IDEMPOTENT MATRICES AND QUADRATIC FORMS IN THE GENERAL LINEAR HYPOTHESIS

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1. Introduction. The important role that idempotent matrices play in the general linear hypothesis theory has long been recognized ([1], [2]), but their usefulness seems not to have been fully exploited. The purpose of this paper is to state and prove some theorems about idempotent matrices and to point out how they might be used to advantage in linear hypothesis theory.

2. Notation and Definitions. Throughout this paper an idempotent matrix will mean a symmetric matrix A such that $AA = A$ (for the sake of brevity we will use the word idempotent matrix to indicate a symmetric idempotent matrix unless specifically stated otherwise). The theorems will not necessarily hold for nonsymmetric idempotent matrices. The statement: Y is distributed as $N_p(\mu, V)$, will mean that a $(p \times 1)$ random vector Y has the p-variate normal distribution whose mean is the $(p \times 1)$ vector, μ, and whose covariance matrix is the positive definite symmetric matrix, V. The statement: u is distributed as $\chi^2(n)$ will mean that a scalar random variable u has the Chi-square distribution with n degrees of freedom, and the statement: v is distributed as $\chi'^2(n, \lambda)$ will mean that the scalar random variable v is distributed as the noncentral Chi-square distribution with n degrees of freedom and with noncentrality, λ. The frequency function of v is ([3])

$$f(v) = \sum_{i=0}^{\infty} \frac{e^{-\lambda}\lambda^i}{i!} \cdot \frac{v^{n+2i-2/2}e^{-(v/2)}}{2^{n+2i/2}\Gamma\left(\frac{n+2i}{2}\right)}, \qquad 0 \leqq v < \infty.$$

If $\lambda = 0$, then the noncentral Chi-square distribution degenerates into the central Chi-square distribution.

A' will indicate the transpose of the matrix A, and A^{-1} will indicate the inverse. I_p will indicate the $(p \times p)$ identity matrix and φ will indicate a null matrix. Below is a list of well-known theorems which will be needed in the succeeding sections.

THEOREM A. *If A is an $(n \times n)$ symmetric matrix of rank p, then a necessary and sufficient condition that A is idempotent is that each of p of the characteristic roots of A is equal to unity and the remaining $(n - p)$ characteristic roots are equal to zero.*

THEOREM B. *If A is an idempotent matrix, then the rank of A equals the trace of A.*

THEOREM C. *The only nonsingular idempotent matrix is the identity matrix.*

THEOREM D. *If A is an $(n \times n)$ idempotent matrix of rank p such that $p < n$ $(p = n)$, then A is a positive semidefinite matrix (positive definite matrix).*

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THEOREM E. *If A is an idempotent matrix whose* ith *diagonal element is equal to zero, then every element in the* ith *row and* ith *column of A is equal to zero.*

THEOREM F. *If Y is distributed as $N_n(\mu, I_n)$, then $v = Y'Y$ is distributed as $\chi'^2(n, \lambda)$, where $\lambda = \frac{1}{2}\mu'\mu$.*

THEOREM G. *In Theorem* F, *the moment generating function of v is*

$$m_v(\theta) = (1 - 2\theta)^{-n/2} e^{-\lambda + \lambda(1-2\theta)^{-1}}.$$

THEOREM H. *If Y is distributed as $N_n(\mu, I_n)$, then a necessary and sufficient condition that $Y'B_1Y, Y'B_2Y, \cdots, Y'B_kY$ be jointly independent is that $B_iB_j = \varphi$ for all $i \neq j$.*

THEOREM J. *If $B_1, B_2, \cdots, B_k$ are a set of $(n \times n)$ symmetric matrices, then a necessary and sufficient condition that there exists an orthogonal matrix, P, such that $P'B_1P, P'B_2P, \cdots, P'B_kP$ are each diagonal is that $B_iB_j = B_jB_i$ for all i and j.*

THEOREM K. *Let $B_1, B_2, \cdots, B_m$ be a collection of $(n \times n)$ symmetric matrices such that $\sum_{i=1}^{m} B_i = I_n$. Then any one of the conditions* K_1, K_2, K_3 *is necessary and sufficient for the remaining two.*

K_1 : *Each B_i is an idempotent matrix.*

K_2 : *$B_iB_j = \varphi$ for all $i \neq j$.*

K_3 : *$\sum_{i=1}^{m} n_i = n$ where n_i is the rank of B_i.*

THEOREM L. *If v is distributed as $\chi'^2(n, \lambda)$ and w is distributed as $\chi^2(m)$, and if v and w are independent, then $u = (v/w)\cdot(m/n)$ is distributed as $F'(n, m, \lambda)$ where $F'(n, m, \lambda)$ refers to the noncentral F distribution with n degrees of freedom for numerator and m degrees of freedom for the denominator and noncentrality, λ. The functional form is*

$$f(u) = \sum_{i=0}^{\infty} \frac{\lambda^i e^{-\lambda}}{i!} \frac{\Gamma\left(\frac{m+n+2i}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n+2i}{2}\right)} \left(\frac{n}{m}\right)^{n/2} \frac{u^{i+(n/2)-1}}{\left(1 + \frac{n}{m}u\right)^{i+(n+m)/2}}.$$

This reduces to Snedecor's F if and only if $\lambda = 0$.

