) in [7]). For $0 < \alpha < 2$, (4.6) is merely equation (5.5) of [7].

In addition to $n(\gamma)$ we define

$$m(\gamma) = k \quad \text{if} \quad S_k < \gamma \quad \text{while} \quad S_i \leqq \gamma \quad \text{for} \quad i = 1, \cdots, k-1 \tag{4.7}$$

and

$$\begin{aligned} r(\gamma, -\delta) = k \quad \text{if} \quad & S_k < -\delta \quad \text{or} \quad S_k > \gamma \\ & \text{while} \quad -\delta \leqq S_i \leqq \gamma \quad \text{for} \quad i = 1, \cdots, k-1. \end{aligned} \tag{4.8}$$

$S_{m(\gamma)}$ is the first partial sum smaller than γ, and $S_{r(\gamma,-\delta)}$ is the first partial sum greater than γ or smaller than $-\delta$. Finally, we put

$$\bar{\pi}(A, -B) = P\{S_{r(A,-B)} > A\},$$

$$\underline{\pi}(A, -B) = 1 - \bar{\pi}(A, -B) = P\{S_{r(A,-B)} < -B\}.$$

$\bar{\pi}$ is the probability of crossing the upper boundary before the lower boundary, and similarly for $\underline{\pi}$ with the words upper and lower interchanged.

THEOREM 2. *If* (2.2) *and* (2.3) *are satisfied with* $\alpha = 2$, *then for any fixed* $c \geqq 0$

$$\lim_{A\to\infty} \bar{\pi}(A, -cA) = \lim_{A\to\infty} 1 - \underline{\pi}(A, -cA) = c(1 + c)^{-1}, \tag{4.9}$$

and

$$\lim_{A\to\infty} A^{-1}(1 - p_0(A, -cA))^{-1} = c(1 + c)^{-1}C(2, Q). \tag{4.10}$$

Consequently

$$\lim_{A\to\infty} P\{N_I(A, -cA) \leqq Ac(1 + c)^{-1}C(2, Q)\mu(I)x\} = 1 - e^{-x}. \tag{4.11}$$

Remark. Note that if $\alpha = 2$, $Q = EX^2/2$ and must be real even without the condition (4.1). Related to this is the fact that (4.5) is valid for $\alpha = 2$ even without (4.1), as follows from the proof of (4.5) and Theorem 3.4 in [12] which states that always $EZ < \infty$ whenever $\alpha = 2$. We therefore do not require symmetric distributions in this theorem but can nevertheless use (4.5).

Proof.

$$(4.12)\quad \begin{aligned}(1 - p_0(A))^{-1} &= E\,\{\text{number of indices } k < n(A) \text{ for which } S_k = 0\}\\ &= E\,\{\text{number of indices } k \leqq r(A, -B) \text{ for which } S_k = 0\}\\ &\quad + \underline{\pi}(A, -B)\, E\{\text{number of indices } k \text{ with } r(A, -B) < k < n(A)\\ &\qquad\qquad \text{for which } S_k = 0 \mid S_{r(A,-B)} < -B\}.\end{aligned}$$

Assume now $S_{r(A,-B)} = -C - 1 < -B$, and let

$$(4.13)\quad n'(0) = \text{the first index greater than } r(A, -B) \text{ for which } S_n > -1.$$

In that case $S_{n'(0)} - (-1)$ has the same distribution as $S_{n(C)} - C$. Hence, it follows from (4.5) that

$$(4.14)\quad \lim_{x\to\infty} P\{S_{n'(0)} > x \mid S_{r(A,-B)} < -B\} = 0$$

uniformly in $B > 0$. Since by the definition $S_k \neq 0$ for $r(A, -B) < k < n'(0)$ one has

$$(4.15)\quad \begin{aligned}&E\{\text{number of indices } k \text{ with } r(A, -B) < k < n(A)\\ &\qquad\qquad \text{for which } S_k = 0 \mid S_{r(A,-B)} < -B\}\\ &= \textstyle\sum_{j=0}^{A} P\{S_{n'(0)} = j \mid S_{r(A,-B)} < -B\}\\ &\quad \cdot E\{\text{number of indices } k \text{ with } n'(0) \leqq k < n(A)\\ &\qquad\qquad \text{for which } S_k = 0 \mid S_{n'(0)} = j\}.\end{aligned}$$

