EXISTENCE THEOREMS FOR NONPROJECTIVE COMPLETE ALGEBRAIC VARIETIES

BY

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The purpose of the present paper is to prove the following two theorems:

THEOREM 1. *Let L be a function field over a ground field k. Assume that* $\dim L$ *is not less than* 2. *Assume furthermore that if* $\dim L = 2$, *then k is sufficiently large.*[1] *Then there exists a complete normal abstract variety of L which is not projective.*

THEOREM 2. *If n is a natural number not less than* 3, *then there exists a complete nonsingular variety of dimension n which is not projective*; *more explicitly, there exists a nonsingular complete variety of the rational function field of dimension n, which is defined over the prime field and which is not projective.*

We shall remark that, since Zariski [4] proved that a normal abstract surface can be imbedded in a projective surface (as an open subset) if there exists an affine variety which carries all singular points of the given surface, our results give a complete answer for the imbedding problem in one sense. Therefore it will be an important problem to give some sufficient conditions for a given variety to be projective.[2] It will be also an interesting problem to characterize function fields which have nonsingular complete nonprojective varieties.

1. Two lemmas

LEMMA 1. *Let V and V' be varieties. If V is not projective, then $V \times V'$ is not projective.*

Proof. $V \times V'$ contains a nonprojective subvariety $V \times P'$ $(P' \in V')$, and therefore $V \times V'$ is not projective.

LEMMA 2. *Let V be a normal variety with function field L, and let L' be a finite algebraic extension of L. Let V' be the derived normal variety of V in L'. If V' can be imbedded in a projective variety V'', then V can be imbedded in a projective variety.*

Proof. We may assume that V' is an open subset of V''. Let P be a generic point of V over a ground field k, and let $Z(P)$ be $\sum P'_i$, where P'_i form

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[1] The meaning of "large" will be explained in the course of the proof.

[2] Cf. Chow [2], Chevalley [1], and Weil [3]. On the other hand, the following problem was offered by Chevalley a few years ago:

Assume that a normal variety V satisfies the following condition: For any finite number of points of V, there exists an affine variety which carries them. Can then V be imbedded in a projective variety?

the complete set of conjugates of a generic point of V', which corresponds to P, over $k(P)$. The locus V^* of $Z(P)$ over k, i.e., the Chow variety of $Z(P)$ over k, is a projective variety and has the following properties: (i) the mapping $P \to Z(P)$ induces a regular mapping from V into V^*, and (ii) every point of V corresponds to a point of V^* in a one-to-one way by the regular mapping defined above. Therefore the derived normal variety of V^* in L contains an open subset which is biregular with V. Thus the lemma is proved.

Now, by virtue of these lemmas, in order to prove the theorems it is sufficient to show (1) an example of a normal complete nonprojective variety of the rational function field of dimension 2, and (2) an example of a complete nonsingular nonprojective variety of the rational function field of dimension 3 which is defined over the prime field.

These examples will be constructed in §4 and §6.

2. A general remark on construction of complete abstract varieties

Let V be a complete abstract variety (which may be projective), and let D be a subvariety of V. Let V' be a variety which is birationally equivalent to V, D' a subvariety of V', and assume that there exist open sets V^* and V'^* of V and V' respectively, satisfying the following conditions: (i) D is the set of points of V which correspond to points of D', (ii) $D \subset V^*$, (iii) $D' \subset V'^*$, (iv) V^* dominates V'^*, and (v) $V^* - D$ is biregular with $V'^* - D'$. In this case, we say that D is a *strictly antiregular total transform* of D' (in V).

Under the above assumption, it is easily seen that $V - D + D'$ is a complete abstract variety.[3] Therefore, if mutually disjoint subvarieties $D_1, \cdots, D_n$ of a projective variety V are the strictly antiregular total transforms of $D_1', \cdots, D_n'$ respectively, where the D_i' are not necessarily on the same variety, then we see that the set $V - (\sum D_i) + \sum D_i'$ is a complete variety.