3. Theory. Let an observation vector, Y, be distributed as $N_n(X\beta, \sigma^2 I_n)$, where X is an $(n \times p)$ $(p < n)$ matrix with known elements and rank p, β is a $(p \times 1)$ vector of unknown parameters, and σ^2 is an unknown scalar. Y is often assumed to have this structure in models which are referred to as multiple regression models and in linear models used in the theory of experimental designs. In these models it is often desired to test hypotheses about elements of the vector β. The technique often employed to devise test functions is the technique of analysis of variance. The procedure is to partition the total sum of squares $Y'Y$ of the observation vector, Y, into quadratic forms such that

$$Y'Y = \sum_{i=1}^{k} Y'A_iY \tag{1}$$

and use Cochran's theorem ([5]) to ascertain the independence and distribution of the quantities $Y'A_iY$. This process is quite well known and will not be explained here except to say that to use Cochran's theorem it is necessary to be able to judge the rank of the matrices A_i. It has been pointed out ([2]) that in certain cases, finding the rank of the matrices A_i and using Cochran's theorem is equivalent to showing that $A_iA_j = \varphi$ for all $i \neq j$, or to showing that each A_i is an idempotent matrix. In many cases it is easier to show that a matrix is idempotent than it is to find the rank of the matrix. Therefore, we will prove some theorems which are new, and which enable us to determine the distribution of the quadratic forms in equations similar to (1) without having to find the rank of the A_i.

The first theorem which we shall prove is an algebraic theorem about symmetric matrices which is useful in developing theorems concerning the distribution of quadratic forms.

THEOREM 1. *Let $A_1, A_2, \cdots, A_m$ be a collection of $n \times n$ symmetric matrices where the rank of A_i is p_i, and let $A = \sum_{i=1}^m A_i$ where the rank of A is p. Consider the four conditions*:

C$_1$. *Each A_i is an idempotent matrix.*

C$_2$. *$A_iA_j = \varphi$ for all $i \neq j$.*

C$_3$. *A is an idempotent matrix.*

C$_4$. *$p = \sum_{i=1}^m p_i$; i.e., the rank of the sum of the A_i equals the sum of the ranks of the A_i.*

The following are true:

(a) *Any two of the three conditions* C$_1$, C$_2$, C$_3$ *imply all four of the conditions* C$_1$, C$_2$, C$_3$, C$_4$.

(b) *Conditions* C$_3$ *and* C$_4$ *imply* C$_1$ *and* C$_2$.

Proof. We will first prove (a). To do this we will show that any two of the conditions C$_1$, C$_2$, C$_3$ imply the remaining one in the set C$_1$, C$_2$, C$_3$, and then show that the three conditions C$_1$, C$_2$, and C$_3$ imply C$_4$. We might point out that if $A = I$ then this is essentially the theorem which Craig and Hotelling proved ([2], [4]).

Suppose C$_1$ and C$_3$ are given. Since A is given to be idempotent of rank p, there exists an orthogonal transformation P such that

$$P'AP = \begin{pmatrix} I_p & \varphi \\ \varphi & \varphi \end{pmatrix}.$$

Thus we have

$$P'AP = \begin{pmatrix} I_p & \varphi \\ \varphi & \varphi \end{pmatrix} = \sum_{i=1}^m P'A_iP.$$

Since A_i is idempotent, $P'A_iP$ is also idempotent, and by Theorems D and E, the last $(n - p)$ diagonal elements of each $P'A_iP$ must be zero. This is true since by Theorem D the diagonal elements of an idempotent matrix are nonnegative and since any one of the last $(n - p)$ diagonal elements when summed over the m matrices is zero, each of the last $(n - p)$ diagonal elements must be zero. Then by Theorem E the last $(n - p)$ rows and $(n - p)$ columns of each $P'A_iP$ must be zero. Thus we can write

$$P'A_iP = \begin{pmatrix} B_i & \varphi \\ \varphi & \varphi \end{pmatrix}.$$

Extracting the $(p \times p)$ matrix in the upper left-hand corner of

$$P'AP = \textstyle\sum_{i=1}^m P'A_iP$$

we have $I_p = \sum_{i=1}^m B_i$, where the B_i are idempotent of rank p_i. Theorem K implies that $B_iB_j = \varphi$ for $i \neq j$; therefore $A_iA_j = \varphi$ if $i \neq j$, and the proof is complete that C_1 and C_3 imply C_2.