However, for fixed $j > 0$

$$(4.16)\quad \begin{aligned}&E\{\text{number of indices } k \text{ with } n'(0) \leqq k < n(A)\\ &\qquad\qquad \text{for which } S_k = 0 \mid S_{n'(0)} = j\}\\ &= p_{-j}(A)(1 - p_0(A))^{-1}.\end{aligned}$$

Hence from (4.12), (4.14), (4.15), (2.4), (2.6), and

$$E\{\text{number of indices } k \leqq r(A, -B) \text{ for which } S_k = 0\} = (1 - p_0(A, -B))^{-1}$$

one obtains, by substituting $B = cA$ and multiplying (4.12) with $1 - p_0(A)$,

$$(4.17)\quad \begin{aligned}&\lim_{A\to\infty} [1 - \underline{\pi}(A, -cA) - (1 - p_0(A, -cA))^{-1}(1 - p_0(A))]\\ &= \lim_{A\to\infty} [\bar{\pi}(A, -cA) - (1 - p_0(A, -cA))^{-1}(1 - p_0(A))] = 0.\end{aligned}$$

Repeating the argument with the roles of the upper and lower boundaries interchanged, gives

$$(4.18)\quad \lim_{A\to\infty} [c\underline{\pi}(A, -cA) - (1 - p_0(A, -cA))^{-1}c(1 - p_0(cA))] = 1.$$

Since by (3.8) for $\alpha = 2$

$$\lim_{A\to\infty} (1 - p_0(cA))c(1 - p_0(A))^{-1} = 1,$$

one gets by subtracting (4.18) from (4.17)

$$\lim_{A\to\infty} [\bar{\pi}(A, -cA) - c\underline{\pi}(A, -cA)] = 0.$$

Combining this with

$$\bar{\pi}(A, -cA) + \underline{\pi}(A, -cA) = 1$$

one obtains (4.9). (4.10) follows from (4.17) and (3.8), while (4.11) follows then from Lemma 1.

Even easier than the case $\alpha = 2$ is the case $\alpha = 1$. The remarkable content of the next theorem is that the addition of a second absorbing barrier has asymptotically no influence on the number of visits to I when $\alpha = 1$.

THEOREM 3. *If* (2.2), (2.3), *and* (4.1) *are satisfied with* $\alpha = 1$, *then for any* $c > 0$

$$\lim_{A\to\infty} \log A(1 - p_0(A, -cA)) = \lim_{A\to\infty} \log A(1 - p_0(A)) = \pi Q. \tag{4.19}$$

Consequently

$$\lim_{A\to\infty} P\{N_I(A, -cA) \leqq \log A \cdot C(1, Q)\mu(I)x\} = 1 - e^{-x}. \tag{4.20}$$

Proof. Instead of the quantities π we shall here work with

$$\begin{aligned}\bar{\rho}(A, -B) = P\{&S_{r(A,-B)} > A \text{ and there exists a } k > r(A, -B) \text{ such} \\ &\text{that } S_k = 0 \text{ but } S_i \geqq -B \text{ for } r(A, -B) \leqq i \leqq k\}\end{aligned} \tag{4.21}$$

and

$$\begin{aligned}\underline{\rho}(A, -B) = P\{&S_{r(A,-B)} < -B \text{ and there exists a } k > r(A, -B) \\ &\text{such that } S_k = 0 \text{ but } S_i \leqq A \text{ for } r(A, -B) \leqq i \leqq k\}.\end{aligned} \tag{4.22}$$

