THE MEAN AND VARIANCE OF THE MAXIMUM OF THE ADJUSTED PARTIAL SUMS OF A FINITE NUMBER OF INDEPENDENT NORMAL VARIATES

BY M. E. SOLARI AND A. A. ANIS

Chelsea Polytechnic, London

1. Introduction. In planning the storage capacity of a reservoir it is desirable to avoid in so far as is practicable both the loss of water that occurs if the reservoir overflows and the harm that is done if the reservoir is empty when water is needed. Hurst [1] on the basis of data from a long series of annual totals of river discharges has discussed the relation between the capacity, the inflow and its variability, and the draft from a reservoir. In the present paper the theoretical analysis of the problem as studied by Anis and Lloyd is carried further.

If, for a period of n years, the annual increment of inflow minus draft is represented by the variable X_i ($i = 1, \cdots, n$) and the partial sums of these increments by $S_r = \sum_{i=1}^{r} X_i$ ($r = 1, \cdots, n$), then the maximum U_n over the n-year period of these S_r is the maximum accumulated storage when there is no deficit, their minimum L_n gives the maximum accumulated deficit when there is no storage, and their range $R_n = U_n - L_n$ gives the capacity necessary to avoid the two difficulties mentioned above. Anis and Lloyd [3] have studied the distribution of U_n and R_n for the idealized case in which the X_i are taken as independent standard normal variables and have shown that, for any $n \geqq 2$, the expected value of the maximum is $(2\pi)^{-1/2} \sum_{s=1}^{n-1} s^{-1/2}$ and hence that the asymptotic value of the mean range, which is twice that of the maximum, agrees with the value $2[(2/\pi)n]^{1/2}$ obtained by Feller [2]. Furthermore Anis [4] has shown the second moment about the origin of the maximum to be

$$\frac{n+1}{2} + \frac{1}{2\pi} \sum_{s=2}^{n-1} \sum_{t=1}^{s-1} t^{-\frac{1}{2}}(s - t)^{-\frac{1}{2}}$$

and has obtained [5] a recurrence relation for computing moments of higher order by means of which he has tabulated the values of the first four moments for $n = 2, 3, \cdots, 15$.

However, from both the engineering and the statistical point of view it is sometimes desirable to separate the effect of inflow and draft, since the latter may be controlled in such a way that the former is the decisive random variable. In his paper Hurst considered the effect that would have been obtained by a rule of release which made the annual draft equal to the mean annual inflow for the n-year period, $\bar{X}_n = (1/n)\sum_{i=1}^{n} X_i$, so that the accumulation after r years became the adjusted partial sum $S_r' = \sum_{i=1}^{r} X_i - r\bar{X}_n$. For these adjusted partial sums Hurst and Feller both obtained $[(\pi/2)n]^{1/2}$ for the asymp-

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totic mean range.[1] Statistically the study of these adjusted partial sums is advantageous because, since they are distributed about zero provided merely the individual X_i are distributed about a common though not necessarily zero mean, there is now no loss of generality in taking that common mean to be zero.

In this paper we obtain, for the case in which the X_i are independent normal variates with a common mean and unit variance, the distribution of the maximum of the adjusted partial sums and find, for any $n \geqq 2$, the first and second moments about the origin to be

$$\mu_1'(n) = \frac{1}{2}\sqrt{\frac{n}{2\pi}}\sum_{s=1}^{n-1} s^{-\frac{1}{2}}(n-s)^{-\frac{1}{2}}$$

$$\mu_2'(n) = \frac{1}{6}\left\{\frac{n^2-1}{n} + \frac{\sqrt{n}}{2\pi}\sum_{s=2}^{n-1}\sum_{t=1}^{s-1}\frac{s(2s-n)}{\sqrt{(n-s)t^3(s-t)^3}}\right\}$$

with the asymptotic values $\frac{1}{2}[(\pi/2)n]^{1/2}$ and $(n/2) - n^{1/2}$ respectively. Since the distribution of the minimum of the adjusted partial sums is, as in the case of the unadjusted sums, that of minus the maximum, the mean range is twice the mean of the maximum so that our asymptotic value is seen to agree with that obtained by Feller and Hurst.