Now suppose C_1 and C_2 are given. We have

$$AA = \Big(\sum_{i=1}^m A_i\Big)^2 = \sum_{i=1}^m A_i^2 + \sum_{i \neq j} A_iA_j = \sum_{i=1}^m A_i = A.$$

Thus we have shown that the sum is idempotent and C_3 is satisfied.

Now suppose C_2 and C_3 are given. By Theorem J there exists an orthogonal matrix P such that $P'A_1P, P'A_2P, \cdots, P'A_mP$ are each diagonal (since $A_iA_j = A_jA_i = \varphi$), and since the sum of diagonal matrices is a diagonal matrix it also follows that $P'AP$ is diagonal. By C_2 it follows that $P'A_iPP'A_jP = \varphi$ for all $i \neq j$.

It follows, therefore, that $P'A_iP$ is idempotent, and hence A_i is idempotent for all i, and the proof is complete.

We will now show that C_1, C_2, and C_3 imply C_4. If the three conditions C_1, C_2, and C_3 are true, then this implies that there exists an orthogonal matrix P such that the following are true:

$P'AP = \begin{pmatrix} I_p & \varphi \\ \varphi & \varphi \end{pmatrix}$; $P'A_iP$ are each diagonal matrices with p_i (the rank of A_i) ones on the diagonal and $(n - p_i)$ zeros on the diagonal. Thus since

$$\sum_{i=1}^m P'A_iP = \begin{pmatrix} I_p & \varphi \\ \varphi & \varphi \end{pmatrix},$$

it is quite clear that the total number of ones on the diagonal of $P'A_iP$ $(i = 1, 2, \cdots, m)$ is equal to p and the result follows.

We will now prove (b). Since A is given as idempotent, there exists an orthogonal matrix P such that $P'AP = \begin{pmatrix} I_p & \varphi \\ \varphi & \varphi \end{pmatrix}$. Applying this transformation to the A_i gives $P'A_iP = M_i$ and M_i has rank p_i. Partition M_i such that

$$M_i = \begin{pmatrix} B_i & C_i' \\ C_i & D_i \end{pmatrix}$$

where B_i is a $p \times p$ symmetric matrix. Since $\sum_{i=1}^m M_i = \begin{pmatrix} I_p & \varphi \\ \varphi & \varphi \end{pmatrix}$ we have $\sum_{i=1}^m B_i = I_p$. Clearly the rank of B_i must be less than or equal to the rank of M_i. Therefore, let the rank of B_i equal $p_i - k_i$ where $k_i \geqq 0$. But the rank of the sum of matrices is less than or equal to the sum of the ranks, hence

$$\sum_{i=1}^m (p_i - k_i) \geqq p.$$

This gives $-\sum_{i=1}^m k_i \geqq 0$, so $k_i = 0$ for $i = 1, 2, \cdots, m$, and the rank of B_i is equal to p_i. Applying Theorem K to the equation $\sum B_i = I_p$ it follows that B_i is idempotent ($i = 1, 2, \cdots, m$) and $B_iB_j = \varphi$ for all $i \neq j$. By Theorem J we know that there exists an orthogonal matrix Q such that $Q'B_iQ$ is diagonal for $i = 1, 2, \cdots, m$. Let $Q'B_iQ = E_i$ where E_i is a $p \times p$ diagonal matrix with p_i diagonal elements equal to unity and the remaining diagonal elements equal to zero. Also, it follows that $\sum_{i=1}^m E_i = I_p$, so there is exactly one matrix in the set $E_1, E_2, \cdots, E_m$ whose tth diagonal element (for any $t = 1, 2, \cdots, p$) is equal to unity. All the remaining E_i have the tth diagonal element equal to zero. Since Q is orthogonal we know that

$$\underset{n\times n}{R} = \begin{pmatrix} Q & \varphi \\ \varphi & I_{n-p} \end{pmatrix}$$

is also orthogonal. Using this transformation on the equation

$$\sum_{i=1}^m M_i = \begin{pmatrix} I_p & \varphi \\ \varphi & \varphi \end{pmatrix}$$