The interpretation is again easy. E.g., $\bar{\rho}$ is the probability that the upper boundary is reached first and that afterwards zero is visited before the lower boundary is reached. Instead of (4.15) and (4.16) use now

$$\begin{aligned}&E\{\text{number of indices } k \text{ with } r(A, -B) < k < n(A) \text{ for which } S_k = 0\} \\ &\quad = P\{\text{there exists a first } k \text{ with } r(A, -B) < k < n(A) \text{ for which } S_k = 0\} \\ &\quad\quad \cdot (1 - p_0(A))^{-1} = \underline{\rho}(A, -B)(1 - p_0(A))^{-1}.\end{aligned}$$

Analogous to (4.12) one then has for $c > 0$

$$(1 - p_0(A))^{-1} = (1 - p_0(A, -cA))^{-1} + \underline{\rho}(A, -cA)(1 - p_0(A))^{-1}. \tag{4.23}$$

Changing the role of the upper and lower boundary gives

$$\begin{aligned}&(1 - p_0(cA))^{-1} \\ &\quad = (1 - p_0(A, -cA))^{-1} + \bar{\rho}(A, -cA)(1 - p_0(cA))^{-1}.\end{aligned} \tag{4.24}$$

Subtracting (4.24) from (4.23) and multiplying by $(\log A)^{-1}$, one obtains by (3.9) as $A \to \infty$

$$\lim_{A\to\infty} [\underline{\rho}(A, -cA) - \bar{\rho}(A, -cA)] = 0. \tag{4.25}$$

On the other hand, if $n'(0)$ has the same meaning as in (4.13)

$$\underline{\rho}(A, -cA) \leqq P\{S_{r(A,-cA)} < -cA \text{ and } S_{n'(0)} \leqq A\},$$

while by (4.6)

$$\lim_{A\to\infty} P\{S_{n'(0)} \leqq A \mid S_{r(A,-cA)} = -\tilde{c}A < -cA\}$$
$$= \lim_{A\to\infty} P\{S_{n(\tilde{c}A)} - \tilde{c}A \leqq A\} = F_1(\tilde{c}^{-1}).$$

Consequently

$$\limsup_{A\to\infty} \underline{\rho}(A, -cA) \leqq F_1(c^{-1}). \tag{4.26}$$

Using the obvious inequality

$$\bar{\rho}(A, -cA) \leqq \bar{\rho}(A, -\tilde{c}A) \quad \text{if} \quad \tilde{c} \geqq c$$

one derives from (4.25) and (4.26)

$$\limsup_{A\to\infty} \bar{\rho}(A, -cA) \leqq \limsup_{A\to\infty} \underline{\rho}(A, -\tilde{c}A) \leqq F_1(\tilde{c}^{-1}).$$

As $\lim_{\tilde{c}\to\infty} F_1(\tilde{c}^{-1}) = 0$, it follows that $\limsup_{A\to\infty} \bar{\rho}(A, -cA) = 0$ for each fixed $c > 0$. (4.19) follows then from (4.23) or (4.24) and (3.9). The proof is completed by an application of Lemma 1.

For the case $1 < \alpha < 2$ it seems much harder to get explicit results. We have seen in the proofs of Theorems 2 and 3 that it is useful to know the conditional distribution of $S_{r(A,-B)}$ given $S_{r(A,-B)} < -B$ and the distribution of $S_{n'(0)}$. These will be considered in the next lemmas and Theorem 4.

LEMMA 7. *Let*

$$F_\alpha(x) = \frac{\sin(\pi\alpha/2)}{\pi} \int_0^x t^{-\alpha/2}(1+t)^{-1}\,dt.$$

If $0 < \alpha < 2$, *then for any* $c, d > 0$ *and* $x \geqq 0$

$$\lim_{k\to\infty} \int_0^\infty dF_\alpha(x_1 d^{-1}) \int_0^\infty dF_\alpha(x_2(1+c+x_1)^{-1})$$
$$\cdot \int_0^\infty dF_\alpha(x_3(1+c+x_2)^{-1}) \cdots \tag{4.27}$$
$$\cdot \int_0^\infty dF_\alpha(x_k(1+c+x_{k-1})^{-1})\, F_\alpha(x(1+c+x_k)^{-1})$$

exists and is independent of d. *If this limit is denoted by* $L_\alpha(x; c)$, *then for* $1 < \alpha < 2$, $L_\alpha(x; c)$ *is a proper distribution function giving probability* 1 *to* $[0, \infty)$, *while for* $0 < \alpha \leqq 1$, $L_\alpha(x; c) = 0$ *for all* x.