2. Distribution of the maximum of the adjusted partial sums. In addition to the notation already introduced in Section 1, we shall use throughout $\phi(x)$ to denote the probability density function of a standard normal variate, i.e., $\phi(x) = (2\pi)^{-1/2}\exp(-x^2/2)$. In this connection it should be noted that in accordance with the remarks above, our results will be valid if the annual increments are independent and normally distributed about a common mean with unit variance since reduction to the standard normal variates X_i will not affect the S_r'.

We shall also use $P_n(u)$ and $p_n(u)$ to denote respectively the distribution function and the density function of the maximum over r, U_n', of the adjusted partial sums S_r'. Since by definition S_n' is zero, we consider the $n-1$ sums S_r' $(r = 1, \cdots, n-1)$ and let their maximum be V_n so that $U_n' = \text{Max}\,[V_n, 0]$. Then $P_n(u) = \Pr\{U_n' \leqq u\} = 0$ for $u < 0$ and $P_n(u) = \Pr\{V_n \leqq u\}$ for $u \geqq 0$, so that $P_n(u)$ has a saltus at $u = 0$ and $p_n(u)$ is not defined there. For $u < 0$, $p_n(u) = 0$ and for $u > 0$, $p_n(u) = dP_n(u)/du$.

For any $n \geqq 2$

$$P_n(u) = \int_K (n) \int (2\pi)^{-n/2}\exp(-\tfrac{1}{2}\mathbf{x}'\mathbf{x})\prod_{i=1}^{n} dx_i \qquad (u \geqq 0), \tag{1}$$

[1] However, the values of the range of these adjusted partial sums observed by Hurst appeared to be more nearly proportional to $n^{.73}$. For this reason the authors of this paper thought the exact formula for the mean range of these adjusted partial sums as a function of n would be of interest.

where the region of integration K is defined by

$$K\colon \sum_{i=1}^{r} X_i - r\bar{X}_n \leqq u \qquad (r = 1, \cdots, n-1)$$

and $\mathbf{x}$ is the n-dimensional vector of the observations $X_1, \cdots, X_n$. We introduce the transformation

$$\mathbf{x} = \mathbf{B}\mathbf{y}, \tag{2}$$

where $\mathbf{B}$ is the $n \times n$ matrix given by

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 1 \\ -1 & 1 & 0 & \cdots & 1 \\ 0 & -1 & 1 & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -1 \end{pmatrix}.$$

It is easy to see that the Jacobian J_n of the transformation (2) is given by the recurrence relation

$$J_n = 1 + J_{n-1}$$

and hence that $J_n = n$. Now $\mathbf{x}'\mathbf{x} = \mathbf{y}'\mathbf{B}'\mathbf{B}\,\mathbf{y} = \mathbf{y}'\mathbf{C}\,\mathbf{y}$ where $\mathbf{C}$ is the $n \times n$ matrix given by

$$\mathbf{C} = \begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 2 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & n \end{pmatrix}.$$

Hence $\mathbf{x}'\mathbf{x} = \mathbf{y}'\mathbf{C}\,\mathbf{y} = n\,y_n^2 + \mathbf{Y}'\mathbf{A}\,\mathbf{Y}$, where $\mathbf{A}$ is the $(n-1) \times (n-1)$ matrix obtained from $\mathbf{C}$ by omitting the last column and the last row and $\mathbf{Y}$ is the $(n-1)$-dimensional vector $y_1, y_2, \cdots, y_{n-1}$. Reverting to (1) and (2), we see that $y_n = \bar{X}_n$, $y_r = S_r'$ $(r = 1, \cdots, n-1)$ and hence $P_n(u)$ can be put in the form

$$P_n(u) = \sqrt{n} \int_{-\infty}^{u} (n-1) \int_{-\infty}^{u} (2\pi)^{-[(n-1)/2]} \exp\left(-\tfrac{1}{2}\mathbf{Y}'\mathbf{A}\mathbf{Y}\right) \prod_{i=1}^{n-1} dy_i. \tag{3}$$

