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# A DIRECT CONSTRUCTION OF THE $R$-INVARIANT MEASURE FOR A MARKOV CHAIN ON A GENERAL STATE SPACE

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A theorem due to Tweedie (1974), showing the existence and uniqueness of an $R$-invariant measure for $R$-recurrent Markov chains, is derived by an alternative and direct method.

**0. Introduction.** The basic technique followed in constructing an invariant measure for a recurrent Markov chain on a general measurable state space has been to generalize the technique used in the case of a countable state space. Suitably chosen "small" sets (called $D$-sets by Orey (1971)) will then correspond to points in the countable space. By rather deep methods and long analysis an invariant measure is found to exist (as a Cauchy limit in a Banach space) for the chain on such a $D$-set. This invariant measure is then "lifted" from the $D$-set to the whole of the state space in a way similar to that used in the countable case. The reader is referred to the original paper of Harris (1956) and to Orey (1971) for details of this approach.

The main purpose of the present study is to provide a direct constructive approach to the invariant measure. Since our analysis is identical in the more general $R$-theory context (for which see Tweedie (1974a, b)) we will prove the existence and the uniqueness of an $R$-invariant measure for an $R$-recurrent chain. We shall refer to Orey (1971), Tweedie (1974a) and Tweedie (1974b) as [0], [T1] and [T2] respectively.

**1. Preliminaries and the main result.** Our notation follows [0] closely. Let $S$ be an arbitrary set and $\mathscr{F}$ a $\sigma$-field of subsets of $S$. We consider a Markov chain $\{X_n; n = 0, 1, \cdots\}$ with state space $(S, \mathscr{F})$ and a stationary transition probability function $P: S \times \mathscr{F} \to [0, 1]$ (where we allow the possibility $P(x, S) < 1$). The $n$-step transition probabilities are denoted by $P^n(\cdot, \cdot)$. We denote by $1_A$ the indicator function of the set $A \in \mathscr{F}$ and by $I_A$ the transition probability $I_A(x, B) = 1_{A \cap B}(x)$ $(x \in S, B \in \mathscr{F})$. If $\nu$ is a measure we write $\nu P^n(\cdot)$ for $\int \nu(dx)P^n(x, \cdot)$. For $B \in \mathscr{F}$, let $\bar{B}_j(n) = \{x \in S : P^n(x, B) > j^{-1}\}$, $j, n \in \mathbb{N} = \{1, 2, \cdots\}$, and $\bar{B} = \bigcup_{j,n=1}^{\infty} \bar{B}_j(n)$. A $\sigma$-finite measure $\nu$, not identically zero, is called $r$-subinvariant ($r$-invariant) for $\{X_n\}$ if $\nu \geqq r\nu P$ $(\nu = r\nu P)$.

Our basic assumption is that the chain $\{X_n\}$ is $\varphi$-irreducible for some nontrivial

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$\sigma$-finite measure $\varphi$ on $(S, \mathscr{F})$, i.e.

$$\bar{B} = S \qquad \text{for all} \quad B \in \mathscr{F} \quad \text{such that} \quad \varphi(B) > 0 \, .$$

Let $\tilde{\varphi}$ be any probability measure equivalent to $\varphi$. Then the probability measure $\mu$, defined by

$$\mu = \sum_{n=1}^{\infty} 2^{-n} \tilde{\varphi} P^n \, ,$$

satisfies

(i) $\{X_n\}$ is $\mu$-irreducible, and
(ii) $\mu(B) = 0$ implies $\mu(\bar{B}) = 0$

(see Lemma 1.1 of [T1] for a proof). In the following, $\mu$ denotes any probability measure satisfying (i) and (ii) and $\mathscr{F}^+ = \{A \in \mathscr{F} : \mu(A) > 0\}$.