Proof. Put in (4.27)

$$x_1 = u_1 d, \qquad x_i = u_i(1+c+x_{i-1}) \quad (i = 2, \cdots, k),$$
$$y = u_{k+1}(1+c+x_k).$$

One has then

$$
\begin{aligned}
y &= u_{k+1}(1+c) + u_{k+1}x_k \\
&= u_{k+1}(1+c) + u_{k+1}u_k(1+c) + u_{k+1}u_k x_{k-1} \qquad (4.28)\\
&= (1+c)\textstyle\sum_{i=2}^{k+1}\prod_{j=i}^{k+1} u_j + d\prod_{j=1}^{k+1} u_j ,
\end{aligned}
$$

and the repeated integral in (4.27) represents $P\{y \leqq x\}$ if $u_1, \cdots, u_{k+1}$ are independent random variables for each of which

$$P\{u_i \leqq u\} = F_\alpha(u).$$

Reversing the numbering of the u's, the repeated integral represents

$$(4.29) \quad P\{(1+c)(u_1 + u_1 u_2 + \cdots + u_1 u_2 \cdots u_k) + du_1 \cdots u_{k+1} \leqq x\}.$$

Assume now $1 < \alpha < 2$. Then

$$
\begin{aligned}
E \log u_i &= \frac{\sin(\pi\alpha/2)}{\pi}\int_0^\infty \log t\, t^{-\alpha/2}(1+t)^{-1}\,dt \\
&= \frac{\sin(\pi\alpha/2)}{\pi}\int_1^\infty \log t(1+t)^{-1}[t^{-\alpha/2} - t^{\alpha/2-1}]\,dt < 0,
\end{aligned}
$$

and $(1+c)(u_1 + u_1u_2 + \cdots + u_1u_2\cdots u_k) + du_1\cdots u_{k+1}$ converges with probability 1 to a random variable not depending on d as $k \to \infty$. The limit $L_\alpha(x; c)$ in (4.27) therefore exists and is the distribution function of this limiting random variable.

If, however, $0 < \alpha < 1$, we obtain $E \log u_i > 0$ and for $\alpha = 1$, $E \log u_i = 0$. In both cases one sees

$$\liminf_{k\to\infty} u_1 u_2 \cdots u_k \geqq 1 \qquad \text{with probability 1}$$

(if $\alpha < 1$ by the strong law of large numbers, and if $\alpha = 1$ by Theorem 4 in [5]). Consequently, for $\alpha \leqq 1$ the limit in (4.27) equals zero for every $c \geqq 0$, $d > 0$.

Let (2.3) be satisfied with $0 < \alpha < 2$. The distribution of $S_{r(A,-cA)}$ is then determined by the following two functions:

$$G_\alpha(x; A, c) = P\{-(x+c)A \leqq S_{r(A,-cA)} < -cA\},$$

$$H_\alpha(x; A, c) = P\{A < S_r(A, -cA) \leqq (1+x)A\}.$$

G and H are monotonic in x and bounded. Hence we can find a sequence $A_1 < A_2 < \cdots$ and bounded monotonic increasing functions $G_\alpha(x; c)$ and $H_\alpha(x; c)$ such that

$$(4.30) \qquad \lim_{j\to\infty} G_\alpha(x; A_j, c) = G_\alpha(x; c),$$

$$(4.31) \qquad \lim_{j\to\infty} H_\alpha(x; A_j, c) = H_\alpha(x; c).$$

It is not hard to see that $G_\alpha(x; c)$ and $H_\alpha(x; c)$ have to be continuous, using the definitions of G and H, (4.6), and the fact that $F_\alpha(x)$ is continuous.