It is worth noting here that the integrand in (3) is, except for a constant factor, precisely the integrand in expression (6.1) in Anis and Lloyd [3]. Using the value obtained there for that integral, we deduce immediately that

$$P_n(0) = \frac{1}{n}. \tag{4}$$

Differentiating (3) by using the rule for differentiation of multiple integrals when the integrand does not contain the variable with respect to which we differentiate but the limits of integration do, which is justifiable since our integrand is a well behaved function, we obtain immediately

$$p_n(u) = \sqrt{2\pi n} \sum_{s=0}^{n-2} h_s(u)h_{n-2-s}(u) \qquad (n \geqq 2), \tag{5}$$

where $h_s(u)$ is the integral defined by Anis and Lloyd [3]; i. e., for $s \geq 1$

$$h_s(u) = \int_0^\infty (s) \cdot \int_0^\infty \phi(u - y_1)\phi(y_1 - y_2) \cdots \phi(y_{s-1} - y_s)\phi(y_s)\, dy_1 \cdots dy_s \tag{6}$$

and

$$h_0(u) = \phi(u).$$

Since the probability density functions for the maximums of the unadjusted partial sums are expressible (Anis and Lloyd [3]) as a linear combination of these integrals $h_s(u)$, it is now possible to express $p_n(u)$ in terms of those probability density functions. However, it proves more convenient to obtain the moments of the distribution (5) directly from the properties of the integrals $h_s(u)$.

3. Properties of the integrals $h_s(u)$.

Lemma 1.

$$h_s(0) = (2\pi)^{-1/2}(s + 1)^{-3/2} \qquad (s \geqq 0). \tag{7}$$

This was proved by Anis and Lloyd [3], since $h_s(0) = (2\pi)^{-[(s+1)/2]}c_s$ in their notation, and is repeated here merely for completeness.

Lemma 2.

$$h_s(\infty) = 0 \qquad (s \geqq 0). \tag{8}$$

To prove this we note that, by virtue of its definition (6) as an integral, $h_s(u)$ is non-negative for all values of u. Hence the probability density functions $p_n(u)$ are from (5) the sum of non-negative terms $h_s(u)h_{n-s}(u)$ for all n and s. Since $p_n(\infty)$ is zero, no one of these terms and so no $h_s(u)$ can differ from zero at infinity.

Lemma 3. *For* $s \geqq 1$

$$h_s(u) = \int_0^\infty \phi(u - y)h_{s-1}(y)\, dy, \tag{9}$$

$$\begin{aligned} h_s'(u) &= \int_0^\infty \phi(u - y)yh_{s-1}(y)\, dy - uh_s(u) \\ &= \int_0^\infty \phi(u - y)h_{s-1}'(y)\, dy + h_{s-1}(0)\phi(u), \end{aligned} \tag{10}$$

$$(11) \qquad h_s''(u) = \int_0^\infty \phi(u-y)y^2 h_{s-1}(y)\,dy - 2uh_s'(u) - (u^2+1)h_s(u)$$
$$= \int_0^\infty \phi(u-y)yh_{s-1}'(y)\,dy - uh_s'(u)$$
$$= \int_0^\infty \phi(u-y)h_{s-1}''(y)\,dy + h_{s-1}(0)\phi'(u) + h_{s-1}'(0)\phi(u).$$

To prove this lemma we note that the reduction formula (9) for $h_s(u)$ itself follows immediately from the definition (6) of $h_s(u)$. The reduction formulae (10) and (11) for the derivatives then follow by differentiation of (9) with some rearranging and integration by parts.

LEMMA 4. *For* $s \geqq 1$

$$(12) \qquad h_s(0) = \int_0^\infty h_0(y)h_{s-1}(y)\,dy,$$

$$(13) \qquad h_s'(0) = \int_0^\infty yh_0(y)h_{s-1}(y)\,dy = h_0(0)h_{s-1}(0) + \int_0^\infty h_0(y)h_{s-1}'(y)\,dy,$$

$$(14) \qquad h_s''(0) = \int_0^\infty y^2h_0(y)h_{s-1}(y)\,dy - h_s(0) = \int_0^\infty yh_0(y)h_{s-1}'(y)\,dy$$
$$= h_0(0)h_{s-1}'(0) + \int_0^\infty h_0(y)h_{s-1}''(y)\,dy.$$

These results follow immediately on putting $u = 0$ in the reduction formulae of Lemma 3.

