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TIGHT BOUNDS FOR THE RENEWAL FUNCTION OF A RANDOM WALK

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It is shown that for a random walk $\{S_n\}$ starting at the origin having generic step random variable X with finite second moment and positive mean $\lambda^{-1} = EX$, the renewal function $U(y) = E\#\{n = 0, 1, \cdots : S_n < y\}$ satisfies for $y \geq 0$

$$|U(y) - \lambda y - \tfrac{1}{2}\lambda^2 EX^2| \leq \tfrac{1}{2}\lambda^2 EX^2 - \lambda EM \leq \tfrac{1}{2}\lambda^2 EX_+^2$$

where $M = -\inf_{n \geq 0} S_n$. Both the upper and lower bounds are attained by simple random walk. Bounds are also given for $U(-y)(y \geq 0)$ and for the renewal function of a transient renewal process when $\Pr\{X \geq 0\} = 1 > \Pr\{0 \leq X < \infty\}$. The proof uses a Wiener-Hopf like identity relating U to the renewal functions of the ascending and descending ladder processes to which Lorden's tight bound for the renewal process case is applied.

1. Introduction. The main result of this paper concerns bounds for the so-called renewal function

$$U(y) = \Sigma_0^\infty F^{n*}(y) = EN(y) \equiv E\#\{n = 0, 1, \cdots : S_n \leq y\} \tag{1.1}$$

of a random walk $\{S_n\}$ generated by the independent identically distributed (i.i.d.) steps $X, X_1, X_2, \cdots$ with finite second moment, positive mean $\lambda^{-1} = EX$, distribution function (df) F, $S_0 = 0$ and $S_n = X_1 + \cdots + X_n$.

THEOREM 1. *For a random walk $\{S_n\}$ as above, the renewal function U at* (1.1) *satisfies*

$$|U(y) - \lambda y - \tfrac{1}{2}\lambda^2 EX^2| \leq \tfrac{1}{2}\lambda^2 EX_+^2 \qquad \textit{all } y \geq 0. \tag{1.2}$$

It will emerge from the proof in section 3 below that other, more complicated, bounds for U can be given, and for all y, but involving in addition to the moments of X and X_+ the random variable (rv)

$$M \equiv -\inf_{n \geq 0} S_n. \tag{1.3}$$

In section 4 we complement the bounds on U by giving bounds for a transient renewal function.

Stone's (1972) original form of the inequality for U is that

$$0 \leq U(y) - \lambda y_+ \leq C \cdot \lambda^2 EX^2 \qquad \text{all } y \tag{1.4}$$

for some constant C in $1 \leq C < 3$, and in subsequent work (Daley, 1978) the Fourier methods he used were refined to show that $1 \leq C < 2.081$. It follows from

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(1.2) that $C = 1$, and it is of some interest that, just as in the Berry-Esseen problem where by using Fourier methods the best bound so far established is of the order of twice the best possible bound, so here Fourier methods produced a bound that is about twice the best possible. Below we adapt Lorden's (1970) more powerful techniques which he used to prove (1.2) in the case that $X \geqslant 0$ a.s., so that then $EX_+^2 = EX^2$ and (1.2) is equivalent to (1.4) with $C = 1$. The key step in our work is a Wiener-Hopf like identity (at (3.3) below) relating U and the expectation $U_0(y)$ of the a.s. finite first passage time rv

$$N_0(y) = \inf\{n\colon S_n > y\}. \tag{1.5}$$

The identity involves the renewal functions of both the ascending and descending ladder variables associated with $\{S_n\}$, and its derivation relies on various identities including the duality lemma that may be found in Feller (1966).

2. Further notation and known results. The argument is simplified by the use of

Assumption A. The rv's $\{X_n\}$ are such that

$$\Pr\{S_n = 0 \quad \text{for any} \quad n \geqslant 1\} = 0.$$

This assumption entails no loss of generality, for irrespective of the nature of the $\{X_n\}$, the modified rv's $\{X_n^h\}$ defined by $X_n^h = X_n + \Delta_n^h$ where $\{\Delta_n^h\}$ are independent, independent of $\{X_n\}$, and uniformly distributed on $(0, h)$, do satisfy Assumption A. Now for every fixed $y > 0$,

$$U^h(y) \equiv E\#\{n = 0, 1, \cdots : X_1^h + \cdots + X_n^h < y\}$$

is monotone increasing as $h\downarrow 0$, with limit $U(y)$ for those y that are continuity points of U. Since $U(y)$ is right-continuous and monotone in y, its continuity points are everywhere dense, so if (1.2) is satisfied on the positive y in this set, it is satisfied for all $y \geqslant 0$. Now as $h\downarrow 0$, $\lambda^h \equiv 1/EX^h\uparrow\lambda$, $E(X^h)^2 \to EX^2$, and $E(X_+^h)^2 \to EX_+^2$, so if Theorem 1 holds for $\{X_n^h\}$, it holds for $\{X_n\}$.