The following solidarity property is due to [T1]: there is a partition $\mathscr{K}$ of $S$ such that for $\mu$-almost all $x$ the power series $G_z(x, A) = \sum_1^\infty P^n(x, A) z^n$ have a common radius of convergence $R$ for $A \in \mathscr{F}^+$ in any element of $\mathscr{K}$, and they all diverge (*R-recurrence*) or $\mu$-almost all converge (*R-transience*) for $z = R$.

The purpose of this paper is to give a direct proof of the following result, which is Theorem 4 in [T1].

THEOREM. *Suppose that $\{X_n\}$ is R-recurrent. Then there exists a unique R-subinvariant measure $\pi$ for $\{X_n\}$. $\pi$ is R-invariant and equivalent to $\mu$.*

REMARK. Orey [0] proves the existence and the uniqueness of an invariant measure by assuming that either

(a) $\{X_n\}$ is $\varphi$-irreducible for some $\varphi$ and $S$ is properly essential (corollary of page 38), or
(b) $S$ is indecomposable and properly essential (Theorem 8.2 (iii)).

A recent result of Tuominen (1975) shows the equivalence of (a), (b) and the assumption

(c) $\{X_n\}$ is 1-recurrent.

Therefore, if $R = 1$, our theorem above and both results of Orey will all coincide.

PROOF OF THE THEOREM. We need only prove the existence and the uniqueness in the case where $\mathscr{F}$ is countably generated because the general case follows by using the technique of admissible $\sigma$-fields, just as in [0], page 34. The existence statement is contained in Proposition 1 and the uniqueness statement in Proposition 2, given in Sections 2 and 3 below.

It remains to be proved that $\pi$ and $\mu$ are equivalent. A simple argument of Harris (1956) shows that $\mu \ll \pi$: Suppose that $B \in \mathscr{F}$ is such that $\pi(B) = 0$. Then for all $n \in \mathbb{N}$, $R^n \int \pi(dx) P^n(x, B) = \pi(B) = 0$ (by $R$-invariance), and therefore $\pi(\bar{B}) = \pi\{x \in S : P^n(x, B) > 0$ for some $n \in \mathbb{N}\} = 0$. Therefore $\mu(B) > 0$ is impossible because $\mu(B) > 0$ and $\mu$-irreducibility together would imply $\bar{B} = S$.

The fact that $\pi \ll \mu$ is proved symmetrically. By Condition I, $\mu(B) = 0$ implies $\mu(\bar{B}) = 0$. On the other hand, just as in the proof of the corollary on page 34 of [0], one may show that $\{X_n\}$ is $\pi$-irreducible. Therefore $\pi(B) > 0$ is impossible, because then $\bar{B} = S$.

**2. The existence of an $R$-invariant measure.**

PROPOSITION 1. *Suppose that $\mathscr{F}$ is countably generated. Then if $\{X_n\}$ is $R$-recurrent there is an $R$-invariant measure for $\{X_n\}$.*

We start with a lemma which is a dual form of Proposition 3.3 in [T1].

LEMMA 1. *Suppose that $\{X_n\}$ is $R$-recurrent and $\pi$ is $R$-subinvariant. Then $\pi$ is $R$-invariant.*

PROOF. The proof follows the lines of an argument given in Neveu (1965), page 198. Let $B \in \mathscr{F}^+$ be such that $\pi(B) < \infty$ and let $n \in \mathbb{N}$ be arbitrary. By $R$-subinvariance,

$$\begin{aligned} 0 &\leqq \textstyle\sum_{k=0}^{n} ((\pi - R\pi P)R^k P^k)(B) \\ &= \pi(B) - R^{n+1}\pi P^{n+1}(B) \\ &\leqq \pi(B) < \infty . \end{aligned}$$

Letting $n \to \infty$,

$$\begin{aligned} 0 &\leqq \textstyle\sum_{k=0}^{\infty} ((\pi - R\pi P)R^k P^k)(B) \\ &= \textstyle\int (\pi - R\pi P)(dx)(1_B(x) + G_R(x, B)) < \infty . \end{aligned}$$

But $G_R(x, B) = \infty$ for all $x \in S$ because $\{X_n\}$ is $R$-recurrent (Theorem 1 of [T1]) and therefore the measures $\pi$ and $R\pi P$ coincide.