4. Moments of the distribution of the maximum of the adjusted partial sums. In this paragraph for simplicity of notation we shall omit the limits of integration which will be from zero to infinity throughout and write h_s only wherever we mean this function to be evaluated at zero. Furthermore we shall consider the distribution

$$p_{n+2}(u) = \sqrt{2\pi(n+2)}\sum_{s=0}^{n} h_s(u)h_{n-s}(u) \qquad (n \geqq 0).$$

For this distribution we write the rth moment about the origin

$$\mu_r'(n+2) = \int u^r p_{n+2}(u)\,du \qquad (r \geqq 0)$$

in the form

$$(15) \qquad \mu_r'(n+2) = \sqrt{2\pi(n+2)}\sum_{s=0}^{n} I_{n,r}(s),$$

where

$$(16) \qquad I_{n,r}(s) = \int u^r h_s(u)h_{n-s}(u)\,du \qquad (0 \leqq s \leqq n).$$

For $I_{n,0}(s)$ we obtain on applying reduction formula (9)

$$I_{n,0}(s) = \int h_{n-s}(u)h_s(u)\,du = \int h_{n-s}(u)\int \phi(u-y)h_{s-1}(y)\,dy\,du \qquad (s \geqq 1)$$

and, reversing the order of integration and using (9) again,

$$\begin{aligned} I_{n,0}(s) &= \int h_{s-1}(y)\int \phi(u-y)h_{n-s}(u)\,du\,dy \\ &= \int h_{s-1}(y)h_{n-s+1}(y)\,dy = I_{n,0}(s-1) \qquad (1 \leqq s \leqq n). \end{aligned}$$

Hence, using (12) and (7),

$$\begin{aligned} I_{n,0}(s) &= I_{n,0}(0) = \int h_0(y)h_n(y)\,dy \\ &= h_{n+1} = (2\pi)^{-1/2}(n+2)^{-3/2} \qquad (0 \leqq s \leqq n, n \geqq 0). \end{aligned} \tag{17}$$

Substituting this result into (15) and noting (4), we obtain

$$\mu_0'(n+2) = \frac{n+1}{n+2} = 1 - P_{n+2}(0)$$

for the zero order moment, as is to be expected when one recalls that $P_n(u)$ is zero for $u < 0$ and has a saltus at $u = 0$.

For $I_{n,1}(s)$ we proceed in the same manner but after reversing the order of integration we apply the first of the two reduction formulae (10) for the derivative to obtain

$$I_{n,1}(s) - I_{n,1}(s-1) = J_n(s-1) \qquad (1 \leqq s \leqq n), \tag{18}$$

where

$$J_n(s) = \int h_s(u)h_{n-s}'(u)\,du \qquad (0 \leqq s \leqq n).$$

Similarly we obtain a difference equation for $J_n(s)$ by applying the reduction formula (9) to $h_s(u)$ in the integrand of $J_n(s)$, reversing the order of integration, applying the second form of formula (10) to $h_{n-s}'(u)$, and simplifying the result by using (12). The resulting difference equation is

$$J_n(s) - J_n(s-1) = -h_s h_{n-s} \qquad (1 \leqq s \leqq n)$$

which when summed over s gives

$$J_n(t) = J_n(0) - \sum_{s=1}^{t} h_s h_{n-s} \qquad (1 \leqq t \leqq n).$$

By Lemma 4 $J_n(0) = I_{n,1}(0) - h_0 h_n$ so we may write

$$J_n(s) = I_{n,1}(0) - \sum_{v=0}^{s} h_v h_{n-v} \qquad (0 \leqq s \leqq n). \tag{19}$$

Returning to (18) we substitute (19) for $J_n(s)$ and sum this difference equation for $I_{n,1}(s)$ over s from 1 to t, reversing the order of summation to obtain

$$I_{n,1}(t) = (t+1)I_{n,1}(0) - \sum_{v=0}^{t-1} (t-v)h_v h_{n-v} \qquad (1 \leqq t \leqq n). \tag{20}$$