We shall add here the following remark:

Let V and V' be birationally equivalent normal projective surfaces. If a curve E on V is the antiregular total transform of a point $P' \in V'$, then E is a strictly antiregular total transform of P', because there exists only a finite number of fundamental points on V' with respect to the birational correspondence with V. From this, we deduce

LEMMA 3. *Let V be a normal projective surface. If an irreducible curve E on V is the antiregular total transform of a point P' of a surface V', then E is a strictly antiregular total transform of a normal point.*

Proof. Let V'' be the derived normal variety of V' on the function field of V. Since V is normal and since the transformation $T': V \to V'$ is regular at each point of E, the mapping $T'': V \to V''$ is also regular at each point of

[3] Here, $V^* - D$ and $V'^* - D'$ are identified by the biregular correspondence.

E. Since E is irreducible and since $T'\{E\}$ is a point, $T''\{E\}$ is a point, say P''. Since $E = T'^{-1}\{P'\}$, we see that $E = T''^{-1}\{P''\}$. Now we have Lemma 3 by the remark stated just before Lemma 3.

3. A remark on the rational mapping defined by a linear system

Let V be a normal projective variety. If L is a linear system of divisors on V, then L defines a rational mapping T from V onto another projective variety V', where V' is defined as follows: Let $D_0, D_1, \cdots, D_n$ be a basis of L, and let f_i be the function on V such that $(f_i) = D_i - D_0$. Then V' is the projective variety with generic point $(1, f_1, \cdots, f_n)$.

Now we want to point out the following well known and elementary facts:

(1) If a point P of V is not a base point of L (i.e., if there exists a member $D \in L$ such that $P \notin D$), then T is regular at P.

(2) If $P, Q \in V$ and if there exist members $D, D' \in L$ such that $P \in D$, $Q \notin D$, $P \notin D'$, $Q \in D'$, then $T(P) \neq T(Q)$.

(3) Let E be an irreducible subvariety of V. If there exists a member $D \in L$ such that E does not meet D, then $T\{E\}$ is a point.

4. An example of a nonprojective rational surface

EXAMPLE 1. *Let C and D be independent generic curves of degree 3 and 4 respectively in the projective plane S, and let $P_1, \cdots, P_{12}$ be their intersections. Let E be the most general cubic curve among those which go through P_1, P_2 and P_3, and let $Q_1, \cdots, Q_9$ be the intersections of D and E other than P_1, P_2, P_3. Now let S' be the quadratic transform of S with centers $P_1, \cdots, P_{12}, Q_1, \cdots, Q_9$, and let C' and D' be the proper transforms of C and D respectively. Then, C' and D' are strictly antiregular total transforms of normal points, say C^* and D^* respectively, and the complete normal variety $S^* = S' - (C' + D') + C^* + D^*$ is not projective.*

Proof. (1) We shall show at first that C' is the strictly antiregular total transform of a normal point. Let p_i, q_j be the total transform of the points P_i, Q_j respectively in S'. We shall denote in general by l a projective line (hyperplane) in S and by $T\{l\}$ the total transform of l in S'. Since $l + C \sim D$, we have $T\{l\} + C' + \sum p_i \sim D' + \sum p_i + \sum q_j$; hence $T\{l\} + C' \sim D' + \sum q_j$. Let V be the projective variety defined by the complete linear system $| T\{l\} + C' |$ on S'. Since C' does not meet the member $D' + \sum q_i$ of $| T\{l\} + C' |$, it follows by (3) in §3 that C' is mapped to a point on V, say C''. By (2) in §3, we see now easily that C' is the antiregular total transform of C''. Therefore by Lemma 3, C' is the strictly antiregular total transform of a normal point.

(2) That D' is the strictly antiregular total transform of a normal point can be proved by a method similar to that above. Namely, we consider, instead of l, curves l'' of degree 2 on S which go through P_1, P_2, P_3; instead

of $T\{l\}$, we consider the cycle: [total transform of l'' in S'] $- p_1 - p_2 - p_3$. Then using the fact that $D + l'' \sim C + E$, we see that D' is the strictly antiregular total transform of a normal point.

(3) Before proving that S^* is not projective, we shall make some remarks on the points $P_1, \cdots, P_{12}, Q_1, \cdots, Q_9$.

We shall denote by π the prime field.