Since (1.2) has been proved by Lorden (1970) for the case $X \geqslant 0$ a.s., we may as well and do make

Assumption B.

$$\Pr\{X < 0\} > 0.$$

Then $M \not\equiv 0$ a.s.

Introduce the first ladder epoch rv's

$$\nu^+ = \inf\{n\colon S_n > 0\} = N_0(0), \tag{2.1}$$

$$\begin{aligned} \nu^- &= \inf\{n\colon S_n < 0\} \qquad \text{if finite,} \\ &= 0 \qquad \text{otherwise.} \end{aligned} \tag{2.2}$$

Let the generic ascending ladder rv Y be defined by

$$Y = S_{\nu^+} \tag{2.3}$$

with df

(2.4) $$G_+(y) = \Pr\{Y \leqslant y\}.$$

Similarly, let

(2.5) $$Z' = -S_{\nu^-}$$

and introduce the improper df

(2.6) $$G_-(y) = \Pr\{0 < Z' \leqslant y\}$$

with defect

(2.7) $$1 - \pi \equiv 1 - G_-(\infty) = \Pr\{\nu^- = 0\} = \Pr\{\inf_{n \geqslant 0} S_n = 0\}.$$

We shall need the renewal functions associated with both G_+ and G_-, namely

(2.8a) $$H_+(y) = \Sigma_0^\infty G_+^{n*}(y),$$

(2.8b) $$H_-(y) = \Sigma_0^\infty G_-^{n*}(y),$$

the latter being related to the df of the rv M at (1.3) by the identity (see XI. (6.3) of Feller (1966))

(2.9) $$W(y) \equiv \Pr\{M \leqslant y\} = (1 - \pi)H_-(y).$$

Introduce the sequence of rv's $\{W_n\}$ by

(2.10) $$W_0 = 0,\ W_n = (W_{n-1} - X_n)_+ \qquad n = 1, 2, \cdots$$

and recall the fact, familiar in queueing theory and to be found in a more general context in VI Section 9 of Feller (1966), that the distribution of W_n converges to that of M (note that Feller's X_n = our $-X_n$, so S_n in (9.5) of Feller corresponds to our $-S_n$: cf. also (1.3) above). By Assumptions A and B, those integers n for which $W_n = 0$ are precisely the ascending ladder epochs of the walk $\{S_n\}$, so the generic ascending ladder rv Y of (2.3) has

(2.11a) $$EY = E(X - M \mid X - M > 0) = EXE\nu^+ = \lambda^{-1}E\nu^+,$$

(2.11b) $$EY^2 = E\big((X - M)^2 \mid X - M > 0\big) = E(X - M)_+^2/\Pr\{X > M\}$$

where X and M are independent. Now M and $(M - X)_+$ have the same distribution, so using Assumption A, $\Pr\{X > M\} = \Pr\{M = 0\}$, while using (2.9) and the ascending ladder variable argument,

(2.12) $$\Pr\{M = 0\} = 1 - \pi = 1/E\nu^+,$$

so (2.11a) becomes

(2.11a′) $$EY = 1/\lambda(1 - \pi).$$

Kingman (1962, 1970) used the equidistribution of M and $(M - X)_+$, by equating their squares and taking expectations, to deduce that

(2.13) $$2EX \cdot EM = EX^2 - E(X - M)_+^2,$$

so by using (2.11) and the succeeding argument we have

(2.14) $$EM = \tfrac{1}{2}(EX^2/EX - EY^2/EY).$$

A consequence of the definition of $U_0(y)$, the ladder variable argument, and (2.12), is that

$$U_0(y) \equiv EN_0(y) = (1-\pi)^{-1}H_+(y). \tag{2.15}$$

Finally, an application of Lorden's (1970) inequality to the renewal process of ascending ladder variables yields $H_+(y) - y_+/EY \leqslant EY^2/(EY)^2$, which with (2.15) and (2.11) gives

$$\begin{aligned} U_0(y) - \lambda y_+ &\leqslant \lambda EY^2/EY = \lambda^2 E(X-M)_+^2 \\ &\leqslant \lambda^2 EX_+^2. \end{aligned} \tag{2.16}$$