We note that this expression is also valid for $t = 0$ if we write the range of summation for v from 0 to t. Since $I_{n,1}(s) = I_{n,1}(n-s)$ by definition (16), we have upon putting $t = n$ in this extended form of (20)

$$I_{n,1}(0) = (n+1)I_{n,1}(0) - \sum_{v=0}^{n} (n-v)h_v h_{n-v}.$$

But

$$\sum_{v=0}^{n} vh_v h_{n-v} = \sum_{v=0}^{n} (n-v)h_v h_{n-v} = n\sum_{v=0}^{n} h_v h_{n-v} - \sum_{v=0}^{n} vh_v h_{n-v},$$

hence

$$\sum_{v=0}^{n} (n-v)h_v h_{n-v} = \frac{n}{2}\sum_{v=0}^{n} h_v h_{n-v}$$

and

$$I_{n,1}(0) = \frac{1}{2}\sum_{v=0}^{n} h_v h_{n-v}. \tag{21}$$

Substituting (21) into the extended form of (20) and summing over t, we have after reversing the order of summation

$$\sum_{t=0}^{n} I_{n,1}(t)$$
$$= \frac{(n+1)(n+2)}{4}\sum_{v=0}^{n} h_v h_{n-v} - \sum_{v=0}^{n} \frac{(n-v)(n-v+1)}{2} h_v h_{n-v} \qquad (n \geqq 0).$$

In this last expression we note that the coefficient of $h_v h_{n-v}$ is

$$2\,\frac{(n+1)(n+2)}{4} - \frac{(n-v)(n-v+1)}{2} - \frac{v(v+1)}{2}$$
$$= (v+1)(n-v+1) \qquad (0 \leqq v \leqq n)$$

so that, using Lemma 1, we may write

$$\sum_{s=0}^{n} I_{n,1}(s) = \frac{1}{2}\sum_{s=0}^{n} (s+1)(n-s+1)h_s h_{n-s}$$
$$= \frac{1}{4\pi}\sum_{s=1}^{n+1} s^{-1/2}(n-s+2)^{-1/2} \qquad (n \geqq 0).$$

Substituting this into (15) we obtain

$$\mu_1'(n+2) = \frac{1}{2}\sqrt{\frac{n+2}{2\pi}}\sum_{s=1}^{n+1} s^{-1/2}(n+2-s)^{-1/2} \qquad (n \geqq 0) \tag{22}$$

for the first moment of the maximum and, hence, twice this value for the mean range of the adjusted partial sums.

For $I_{n,2}(s)$ the method is similar though more tedious. Applying (9) to $h_s(u)$ in the integrand of $I_{n,2}(s)$, reversing the order of integration, and applying the first form of (11) to the inner integral we have

$$\begin{aligned} I_{n,2}(s) = I_{n,2}(s-1) + K_n(s-1) + 2L_n(s-1) \\ + I_{n,0}(s-1) \quad (1 \leqq s \leqq n), \end{aligned} \tag{23}$$

where

$$\left.\begin{aligned} K_n(s) &= \int h_s(u)h''_{n-s}(u)\,du, \\ L_n(s) &= \int uh_s(u)h'_{n-s}(u)\,du \end{aligned}\right\} \qquad (0 \leqq s \leqq n).$$

We obtain, in the same way as was done for $J_n(s)$ [but using the second and third forms of (11)] difference equations for $L_n(s)$ and $K_n(s)$ respectively

$$\left.\begin{aligned} L_n(s) - L_n(s-1) &= K_n(s-1), \\ K_n(s) - K_n(s-1) &= h_{n-s}h_s' - h_sh_{n-s}' \end{aligned}\right\} \qquad (1 \leqq s \leqq n).$$

Using Lemma 4 to evaluate $L_n(0)$ and $K_n(0)$, we find the solutions of these equations to be, for $0 \leqq s \leqq n$,

$$L_n(s) = (s+1)[I_{n,2}(0) - I_{n,0}(0)] + \sum_{v=0}^{s}(s-v)[h_{n-v}h_v' - h_v h_{n-v}'],$$

$$K_n(s) = I_{n,2}(0) - I_{n,0}(0) + \sum_{v=0}^{s}[h_{n-v}h_v' - h_v h_{n-v}'].$$

Substituting these expressions into (23) we have after summing over s and rearranging

$$\begin{aligned} I_{n,2}(t) = (t+1)^2 I_{n,2}(0) - t(t+1)I_{n,0}(0) \\ + \sum_{v=0}^{t}(t-v)^2[h_{n-v}h_v' - h_v h_{n-v}'] \qquad (0 \leqq t \leqq n). \end{aligned} \tag{24}$$