LEMMA 8. *If* (2.3), (4.1), (4.30), *and* (4.31) *are satisfied with* $0 < \alpha < 2$, *then*

$$G_\alpha(x; c) = \textstyle\sum_{k=0}^\infty G_\alpha^{(k)}(x; c) + gL_\alpha(x; c), \tag{4.32}$$

$$H_\alpha(x; c) = \textstyle\sum_{k=0}^\infty H_\alpha^{(k)}(x; c) + hL_\alpha(x; c), \tag{4.33}$$

where[3]

$$\begin{aligned} G_\alpha^{(k)}(x; c) &= \\ &\int_0^\infty dF_\alpha(x_1 c^{-1}) \int_0^\infty dF_\alpha(x_2(1 + c + x_1)^{-1}) \cdots \int_0^\infty dF_\alpha(x_{2k}(1 + c + x_{2k-1})^{-1}) \\ &\qquad \cdot F_\alpha(x(1 + c + x_{2k})^{-1}) \\ &- \int_0^\infty dF_\alpha(x_1) \int_0^\infty dF_\alpha(x_2(1 + c + x_1)^{-1}) \cdots \int_0^\infty dF_\alpha(x_{2k+1}(1 + c + x_{2k})^{-1}) \\ &\qquad \cdot F_\alpha(x(1 + c + x_{2k+1})^{-1}) \end{aligned}$$

and

$$\begin{aligned} H_\alpha^{(k)}(x; c) &= \\ &\int_0^\infty dF_\alpha(x_1) \int_0^\infty dF_\alpha(x_2(1 + c + x_1)^{-1}) \cdots \int_0^\infty dF_\alpha(x_{2k}(1 + c + x_{2k-1})^{-1}) \\ &\qquad \cdot F_\alpha(x(1 + c + x_{2k})^{-1}) \\ &- \int_0^\infty dF_\alpha(x_1 c^{-1}) \int_0^\infty dF_\alpha(x_2(1 + c + x_1)^{-1}) \cdots \int_0^\infty dF_\alpha(x_{2k+1}(1 + c + x_{2k})^{-1}) \\ &\qquad \cdot F_\alpha(x(1 + c + x_{2k+1})^{-1}). \end{aligned}$$

Proof. We shall prove (4.32). Practically the same proof applies to (4.33). Let

$$s(A, -cA) = \text{the first index } s > n(A) \text{ for which } S_s < -cA.$$

Note that $s(A, -cA) = m(-cA)$ only if the upper boundary is reached

---

[3] $$G_\alpha^{(0)}(x; c) = F_\alpha(xc^{-1}) - \int_0^\infty dF_\alpha(x_1)F_\alpha(x(1 + c + x_1)^{-1})$$

and

$$H_\alpha^{(0)}(x; c) = F_\alpha(x) - \int_0^\infty dF_\alpha(x_1 c^{-1})F_\alpha(x(1 + c + x_1)^{-1}).$$

before the lower boundary, i.e., if $S_{r(A,-cA)} = S_{n(A)} > A$. We have then

$$\begin{aligned} G_\alpha(x; A, c) &= P\{-(x+c) \leqq S_{m(-cA)} < -cA\} \\ &\quad - P\{r(A, -cA) = n(A) < m(-cA) = s(A, -cA) \text{ and } \\ &\qquad\qquad -(x+c)A \leqq S_{s(A,-cA)} < -cA\} \\ &= P\{-(x+c)A \leqq S_{m(-cA)} < -cA\} \\ &\quad - P\{-(x+c) \leqq S_{s(A,-cA)} < -cA\} \\ &\quad + P\{r(A, -cA) = m(-cA) < n(A) < s(A, -cA) \text{ and } \\ &\qquad\qquad -(x+c)A \leqq S_{s(A,-cA)} < A\}. \end{aligned} \tag{4.34}$$