Let L_4 be the trace of the linear system of curves of degree 4 on C. Then L_4 has degree 12 and dimension 11, and hence is complete, because C is of genus 1. Now, since D is generic, we see that 11 of the points $P_1, \cdots, P_{12}$ are independent generic points of C over $\pi(C)$. From this we deduce the following:

(i) *If a curve F is such that $F \cdot C = \sum a_i P_i$, then $a_1 = a_2 = \cdots = a_{12}$.*

Proof. Assume, for instance, that $a_1 \leqq a_j$ for any j. Then $(F - a_1 D) \cdot C = \sum b_i P_i$ with $b_1 = 0$ and $b_j = a_j - a_1 \geqq 0$. Therefore there exists a curve F' of degree equal to $(\deg F - 4a_1)$ such that $F' \cdot C = \sum b_i P_i$ $(b_1 = 0)$. Since $P_2, \cdots, P_{12}$ are independent generic points of C, and since C is of positive genus, this is impossible, unless all the $b_i = 0$. Therefore $a_1 = a_2 = \cdots = a_{12}$.

Next we consider the fields of definition of S^*. S^* is defined over any field k such that C, D, E, and $P_1 + P_2 + P_3$ are rational over k. Let k_0 be the smallest common field of definition of C, D, E, and $P_1 + P_2 + P_3$: $k_0 = \pi(C, D, E, P_1 + P_2 + P_3)$. Since E is generic over $\pi(C, D, P_1 + P_2 + P_3)$, we see that $\sum Q_i$ is prime rational over k_0. Thus we have

(ii) *k_0 is a field of definition of S^* and $\sum Q_i$ is prime rational over k_0. Furthermore, C^* and D^* are rational over k_0.*

(4) Now we shall prove that S^* is not projective. In order to do so, it is sufficient to prove that any divisorial closed set F^* of S^* must go through either C^* or D^*.[4] Assume the contrary, namely, assume that there exists an irreducible divisor F^* which does not go through any of C^* and D^*. Let K be a field of definition of F^* containing k_0 given above. Let K' be a maximal purely transcendental extension of k_0 contained in K. Then $\sum Q_j$ is still prime rational over K'. Let F^{**} be the prime rational divisorial closed set over K' such that F^* is its component. Since C^* and D^* are rational points over K', F^{**} does not go through any of C^* and D^*. Now, F^{**} must be the proper transform of a prime rational divisorial closed set F of S over K'. We regard F as a prime rational cycle over K'. Since F^{**} does not go through C^*, we see that (i) F and C have no common point outside of $\sum P_i$, and (ii) F and C have no common tangential direction at each P_i. Therefore $F \cdot C = \sum a_i P_i$, and the coefficient a_i is the multiplicity of the point P_i on F. By a remark in (3), we have $a_1 = \cdots = a_{12}$. Thus $F \cdot C = a(\sum P_i)$.

[4] This shows that there exists no nonconstant function which is defined at both C^* and D^*.

Since F^{**} does not go through D^*, we see that (i) F and D have no common point outside of $\sum P_i + \sum Q_j$, and (ii) F and D have no common tangential direction at each P_i, Q_j. Therefore $F \cdot D = \sum c_i P_i + \sum b_j Q_j$, and c_i is the multiplicity of P_i on F. Therefore $c_i = a$. Since F and $\sum Q_j$ are prime rational over K', and since $F \cdot D = a(\sum P_i) + \sum b_j Q_j$, we have $b_1 = \cdots = b_9$. Thus $F \cdot D = a(\sum P_i) + b(\sum Q_j)$. Therefore $(F - aC) \cdot D = b(\sum Q_j)$; hence $(bE + aC - F) \cdot D = b(P_1 + P_2 + P_3)$. Since P_1, P_2, and P_3 are independent generic points of D over $\pi(D)$, and since D is of positive genus, we see that $b = 0$ (cf. the proof of (i) in (3) above). Then we have $F \cdot D = F \cdot C$, which is obviously a contradiction because $\deg D \neq \deg C$. Thus the proof is completed.

5. A lemma on product varieties and an application

Let V_1 be a nonsingular projective variety,[5] and let C be the projective line. Let D_1 be a hyperplane section of V_1 which is also nonsingular,[5] and let P be a point of C. Set $V = V_1 \times C$, $W = D_1 \times P$, $D = D_1 \times C$, $A = V_1 \times P$. Let V' be the monoidal transform of V with the center W, and let A' be the proper transform of A in V'. Then

LEMMA 4. *A' is the strictly antiregular total transform of the vertex of the representative cone of V_1 (i.e., the cone with base variety V_1).*

Remark. As will be seen from the proof below, V' dominates the cone K with the base variety V_1, and the behaviour of the correspondence is as follows: (i) If $Q \in D_1$, then the proper transform of $Q \times C$ in V' is mapped into a point; the proper transform D' of D is mapped to a divisor of a base variety; (ii) A' is mapped to the vertex; and (iii) the correspondence is biregular at each point of $V' - A' - D'$.