3. **Proof of Theorem 1.** The proof makes essential use of the identity

$$\begin{aligned} U(y) &= U_0(y) + \textstyle\int_y^\infty [H_-(\infty) - H_-(u-y)]\,dH_+(u) \\ &= U_0(y) + \textstyle\int_{0+}^\infty [H_+(y+v) - H_+(y)]\,dH_-(v) \end{aligned} \tag{3.1}$$

where the equivalence of the two expressions follows from an integration by parts, recalling where necessary that U_0, H_+ and H_- all vanish for negative arguments. The identity can be deduced from Problems 18 and 19 of Chapter XII of Feller (1966), or else by taking the expectation of $N(y)$, the number of visits of $\{S_n\}$ to $(-\infty, y)$, when expressed as the sum of $N_0(y)$ and of any subsequent visits to the half-line between consecutive ascending ladder epochs in (y, ∞). In taking these expectations, $dH_+(u)$ appears as the probability that there is an ascending ladder epoch in $(u, u+du)$, while by the duality lemma (e.g., Chapter 12 Section 2 of Feller (1966)) $H_-(\infty) - H_-(u-y)$ for $u > y$ equals the expected number of visits of the walk $\{S_n\}$ to the half-line $(-\infty, -(u-y))$ prior to any visit at epochs $n \geqslant 1$ to the half-line $(0, \infty)$.

Substituting in (3.1) from (2.9) and (2.15) and subtracting a linear term yields

$$U(y) - \lambda y_+ = U_0(y) - \lambda y_+ + \textstyle\int_{0+}^\infty [U_0(y+v) - U_0(y)]\,dW(v) \tag{3.2}$$

$$= (1-\pi)[U_0(y) - \lambda y_+] + \textstyle\int_{0+}^\infty [U_0(y+v) - \lambda y_+]\,dW(v). \tag{3.3}$$

Substitution in (3.3) of Lorden's bound at (2.16) and use of (2.14) yields for $y > 0$

$$U(y) - \lambda y \leqslant \lambda(1-\pi)EY^2/EY + \textstyle\int_{0+}^\infty (\lambda v + \lambda EY^2/EY)\,dW(v)$$

$$= \lambda EY^2/EY + \lambda EM = \lambda^2 EX^2 - \lambda EM \tag{3.4}$$

$$= \tfrac{1}{2}\lambda(EX^2/EX + EY^2/EY). \tag{3.5}$$

Use of the second inequality at (2.16) yields the upper bound at (1.2), while appealing to the nonnegativity of $U_0(y) - \lambda y_+$ and replacing the integrand at (3.3) by λv yields the lower bound. Theorem 1 is proved.

In the case of negative y, Lorden's inequality (2.16) when substituted into (3.3) yields

$$0 \leqslant U(y) - \lambda E(M+y)_+ \leqslant (\lambda^2 EX^2 - 2\lambda EM)\Pr\{M > -y\} \qquad y < 0. \tag{3.6}$$

Reference to (3.4) shows (3.6) is in fact valid for all y.

The bounds at (3.5) (or, at (1.2) with $\frac{1}{2}\lambda^2 EX_+^2$ replaced by λEM) are tight. For, consider a random walk with $X = 1$ or -1 with probabilities p and $1 - p$ respectively for some $\frac{1}{2} < p < 1$. This walk has $\lambda = 1/(2p - 1)$, $EX^2 = 1$, $EM = \lambda(1 - p)$, and the infimum and supremum of $U(y) - \lambda y$ in $y \geqslant 0$ are realized by letting $y \uparrow 1$ and $\downarrow 0$ respectively. It is easily checked that $U(0) = U(1 - 0) = \lambda^2 p$, and hence the bounds are tight as asserted.

The random variable M appears much in queueing theory where the work directed towards bounding EM (see e.g., Daley and Trengove (1977)) is concerned more with upper bounds rather than lower bounds as required in going from (3.4) to (1.2).

4. **Bounds for transient renewal functions.** The inequality to be established in this section was prompted by work of Köllerström (1978) who gave the lower bound but had a weaker upper bound. The techniques used are those of Lorden (1970).

For this section, let the rv's $\{X_n\}$ underlying $\{S_n\}$ be nonnegative, and for some p in $0 < p < 1$, let the rv N_1 be independent of $\{X_n\}$ and geometrically distributed on $\{1, 2, \cdots\}$ with

$$\Pr\{N_1 = k\} = (1 - p)p^{k-1} \qquad k = 1, 2, \cdots.$$

Let

$$W_1 = S_{N_1} \tag{4.1}$$

and set

$$N_1(y) = \min\{N_1, \inf\{n: S_n > y\}\} \tag{4.2}$$

with expectation

$$U_1(y) = EN_1(y) = \Sigma_0^\infty p^n F^{n*}(y) = \Sigma_0^\infty G^{n*}(y) \uparrow (1 - p)^{-1} \qquad y \to \infty \tag{4.3}$$