By (4.6)

$$\lim_{A\to\infty} P\{-(x+c)A \leqq S_{m(-cA)} < -cA\} = F_\alpha(xc^{-1}),$$

and similarly

$$\lim_{A\to\infty} P\{-(x+c)A \leqq S_{s(A,-cA)} < -cA\} = \int_0^\infty dF_\alpha(x_1)F_\alpha(x(1+c+x_1)^{-1}).$$

The last term, for $A = A_j$ can be written as

$$\int_0^\infty dG_\alpha(x_1; A_j, c) \int_0^\infty dP\{S_{n(A_j(1+c+x_1))} - A_j(1+c+x_1) \leqq x_2 A_j\}$$
$$\cdot P\{-S_{m(-A_j(1+c+x_2))} - A_j(1+c+x_2) \leqq xA_j\}$$

and tends as $j \to \infty$ to

$$\int_0^\infty dG_\alpha(x_1; c) \int_0^\infty dF_\alpha(x_2(1+c+x_1)^{-1})F_\alpha(x(1+c+x_2)^{-1}).$$

Combining these results one sees from (4.34) by letting $A \to \infty$ through the values A_j, that

$$\begin{aligned} G_\alpha(x;c) &= F_\alpha(xc^{-1}) - \int_0^\infty dF_\alpha(x_1)F_\alpha(x(1+c+x_1)^{-1}) \\ &\quad + \int_0^\infty dG_\alpha(x_1; c) \int_0^\infty dF_\alpha(x_2(1+c+x_1)^{-1})F_\alpha(x(1+c+x_2)^{-1}). \end{aligned}$$

Iterating this equation one obtains

$$\begin{aligned} G_\alpha(x;c) &= \sum_{k=0}^{N-1} G_\alpha^{(k)}(x;c) + \int_0^\infty dG_\alpha(x_1; c) \int_0^\infty dF_\alpha(x_2(1+c+x_1)^{-1}) \\ &\quad \cdots \int_0^\infty dF_\alpha(x_{2N}(1+c+x_{2N-1})\, F_\alpha(x(1+c+x_{2N})^{-1}). \end{aligned} \tag{4.35}$$

Since $F_\alpha(x)$ decreases as x decreases, it follows from the definition that $G_\alpha^{(k)}(x; c) \geqq 0$ (this is also obvious from the interpretations (4.28) and (4.29)). Hence, letting $N \to \infty$ in (4.35) and using Lemma 7 one gets

$$G_\alpha(x; c) = \sum_{k=0}^{\infty} G_\alpha^{(k)}(x; c) + \int_0^\infty dG_\alpha(x_1; c) L_\alpha(x; c),$$

This proves (4.32) with $g = 0$ if $\alpha \leqq 1$, and if $1 < \alpha < 2$ with

$$g = \int_0^\infty dG_\alpha(x_1; c) = \lim_{x \to \infty} G_\alpha(x; c). \tag{4.36}$$

(Since G_α was already known to be bounded, the proof shows at the same time that

$$\sum_{k=0}^{\infty} G_\alpha^{(k)}(x; c)$$

converges and that g is finite, so that (4.32) makes sense.)

Notice that g and h may still depend on the sequence $\{A_j\}$. That this is not so will be proved in Lemma 10.

Let us consider again the quantities $p_{-b}(A)$ = probability of reaching $-b$ before any partial sum exceeds A.

It seems reasonable that $p_{-b}(A)$ has a limit if $A, b \to \infty$, such that $bA^{-1} \to y$ and that this limit is continuous in y. Since the proof of this fact is slightly tedious and not enlightening, we do not reproduce it, but rather compute the value of this limit.