Proof. Let $(x_0, \cdots, x_n)$ be strictly homogeneous coordinates of a generic point of V_1, and let C_i be the hyperplane section of V_1 defined by $x_i = 0$. We may assume that D_1 is different from any of the C_i. Let W' be the total transform of W in V'. Let E_i' be the proper transform of $E_i = C_i \times C$ for each i. Since $D \sim E_i$, we have $D' + W' \sim E_i'$, where D' is the proper transform of D in V'. Let R be a point of C which is different from P, set $B = V_1 \times R$, and let B' be the proper transform of B in V'. Then since $A \sim B$, we have $A' + W' \sim B'$. Therefore, on account of the relation $D' + W' \sim E_i'$, we have $A' + E_i' \sim D' + B'$. Now let L be the linear system spanned by $D' + B'$ and the $A' + E_i'$ $(i = 0, 1, \cdots, n)$, and let K be the variety defined by L. By a property of monoidal transformation we see easily that A' and D' have no common point. Therefore we see easily that L has no base point; hence K is dominated by V'. Let $(x_0'', \cdots, x_n'', w'')$ be a generic point of K, where $(x_i''/x_0'') = (A' + E_i') - (A' + E_0')$, $(w''/x_0'') = (D' + B') - (A' + E_0')$. Then we see that $x_i''/x_j'' = x_i/x_j$ for

[5] The assumption of nonsingularity for V_1 and D_1 can be weakened.

any i, j. Furthermore, since the trace of L on the proper transform of $Q \times C$ ($Q \in V$, $Q \notin D_1$) in V' is of degree 1 and has no base point, w''/x_i'' (for each i) generates the function field of C over the function field of V_1. Therefore K is birational with V. This implies, incidentally, that w'' is transcendental over $k(x_0'', \cdots, x_n'')$, where k is a ground field. Therefore, on account of the fact that $x_i''/x_j'' = x_i/x_j$ for any i, j, we see that K is the cone with base variety V_1 and vertex $x_0'' = x_1'' = \cdots = x_n'' = 0$. Since A' has no common point with $D' + B'$, and since A' is contained in $A' + E_i'$, it follows that if P' is any point of A', then at the corresponding point of K we must have $x_i'' = 0$ (for each i) and $w'' \neq 0$. Thus A' is mapped to the vertex of K. (If $Q \in D_1$, then there exists a member of L which does not meet the proper transform Q^* of $Q \times C$, hence Q^* is mapped into a point; this statement is not necessary for the proof of Lemma 4, but is necessary for the proof of the remark.) We shall next show that the mapping from V' to K is biregular at every point of $V' - A' - D'$. Since K is normal outside of the vertex, it is sufficient to show that the points of $V' - A' - D'$ correspond in a one-to-one way with points of K (observe that no point of V', outside of A', corresponds to the vertex of K, as is easily seen from the nature of L). Since K is dominated by V', it is sufficient to show that if Q_1' and Q_2' are distinct points of $V' - A' - D'$, then the corresponding points Q_1^* and Q_2^* are distinct. Let $Q_i \times P_i$ ($Q_i \in V_1$, $P_i \in C$) be the point of V which corresponds to Q_i' ($i = 1, 2$). (i) If $Q_1 \neq Q_2$, then there are hyperplane sections of V_1 which go through one of Q_i and not through the other. Therefore Q_1' and Q_2' are separated by members of L which contain A'.[6] Therefore $Q_1^* \neq Q_2^*$ in this case. (That D' is mapped to a divisor on the base variety can be proved in the same way as here.) (ii) Assume now that $Q_1 = Q_2$. Let l' be either the total transform of $Q_1 \times P$ or the proper transform of $Q_1 \times C$ according to whether $Q_1 \in D_1$ or $Q_1 \notin D_1$. Then the Q_i' are points of l'. The trace of L on l' is a linear system of degree 1 and has no base point. Therefore $Q_1^* \neq Q_2^*$ also in this case. Thus Lemma 4 (and also the remark) is proved completely.