PROOF OF PROPOSITION 1. Denote by $\mathscr{M}$ the set of all nonnegative $\sigma$-finite measures on $(S, \mathscr{F})$. Let $B$ be some $R$-recurrent set (see [T1], Proposition 2.2) and let $C$ be a fixed $C$-set in $B$ (cf. [0], pages 7–10), so that there exists $n \in \mathbb{N}$, $\alpha > 0$ with

$$\inf_{(x,y) \in C \times C} p^n(x, y) = \alpha > 0 ,$$

where $p^n(x, y)$ is the density of $P^n(x, \cdot)$ with respect to $\mu$. For all $r \in (0, R)$ define the measure $\pi_r$ by

$$\pi_r(\cdot) = \frac{\mu I_C G_r(\cdot)}{\mu I_C G_r(C)} .$$

The denominator is finite: by the definition of an $R$-recurrent set (see [T1]), for $\mu$-almost all $x \in S$

$$\begin{aligned} \infty > G_r(x, C) &\geqq r^n \textstyle\int_S P^n(x, dy) G_r(y, C) \\ &\geqq r^n \textstyle\int_C p^n(x, y)\mu(dy) G_r(y, C) \\ &\geqq r^n \alpha \mu I_C G_r(C) . \end{aligned}$$

$\pi_r$ is easily seen to be $r$-subinvariant. Let $A \in \mathscr{F}_C$ be arbitrary. Then

$$\begin{aligned} \mu I_C G_r(A) &\geqq r^n \textstyle\int_S \mu I_C G_r(dx) P^n(x, A) \\ &\geqq r^n \textstyle\int_C \mu I_C G_r(dx) \int_A p^n(x, y)\mu(dy) \\ &\geqq r^n \alpha \mu I_C G_r(C)\mu(A) \end{aligned}$$

so that

$$\pi_r \mid \mathscr{F}_C \geqq r^n \alpha\mu \mid \mathscr{F}_C \qquad \text{for all} \quad r \in (0, R)\,.$$

Trivially $\pi_r(C) = 1$ for all $r \in (0, R)$ and so from Proposition 8.1 of [T2], $\pi_r(\bar{C}_j(n)) \leqq jr^{-n}$ for all $r \in (0, R)$. Any (even uncountable) family of measures in $\mathscr{M}$ is known to have a well-defined infimum in $\mathscr{M}$ with respect to the usual lattice order. Define

$$\bar{\pi}_r = \inf\nolimits_{s \in [r, R)} \pi_s\,;$$

the family $\{\bar{\pi}_r;\, r < R\}$ is nondecreasing in $\mathscr{M}$, satisfies

$$\bar{\pi}_r \mid \mathscr{F}_C \geqq r^n \alpha\mu \mid \mathscr{F}_C \qquad \text{and} \qquad \bar{\pi}_r(\bar{C}_j(n)) \leqq jr^{-n}\,,$$

and the limit measure

$$\pi = \lim\nolimits_{r \uparrow R} \uparrow \bar{\pi}_r = \lim\nolimits_{r \uparrow R} \uparrow (\inf\nolimits_{s \in [r, R)} \pi_s)$$

exists and satisfies

$$\pi \mid \mathscr{F}_C \geqq R^n \alpha\mu \mid \mathscr{F}_C\,, \qquad \pi(\bar{C}_j(n)) \leqq jR^{-n}$$

which shows that $\pi$ is $\sigma$-finite and nontrivial. Finally $\pi$ is $R$-subinvariant, since