Since $I_{n,2}(s) = I_{n,2}(n-s)$, we now evaluate $I_{n,2}(0)$ as before obtaining

$$I_{n,2}(0) = \frac{n+1}{n+2} I_{n,0}(0) + \frac{1}{n+2}\sum_{v=0}^{n} v[h_{n-v}h_v' - h_v h_{n-v}'],$$

after we have observed that, as for $I_{n,1}(0)$, the summations involved satisfy certain identities, in particular

$$\sum_{v=0}^{n} (n - v)^2[h_{n-v} h_v' - h_v h_{n-v}'] = -n \sum_{v=0}^{n} v[h_{n-v} h_v' - h_v h_{n-v}'].$$

Substituting the expression for $I_{n,2}(0)$ into (24) and summing over t, we have

$$\begin{aligned}\sum_{t=0}^{n} I_{n,2}(t) &= \frac{(n+1)(n+3)}{6} I_{n,0}(0) + \frac{1}{6}\sum_{v=0}^{n} [(n+1)(2n+3)v \\ &\quad + (n-v)(n-v+1)(2n-2v+1)]\,[h_{n-v} h_v' - h_v h_{n-v}'] \\ &= \frac{(n+1)(n+3)}{6} I_{n,0}(0) + \frac{1}{3}\sum_{v=1}^{n} (v+1)(n-v+1)(2v-n)h_{n-v} h_v'.\end{aligned}$$

By (13) $h_v' = I_{v-1,1}(0)$ so that, using (7), (17) and (21), we may write

$$\begin{aligned}&\sum_{s=0}^{n} I_{n,2}(s) \\ &= \frac{1}{6}\left[\frac{(n+1)(n+3)}{\sqrt{2\pi(n+2)^3}} + \frac{1}{(2\pi)^{3/2}} \sum_{s=2}^{n+1}\sum_{t=1}^{s-1} \frac{s(2s-n-2)}{\sqrt{(n+2-s)t^3(s-t)^3}}\right] \qquad (n \geqq 0)\end{aligned}$$

provided we interpret the summation as zero when $n = 0$. Hence from (15) the second moment of the maximum is

$$\begin{aligned}\mu_2'(n+2) &= \frac{1}{6} \\ &\cdot\left[\frac{(n+1)(n+3)}{n+2} + \frac{\sqrt{n+2}}{2\pi}\sum_{s=2}^{n+1}\sum_{t=1}^{s-1}\frac{s(2s-n-2)}{\sqrt{(n+2-s)t^3(s-t)^3}}\right] \quad (n \geqq 0).\end{aligned} \tag{25}$$

A table of values of μ_1', μ_2' and σ for samples of size 10, 20, $\cdots$, 150, was computed from formulae (22) and (25), (see Table 1).

5. Asymptotic values for the first and second moments. Feller [2] has considered the asymptotic distribution of the adjusted range for large n and found the asymptotic value of the mean adjusted range to be $[(\pi/2)n]^{1/2}$. That this is in agreement with our result can be seen by approximating to the sum in our formula (22)

$$\mu_1'(n) = \frac{1}{2}\sqrt{\frac{n}{2\pi}} \sum_{s=1}^{n-1} s^{-1/2}(n-s)^{-1/2}$$

by the integral

$$\int_1^{n-1} s^{-1/2}(n-s)^{-1/2}\, ds.$$

On making the substitution $n\theta = s$, this integral becomes

$$\int_{1/n}^{1-1/n} \theta^{-1/2}(1-\theta)^{-1/2}\, d\theta,$$

which approaches $\mathrm{B}(\frac{1}{2}, \frac{1}{2}) = \pi$ as n becomes large. Thus the asymptotic value for the mean of the maximum of the adjusted partial sums is

$$\mu_1'(n) \sim \frac{1}{2}\sqrt{\frac{\pi n}{2}} \doteq 0.6267n^{1/2} \tag{26}$$

and the asymptotic mean range is $[(\pi/2)n]^{1/2}$ as obtained by Hurst and Feller.