Now we shall apply the above result for a special variety: Let C, C', and C'' be projective lines, and set $V_1 = C' \times C''$, $V = V_1 \times C$. We remark that V_1 is the surface defined by $xy = zw$. We apply the above construction to V; then we get the cone K defined by $xy = zw$ (and with the homogeneous coordinates $(x, y, z, w, 1)$). A plane section of K which does not go through the vertex A^* is the proper transform of $C' \times C'' \times Q$ with $Q \in C$, $Q \neq P$; it can be identified naturally with $C' \times C''$, and we may assume that $x = z = 0$ is a line $C' \times R''$ ($R'' \in C''$). Now we consider the linear pencil L'' on K spanned by the divisors $x = z = 0$ and $w = y = 0$ and let $\bar{L}$ be the minimal sum of L'' and the linear system of plane sections on K. The projective variety $\bar{K}$ defined by $\bar{L}$ certainly dominates K. Since L'' has only one base point A^*, the vertex of K, the same is true of $\bar{L}$, and hence the vertex A^*

[6] Observe that if $Q_i \in D_1$, and if a plane section C' of V_1 goes through Q_i, and if D_1 is not contained in C', then the proper transform of $C' \times C$ contains the total transform of $Q_i \times P$.

of K is the unique fundamental point with respect to $\bar{K}$. The local ring of any points of $\bar{K}$ which corresponds to A^* is a ring of quotients of one of the two rings $k[x, y, z, w, w/x]$ and $k[x, y, z, w, x/w]$ (with respect to a prime ideal containing the elements x, y, z, w). Since $y/z = w/x$, we have $k[x, y, z, w, w/x] = k[x, z, w/x]$ and $k[x, y, z, w, x/w] = k[y, w, x/w]$; these are polynomial rings. Therefore any point of $\bar{K}$ which corresponds to A^* is a simple point. Since K has no singular point other than A^*, we see that $\bar{K}$ is a nonsingular variety. It is easy to see that the total transform of A^* is a projective line, say $\bar{C}$.

Now we consider on the variety V' the linear system L''' spanned by the transforms of $C' \times P'' \times C$, $C' \times Q'' \times C$ (P'', $Q'' \in C''$), which corresponds to L'' on K. Let $\bar{L}'$ be the minimal sum of L''' and the linear system L. Since L corresponds to the system of plane sections of K, the projective variety defined by $\bar{L}'$ is nothing but $\bar{K}$, and the divisor A' is the strictly antiregular total transform of $\bar{C}$; this is easily seen from the nature of $\bar{L}'$. Furthermore, identifying A' naturally with $C' \times C''$ and $\bar{C}$ with C'', we see easily from the nature of $\bar{L}'$ that the mapping from A' to $\bar{C}$ is nothing but the projection, i.e., two points of A' are mapped to the same point if and only if there exists a member of L''' which contains these points.

6. An example of a complete nonsingular nonprojective variety

EXAMPLE 2. *Let C, C', C'' be projective lines, and set $V_1 = C' \times C''$, $V = V_1 \times C$. Let D_1 be an irreducible plane section of V_1. (Observe that V_1 is defined by $xy = zw$, hence we can take D_1 such that it is defined over the prime field and also such that D_1 is nonsingular.) Let P, Q be points of C ($P \neq Q$); they can be chosen to be rational over the prime field. Set $W_1 = D_1 \times P$, $W_2 = D_1 \times Q$. Let V_2 be the monoidal transform of V with the centers W_1 and W_2, and let A', B' be the proper transforms of $A = V_1 \times P$, $B = V_1 \times Q$ respectively. Then by the observation in §5, A' and B' are strictly antiregular total transforms of projective lines l and l' on nonsingular varieties which are birationally equivalent to V. Therefore we have a complete nonsingular abstract variety $V^{**} = V_2 - A' - B' + l + l'$. Here, A' and B' are naturally identified with $C' \times C''$, and the deformation observed in §5 can be done symmetrically with respect to C' and C''. Therefore we deform A' to C'' and B' to C' (i.e., l is identified naturally with C'', and l' is identified naturally with C'; see the observation at the end of §5). Then the variety V^{**} is not projective.*