$$\begin{aligned} R\pi P &= R(\lim\nolimits_{r \uparrow R} \uparrow (\inf\nolimits_{s \in [r, R)} \pi_s))P \\ &= R \lim\nolimits_{r \uparrow R} \uparrow ((\inf\nolimits_{s \in [r, R)} \pi_s)P) \quad \text{(by monotonicity)} \\ &\leqq R \lim\nolimits_{r \uparrow R} \uparrow (\inf\nolimits_{s \in [r, R)} (\pi_s P)) \quad \text{(because } (\inf\nolimits_{s \in [r, R)} \pi_s)P \leqq \pi_t P \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for all } t \in [r, R)) \\ &\leqq \lim\nolimits_{r \uparrow R} \uparrow (\inf\nolimits_{s \in [r, R)} (s\pi_s P)) \quad \text{(by linearity)} \\ &\leqq \lim\nolimits_{r \uparrow R} \uparrow (\inf\nolimits_{s \in [r, R)} \pi_s) \quad \text{(because } \pi_s \text{ is } s\text{-subinvariant)} \\ &= \pi \quad \text{(by the definition of } \pi)\,. \end{aligned}$$

By Lemma 1 $\pi$ is $R$-invariant.

REMARK. Our referee has pointed out that the technique used in the proof of Proposition 1 is not entirely new. In Harris (1963) on pages 78–79 there is a very similar approach which, however, calls for some extra conditions. Also Tweedie's argument in the proof of Theorem 3 of [T1] is similar to ours, but it involves $R$-invariant functions instead of measures.

### 3. The uniqueness of an $R$-invariant measure.

PROPOSITION 2. *Suppose that the chain $\{X_n\}$ is $R$-recurrent and possesses an $R$-invariant measure $\pi$. Then $\pi$ is unique.*

For the proof let the measure $\pi$ be $R$-invariant for $\{X_n\}$ and $A \in \mathscr{F}^+$ such that $\pi(A) < \infty$. Define the transition function ${}_AG_R$ by

$$ {}_AG_R = \sum_{n=0}^{\infty} R^n(PI_{A^c})^n $$

and the measure $\pi_A$ by

$$ \pi_A = \pi I_A \, {}_AG_R \, . $$

(Tweedie's definitions of ${}_AG_R$ and $\pi_A$ (the latter was denoted by $Q_A$ in [T1]) are slightly different from those above and the same as our ${}_AG_RP$ and $\pi_AP$. Also Harris (1956) in his original paper and Orey [0] used similar definitions to Tweedie's. The present definitions will, however, dramatically simplify the proof of Lemma 2 (ii) below—without complicating anything else.)

LEMMA 2. (i) $\pi_A \leqq \pi$

(ii) $\pi_A \,|\, \mathscr{F}_A = \pi \,|\, \mathscr{F}_A$

(iii) *if $\{X_n\}$ is R-recurrent, then $\pi_A$ is R-invariant.*

PROOF. (i) We prove the identity $\pi = \sum_{m=0}^{n} R^m \pi I_A(PI_{A^c})^m + R^n \pi I_{A^c}(PI_{A^c})^n$ ($n \geqq 0$) by induction (cf. equation (7.4) of [0] and Lemma 3.1 (ii) of [T1]). The identity is obviously valid for $n = 0$. Suppose that it holds for some $n$. Then it also holds for $n + 1$ since

$$ \begin{aligned} R^n \pi I_{A^c}(PI_{A^c})^n &= R^n(R\pi P)I_{A^c}(PI_{A^c})^n & (\pi \text{ is } R\text{-invariant}) \\ &= R^{n+1}\pi I_A(PI_{A^c})^{n+1} + R^{n+1}\pi I_{A^c}(PI_{A^c})^{n+1} \, . \end{aligned} $$

(ii) Let $B \in \mathscr{F}_A^{\,+}$. Then

$$ \pi_A(B) = \sum_{n=0}^{\infty} R^n \pi I_A(PI_{A^c})^n(B) = \pi I_A(B) = \pi(B) $$

since $(PI_{A^c})^n(B) = 0$ for all $n \geqq 1$.