Similarly we obtain the asymptotic value of the second moment of the maximum from our formula (25)

$$\mu_2'(n) = \frac{1}{6}\left[\frac{n^2-1}{n} + \frac{\sqrt{n}}{2\pi}\sum_{s=2}^{n-1}\sum_{t=1}^{s-1}\frac{s(2s-n)}{\sqrt{(n-s)t^3(s-t)^3}}\right]$$

by approximating to the double sum by the double integral

$$I = \int_{s=2}^{n-1}\int_{t=1}^{s-1}\frac{s(2s-n)}{\sqrt{(n-s)t^3(s-t)^3}}\,dt\,ds.$$

Integrating first with respect to t by means of the substitution $t = (sz)^{-1}$, we have

$$I = \tfrac{1}{4}\int_2^{n-1}\frac{(2s-n)(s-2)}{s\sqrt{(n-s)(s-1)}}\,ds =$$

$$\tfrac{1}{4}\int_2^{n-1}\left[\sqrt{\frac{s-1}{n-s}} - \sqrt{\frac{n-s}{s-1}} - \frac{3}{\sqrt{(n-s)(s-1)}} + \frac{2n}{s\sqrt{(n-s)(s-1)}}\right]ds.$$

Applying the substitution $s = n\sin^2\theta + \cos^2\theta$ to the first three terms and the substitution $s = 2n[n+1+(n-1)\sin\theta]^{-1}$ to the last term of this integrand,

TABLE 1

Values of $\mu_1'(n)$, $\mu_2'(n)$, σ_n and the asymptotic approximations for $\mu_1'(n)$ and σ_n

n	Exact values			Asymptotic approximation	
	$\mu_1'(n)$	$\mu_2'(n)$	σ_n	$\mu_1' \sim 0.6267\ n^{1/2}$	$\sigma_n \sim 0.3276\ n^{1/2}$
10	1.3948	3.019	1.0358	1.9817	1.0359
20	2.2178	7.068	1.4660	2.8025	1.4649
30	2.8483	11.336	1.7952	3.4323	1.7942
40	3.3796	15.718	2.0726	3.9633	2.0717
50	3.8477	20.173	2.3170	4.4311	2.3162
60	4.2707	24.680	2.5378	4.8541	2.5373
70	4.6597	29.226	2.7409	5.2430	2.7406
80	5.0218	33.803	2.9298	5.6050	2.9298
90	5.3619	38.405	3.1072	5.9450	3.1076
100	5.6835	43.028	3.2749	6.2666	3.2757
110	5.9894	47.667	3.4342	6.5724	3.4356
120	6.2817	52.322	3.5863	6.8647	3.5883
130	6.5620	56.989	3.7321	7.1450	3.7349
140	6.8318	61.667	3.8721	7.4147	3.8758
150	7.0920	66.351	4.0067	7.6749	4.0118

we obtain

$$I = -24 \sin^{-1}\left(1 - \frac{2}{n-1}\right) + 8\sqrt{n} \sin^{-1}\left[1 - \frac{2}{(n-1)^2}\right] - 8\sqrt{n} \sin^{-1}\left(\frac{1}{n-1}\right),$$

which approaches $-12\pi + 4\pi\sqrt{n}$ for large n. Thus the asymptotic value for the second moment about the origin is

$$\mu_2'(n) \sim \frac{n}{2} - \sqrt{n}$$

and, using the asymptotic value obtained for $\mu_1'(n)$ in (26), we find the asymptotic value of the variance of the maximum of the adjusted partial sums is

$$\sigma_n^2 \sim \left(\frac{1}{2} - \frac{\pi}{8}\right) n \doteq 0.1073n. \tag{27}$$

Comparing this asymptotic value with that obtained by Anis [4] for the variance of the maximum of the unadjusted sums which was $[1 - (2/\pi)]n \doteq 0.3634n$, we see that Feller's comment on the greater stability of the adjusted partial sums is well borne out by our results.

In Table 1 we note that:

1) the series for $\mu_1'(n)$ converges very slowly so that the asymptotic approximation (26) should not be used for values of n within the range of this table,
2) the series for $\mu_2'(n)$ converges even more slowly, but
3) the asymptotic approximation (27) gives very good values for σ_n even within the range of the table because the errors in the approximations to $\mu_1'(n)$ and $\mu_2'(n)$ are in the same direction and largely cancel.

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