if we define G to be the improper df equal to pF, and so relate U_1 to V at (1.5). We consider the modified residual lifetime rv

$$R(y) = (S_{N_1(y)} - y)_+ \tag{4.4}$$

which has expectation

$$\begin{aligned} ER(y) &= ES_{N_1(y)} - y + E(y - S_{N_1(y)})_+ \\ &= U_1(y)EX - y + E(y - W_1)_+ \end{aligned} \tag{4.5}$$

where the replacement of $S_{N_1(y)}$ by $W_1 = S_{N_1(\infty)}$ is permitted because either $S_{N_1(y)} < y$, which implies that $N_1(y) = N_1(\infty) = N_1$ and $W_1 = S_{N_1(y)}$, or else $W_1 \geqslant S_{N_1(y)} > y$ and thus $(y - S_{N_1(y)})_+ = (y - W_1)_+ = 0$.

Hence

$$\lambda ER(y) = U_1(y) - \lambda[EW_1 - E(W_1 - y)_+] \to (1 - p)^{-1} - EW_1 \qquad 0 < y \to \infty. \tag{4.6}$$

THEOREM 2. *For the transient renewal function* $U_1(\cdot)$ *at* (4.3) *and with* W_1 *as at* (4.1),

$$0 \leqslant U_1(y) - \lambda[EW_1 - E(W_1 - y)_+] \tag{4.7}$$

$$\leqslant \lambda(1 + y/\lambda EX^2)^{-1}(\lambda EX^2 + EW_1 - E(W_1 - y)_+)$$

$$= \lambda^2 EX^2[1 - E(y - W_1)_+/(y + \lambda EX^2)] \leqslant \lambda^2 EX^2 \tag{4.7a}$$

and

$$U_1(y) \to \lambda EW_1 = (1 - p)^{-1} = EN_1 \qquad as\ y \to \infty.$$

PROOF. The lower bound at (4.7) follows trivially from (4.6) and the nonnegativity of $R(\cdot)$. The inequality at (4.7a) is an easy consequence of the finiteness of EX^2 and EW_1; in fact it can be checked by differentiation that

$$0 \leqslant E(y - W_1)_+/(y + \lambda EX^2)\uparrow 1 \qquad 0 \leqslant y\uparrow\infty.$$

For the more substantial part of (4.7), consideration of sample paths shows that for $x, y \geqslant 0$,

$$R(x + y) \leqslant_d R'(x) + R''(y) \tag{4.8}$$

where R' and R'' are i.i.d. like R and $\leqslant_d$ denotes inequality in distribution. Taking expectations, it follows that $ER(\cdot)$ is subadditive as in Lorden (1970), that is,

$$ER(x + y) \leqslant ER(x) + ER(y) \qquad x, y \geqslant 0. \tag{4.9}$$

Also, as in Lorden, integration of $R(u)$ on $0 \leqslant u \leqslant y$ yields

$$2\textstyle\int_0^y R(u)\,du = \sum_1^{N_1(y)} X_n^2 - R^2(y) \tag{4.10}$$

whence, on taking expectations, using (4.9) on the left-hand side and using the stopping time property of $N_1(y)$ on the right-hand side,

$$yER(y) \leqslant \textstyle\int_0^y [ER(u) + ER(y - u)]\,du = EN_1(y)\cdot EX^2 - ER^2(y) \tag{4.11}$$

$$\leqslant U_1(y)EX^2 - (ER(y))^2.$$

Use of (4.6) and rearranging the inequality leads to

$$(ER(y))^2 + (y - \lambda EX^2)ER(y) - y\lambda EX^2 \leqslant -E(y - W_1)_+\cdot\lambda EX^2. \tag{4.12}$$

The left-hand side is expressible as $(ER(y) + y)(ER(y) - \lambda EX^2)$ which, being negative by (4.12), implies that $ER(y) \leqslant \lambda EX^2$. This upper bound is derived by ignoring the negative term on the right-hand side of (4.12); inclusion of this term yields the tighter bound

$$ER(y) \leqslant \tfrac{1}{2}\left\{\lambda EX^2 - y + \left[(\lambda EX^2 - y)^2 + 4(y - E(y - W_1)_+)\lambda EX^2\right]^{\frac{1}{2}}\right\}$$

(4.13)

$$= \tfrac{1}{2}\left\{\lambda EX^2 - y + (\lambda EX^2 + y)\left[1 - 4\lambda EX^2\cdot E(y - W_1)_+/(\lambda EX^2 + y)^2\right]^{\frac{1}{2}}\right\}.$$

Using the inequality $(1 - 2\xi)^{\frac{1}{2}} \leqslant 1 - \xi$, valid for $0 < \xi < \frac{1}{2}$, now yields (4.7).

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