LEMMA 9. *Let* (2.2), (2.3), *and* (4.1) *be satisfied with* $1 < \alpha < 2$, *and let* $p_{-b}(A) \to p_\alpha(y)$ *if* $A, b \to \infty$ *such that* $bA^{-1} \to y(1 - y)^{-1}$ $(0 < y < 1)$. *Then*

$$\begin{aligned} p_\alpha(y) &= -y \frac{d}{dy} \int_y^1 (t^{-\alpha/2} - t^{\alpha/2-1})(t - y)^{\alpha/2-1}\, dt \\ &= -y \frac{d}{dy} \int_1^{y^{-1}} (w^{-\alpha/2} - w^{\alpha/2-1} y^{\alpha-1})(w - 1)^{\alpha/2-1}\, dw \\ &= (\alpha - 1) y^{\alpha-1} \int_y^1 w^{-\alpha}(1 - w)^{\alpha/2-1}\, dw. \end{aligned}$$

Proof. Analogous to (4.23) one has for $0 < y < 1$

$$\begin{aligned} (1 - p_0(A))^{-1} &= (1 - p_0(yA))^{-1} + E\,\{\text{number of indices } k \text{ with } \\ &\qquad n(yA) < k < n(A) \text{ for which } S_k = 0\} \\ &= (1 - p_0(yA))^{-1} + \int_y^1 dP\{S_{n(yA)} \leqq tA\} \cdot p_{-tA}((1 - t)A) \cdot (1 - p_0(A))^{-1}. \end{aligned}$$

Multiplying by $1 - p_0(A)$ and using (3.8) and (4.6) one gets as $A \to \infty$

$$1 = y^{\alpha-1} + \frac{\sin(\pi\alpha/2)}{\pi} \int_y^1 (t - y)^{-\alpha/2} y^{\alpha/2} t^{-1} p_\alpha(t)\, dt,$$

or

$$[y^{-\alpha/2} - y^{\alpha/2-1}] \frac{\pi}{\sin(\pi\alpha/2)} \int_y^1 [t^{-1} p_\alpha(t)](t - y)^{-\alpha/2}\, dt.$$

This is an integral equation of Abel's type [2] for $t^{-1}p_\alpha(t)$. Solving it along the standard lines gives the lemma (cf. [2], pp. 8–10).

LEMMA 10. *If* (2.3) *and* (4.1) *are satisfied with* $0 < \alpha < 2$, *then*

$$\lim_{A\to\infty} G_\alpha(x; A, c) = G_\alpha(x; c) \tag{4.37}$$

and

$$\lim_{A\to\infty} H_\alpha(x; A, c) = H_\alpha(x; c) \tag{4.38}$$

exist, and

$$G_\alpha(x; c) = \sum_{k=0}^{\infty} G_\alpha^{(k)}(x; c) + gL_\alpha(x; c), \tag{4.39}$$

$$H_\alpha(x; c) = \sum_{k=0}^{\infty} H_\alpha^{(k)}(x; c) + hL_\alpha(x; c), \tag{4.40}$$

where g *and* h *for* $1 < \alpha < 2$ *are determined by*[4]

$$g + h = 1 \tag{4.41}$$

and

$$\begin{aligned} 1 - c^{\alpha-1} = &\int_0^\infty dG_\alpha(x_1; c) \int_0^1 p_\alpha(x_2)\, dF_\alpha(x_2(c + x_1)^{-1}) \\ &- c^{\alpha-1} \int_0^\infty dH_\alpha(x_1; c) \int_0^c p_\alpha(x_2 c^{-1})\, dF_\alpha(x_2(1 + x_1)^{-1}). \end{aligned} \tag{4.42}$$

For $0 < \alpha \leqq 1$, $g = h = 0$. *Furthermore, if* $1 < \alpha < 2$

$$g = \lim_{A\to\infty} P\{S_{r(A,-cA)} < -cA\}, \qquad h = \lim_{A\to\infty} P\{S_{r(A,-cA)} > A\}. \tag{4.43}$$

Proof. Let $\{A_j\}$ be any sequence such that (4.30) and (4.31) are satisfied for some G_α and H_α. For $0 < \alpha \leqq 1$ it was proved in Lemma 8 that (4.39) and (4.40) have to be satisfied with $g = h = 0$. If $1 < \alpha < 2$, then

$$\lim_{j\to\infty} P\{S_{r(A_j,-cA_j)} < -cA_j\} = \lim_{x\to\infty} G_\alpha(x; c) = g$$