Proof. If V^{**} is projective, then there exists a divisorial closed set F^{**} which meets properly both l and l'. We shall show that this is impossible. Assume the existence of F^{**}. F^{**} must be the proper transform of a divisorial closed set F on V. We regard F to be a cycle and consider the intersection cycle $F \cdot A$. (i) If E is a component of $F \cdot A$, and if E is neither W_1 nor $C' \times P'' \times P$ ($P'' \in C''$), then $\mathrm{proj}_{C''} E = C''$ and therefore F^{**} must contain l, which is a contradiction. (ii) Assume that $F \cdot A = mW_1$. Then denoting by F' and W' the proper transforms of F in V_2 and of W_1 in A' respectively, we have either F' contains W' or F' does not meet A'. (For F' and A' cannot have a common point outside of W'; if there exists a common point, then the intersection must be a curve, hence it must be W'.) But, each of these cases is impossible because F^{**} meets properly l. By the observations (i) and (ii), we see that $F \cdot A$ must be of the form $mW_1 + \sum C' \times P_i'' \times P$ ($P_i'' \in C''$), and this second term is actually present. Since W_1 is linearly equivalent to $P' \times C'' \times P + C' \times P'' \times P$ ($P' \in C'$, $P'' \in C''$) on A, we have that $F \cdot A$ is linearly equivalent to $a(P' \times C'' \times P) + b(C' \times P'' \times P)$ on A with $b > a$. Symmetrically, the intersection cycle $F \cdot B$ is linearly equivalent to $a'(P' \times C'' \times Q) + b'(C' \times P'' \times Q)$ on B with $a' > b'$. On the other hand, since C is the projective line, F is translated along C to a linearly equivalent divisor F_1 so that P corresponds to Q. Then $F_1 \cdot B$ is linearly equivalent to $a(P' \times C'' \times Q) + b(C' \times P'' \times Q)$ on B. Since $F \sim F_1$, we have $a(P' \times C'' \times Q) + b(C' \times P'' \times Q)$ is linearly equivalent to $a'(P' \times C'' \times Q) + b'(C' \times P'' \times Q)$ on B. Therefore $a = a'$, $b = b'$. (For by considering the intersection number with $P' \times C'' \times Q$, we have $b = b'$; similarly $a = a'$.) Therefore the inequalities $b > a$, $a' > b'$ give a contradiction. Thus the proof is completed.

Remark 1. In the above construction, if we deform A' and B' to l and l' so that both l and l' can be naturally identified with C' (on A' and B' respectively), then the new variety is projective; if A' and B' are deformed to normal points, then the new variety is also projective.

Remark 2. The following question was asked by Takahashi and also by Serre:

Assume that a normal complete variety V of dimension n can be covered by $n + 1$ affine varieties. Is then V a projective variety?

Our Example 2 shows that the answer to this question is negative even if V is nonsingular.

Proof. Take the variety V^{**} in Example 2. $V^{**} - l$ is an open subset of a projective variety, say V_3 (by Remark 1, or it can easily be seen directly). Set $G = V_3 - (V^{**} - l)$, and let L_1 and L_2 be sufficiently general hypersurface sections on V_3 which contain G, and set $A_1 = V_3 - L_1$, $A_2 = V_3 - L_2$. Then A_1 and A_2 cover $V^{**} - l - g$ with $g = (V^{**} - l) \cap L_1 \cap L_2$. Since we have chosen L_1 and L_2 to be general, g is a curve on $V^{**} - l$, and g does not meet l' (because l' is a curve). Similarly, there are two affine varieties A_3 and A_4 contained in $V^{**} - l'$ which cover l and g. Therefore V^{**} is covered by A_1, A_2, A_3, and A_4.

Remark 3. It was communicated to the writer that Kodaira proved that our Example 2 gives an example of a non-Kaehlerian algebraic manifold, if it is constructed over the field of complex numbers. Therefore the following problem will be interesting:

Assume that V is a complete algebraic manifold which is Kaehlerian. Is then V projective?

Added in proof. Hironaka recently proved the following:

If V is a nonsingular projective variety of dimension not less than 3, *then there exists a complete nonsingular nonprojective variety V' such that* (1) *V' is birationally equivalent to V, and* (2) *V' dominates V.*

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