(iii) We need only prove that $\pi_A$ is $R$-subinvariant (cf. [T1], equation (3.32)) and $R$-invariance follows by Lemma 1.

$$ \begin{aligned} R\pi_A P &\leqq R\pi_A PI_{A^c} + R\pi PI_A & \text{(by (i))} \\ &= \sum_{n=0}^{\infty} R^{n+1}\pi I_A(PI_{A^c})^{n+1} + \pi I_A & (\pi \text{ is } R\text{-invariant}) \\ &= \pi_A \, . \end{aligned} $$

This leads to the following corollary, which is the key to Proposition 2.

COROLLARY. *Suppose that $\{X_n\}$ is R-recurrent. Then $\pi = \pi_A$.*

PROOF. By Lemma 2, for any $n \in \mathbb{N}$

$$ \pi_A(A) = R^n \pi_A P^n(A) \leqq R^n \pi P^n(A) = \pi(A) = \pi_A(A) \, , $$

hence $R^n(\pi - \pi_A)P^n(A) = 0$. Using exactly the same steps as in the proof of Proposition 8.1 of [T2] we find

$$ \begin{aligned} 0 = R^n(\pi - \pi_A)P^n(A) &\geqq R^n \int_{\bar{A}_j(n)} (\pi - \pi_A)(dx)P^n(x, A) \\ &\geqq R^n j^{-1}(\pi - \pi_A)(\bar{A}_j(n)) \end{aligned} $$

and the assertion follows because $\pi - \pi_A \geqq 0$ and $\bigcup_{n,j=1}^{\infty} \bar{A}_j(n) = S$.

PROOF OF PROPOSITION 2. Suppose that $U$ and $U'$ are $\sigma$-finite and $R$-invariant. We can always find $E \in \mathscr{F}^+$ such that $0 < U(E) < \infty$ and $0 < U'(E) < \infty$. The proposition follows from proving that the measures

$$\pi(\cdot) = \frac{U(\cdot)}{U(E)} \qquad \text{and} \qquad \pi'(\cdot) = \frac{U'(\cdot)}{U'(E)}$$

coincide, and from the corollary it suffices to prove that $\pi$ and $\pi'$ coincide on some $\mathscr{F}_B$ ($B \in \mathscr{F}^+$). Let $(E^+, E^-)$ be a Jordan–Hahn decomposition of the signed measure $\pi - \pi' \mid \mathscr{F}_E$. At least one of the sets $E^+$, $E^-$ is in $\mathscr{F}^+$; denote it by $B$. From the corollary

$$0 = \pi(E) - \pi'(E) = \int_B (\pi - \pi')(dx)\, {}_BG_R(x, E)\,.$$

But ${}_BG_R(x, E) \geqq {}_BG_R(x, B) = \sum_{n=0}^{\infty} R^n(PI_{B^c})^n(x, B) = I(x, B) = 1$ for all $x \in B$. So $\pi = \pi'$ on $\mathscr{F}_B$ as was required and the proposition is proved.

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## REFERENCES

HARRIS, T. E. (1956). The existence of stationary measures for certain Markov processes. *Proc. Third Berkeley Symp. Math. Statist. Prob.* **2** 113–124, Univ. of California Press.

HARRIS, T. E. (1963). *The Theory of Branching Processes.* Springer-Verlag, Berlin.

NEVEU, JACQUES (1965). *Mathematical Foundations of the Calculus of Probability.* Holden-Day, San Francisco.

OREY, STEVEN (1971). *Lecture Notes on Limit Theorems for Markov Chain Transition Probabilities.* Van Nostrand, London.

TUOMINEN, PEKKA (1975). Recurrent Markov chains on a general state space. Thesis, Univ. of Helsinki (in Finnish).

TWEEDIE, RICHARD L. (1974a). $R$-theory for Markov chains on a general state space I: Solidarity properties and $R$-recurrent chains. *Ann. Probability* **2** 840–864.

TWEEDIE, RICHARD L. (1974b). $R$-theory for Markov chains on a general state space II: $r$-subinvariant measures for $r$-transient chains. *Ann. Probability* **2** 865–878.

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