(cf. (4.36)), while similarly

$$\lim_{j\to\infty} P\{S_{r(A_j,-cA_j)} > A\} = h.$$

Hence

$$g + h = 1.$$

In addition, for $x \geqq 0$

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[4] One can also prove that G_α must satisfy the equation

$$p_\alpha(x) = \int_0^\infty (1 - y)^{\alpha-1} p_\alpha(y(1 + y)^{-1})\, dG_\alpha(y(1 - x)^{-1}, x(1 - x)^{-1}).$$

This of course can be used to determine G_α instead of (4.41) and (4.42). The quantities in (4.41) and (4.42), however, have interesting probability interpretations and will be computed in a subsequent paper.

$$\lim_{j\to\infty} P\{S_{r(A_j,-cA_j)} < -cA_j \quad\text{and}\quad S_{n'}(0) \leqq xA\} = \int_0^\infty dG_\alpha(x_1; c) F_\alpha(x(c + x_1)^{-1}).$$

Hence if $\underline{\rho}(A, -cA)$ has the same meaning as in (4.22), then

$$\lim_{j\to\infty} \underline{\rho}(A_j, -cA_j) = \int_0^\infty dG_\alpha(x_1; c) \int_0^1 p_\alpha(x_2)\, dF_\alpha(x_2(c + x_1)^{-1}).$$

Even though Lemma 9 was proved with the help of assumption (2.2), it is easily seen that the above expression remains valid without (2.2); compare also footnote 2. Similarly

$$\lim_{j\to\infty} \bar{\rho}(A_j, -cA_j) = \int_0^\infty dH_\alpha(x_1; c) \int_0^c p_\alpha(x_2 c^{-1})\, dF_\alpha(x_2(1 + x_1)^{-1}).$$

Instead of (4.25), we obtain from (4.23) and (4.24)

$$\begin{aligned}\lim_{j\to\infty} A_j^{1-\alpha}(1 - p_0(A_j))^{-1} - \lim_{j\to\infty} A_j^{1-\alpha}(1 - p_0(cA_j))^{-1} &= \lim_{j\to\infty} \underline{\rho}(A_j, -cA_j) A^{1-\alpha}(1 - p_0(A_j))^{-1} \\ &\quad - \lim_{j\to\infty} \bar{\rho}(A_j, -cA_j) A^{1-\alpha}(1 - p_0(cA_j))^{-1}.\end{aligned}$$

By using (3.8) this reduces to (4.42). Clearly (4.39)–(4.42) determine g and h uniquely, so that g and h cannot depend on the particular sequence $\{A_j\}$, and (4.37), (4.38), (4.43) must hold.

THEOREM 4. *If (2.2), (2.3), and (4.1) are satisfied with* $1 < \alpha < 2$, *then*

$$\lim_{A\to\infty} A^{1-\alpha}(1 - p_0(A, -cA))^{-1} = (1 - \underline{\rho}) C(\alpha, Q) \tag{4.44}$$

where

$$\underline{\rho} = \int_0^\infty dG_\alpha(x_1; c) \int_0^1 p_\alpha(x_2)\, dF_\alpha(x_2(c + x_1)^{-1}). \tag{4.45}$$

Consequently

$$\lim_{A\to\infty} P\{N_I(A, -cA) \leqq A^{\alpha-1}(1 - \underline{\rho}) C(\alpha, Q)\mu(I)x\} = 1 - e^{-x}. \tag{4.46}$$

Proof. By Lemma 10

$$\lim \underline{\rho}(A, -cA)$$

is given by (4.45). (4.44) follows now from (4.23) and (3.8), while (4.46) follows from Lemma 1.

Remark. The solutions for G_α and H_α have recently been obtained by R. M. Blumenthal and R. K. Getoor and independently by H. Widom and will be discussed together with their applications to Toeplitz forms in a subsequent paper. The explicit expression for $1 - \underline{\rho}$ in Theorem 4 turns out to be $(c/(c + 1))^{\alpha-1}$.

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