gives

$$R'M_iR = \begin{pmatrix} E_i & F_i' \\ F_i & G_i \end{pmatrix}$$

and the rank of $R'M_iR$ equals the rank of E_i. But then $(F_i, G_i) = T_i(E_i, F_i')$ where T_i is an $(n - p) \times p$ matrix and $G_i = T_iE_iT_i'$. Let t_i be the first row of T_i. Then the first diagonal element of G_i is $t_iE_it_i'$ which is a sum of squares of some of the elements of t_i and the first row of F_i is t_iE_i which is a vector containing those same elements of t_i and zeros. Hence $\sum G_i = \varphi$ implies that $t_iE_it_i' = 0$ and $t_iE_i = \varphi$. The first row of G_i is $t_iE_iT_i' = \varphi$. Applying this argument to each row of T_i we have $F_i = \varphi$ and $G_i = \varphi$.

Hence

$$R'M_iR = \begin{pmatrix} E_i & \varphi \\ \varphi & \varphi \end{pmatrix}$$

and $R'M_iRR'M_jR = \varphi$ (for all $i \neq j$) and $R'M_iR$ is idempotent. Hence, A_i is idempotent and $A_iA_j = \varphi$ (for all $i \neq j$), and the proof is complete.

It has been pointed out (Craig, [2]) that if Y is distributed as $N_n(\varphi, I)$, then a necessary and sufficient condition that $Y'AY$ be distributed as $\chi^2(p)$ is that A be an idempotent matrix of rank p. We will generalize this result into

THEOREM 2. *If Y is distributed as $N_n(\mu, I)$, then a necessary and sufficient condition that $Y'AY$ is distributed as $\chi'^2(k, \lambda)$* (where $\lambda = \frac{1}{2}\mu'A\mu$) *is that A be an idempotent matrix of rank k.*

Proof. We will first prove sufficiency. Let P be an orthogonal matrix such that $P'AP = \begin{pmatrix} I_k & \varphi \\ \varphi & \varphi \end{pmatrix}$, and let $Z = P'Y$. Then $Z = \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}$ is distributed as $N_n(\alpha, I)$ where $\alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = P'\mu$ and where α_1 and Z_1 are each $k \times 1$ vectors. Z_1 is distributed as $N_k(\alpha_1, I)$. Also $Y'AY = Z'P'APZ = Z_1'Z_1$. Thus by Theorem F, $Y'AY = Z_1'Z_1$ is distributed as $\chi'^2(k, \lambda)$ where $\lambda = \frac{1}{2}\alpha_1'\alpha_1$. This proves sufficiency if we can show that $\alpha_1'\alpha_1 = \mu'A\mu$. To do this let $P = (P_1P_2)$ where P_1 has dimension $n \times k$, then

$$\mu'A\mu = \mu'PP'APP'\mu = \mu'(P_1P_2)P'AP\begin{pmatrix} P_1' \\ P_2' \end{pmatrix}\mu = (\mu'P_1, \mu'P_2)\begin{pmatrix} I_k & \varphi \\ \varphi & \varphi \end{pmatrix}\begin{pmatrix} P_1'\mu \\ P_2'\mu \end{pmatrix}$$

$$= \mu'P_1P_1'\mu = \alpha_1'\alpha_1.$$

To prove necessity, we will assume that $Y'AY$ is distributed as $\chi'^2(k, \lambda)$ and show that this implies that A is an idempotent matrix of rank k. We know that there exists an orthogonal matrix C such that $C'AC = D$ where D is a diagonal matrix where the number of non-zero diagonal elements, d_{ii}, equals the rank of A. Let $Z = C'Y$, then $Y'AY = Z'C'ACZ = Z'DZ = \sum_{i=1}^{n} d_{ii}z_i^2$. Since Z is distributed as $N_n(C'\mu, I)$, we know by Theorem F that z_i^2 is distributed as $\chi'^2(1, \lambda_i)$ where $\lambda_i = [E(z_i)]^2/2$. Since the z_i are independent the moment generating function of $\sum_{i=1}^{n} d_{ii}z_i^2$ is

$$\prod_{i=1}^{n} (1 - 2t\, d_{ii})^{-1/2} e^{-\lambda_i + \lambda_i(1-2d_{ii}t)^{-1}}.$$

Also, since the hypothesis states that $Y'AY$ is distributed as $\chi'^2(k, \lambda)$ (where $\lambda = \frac{1}{2}\mu'A\mu$) the moment generating function of $Y'AY$ is $(1 - 2t)^{-k/2}e^{-\lambda+\lambda(1-2t)^{-1}}$. Since $Y'AY = \sum d_{ii}z_i^2$, the moment generating functions are equal, and we get

$$(2) \qquad (1 - 2t)^{-k/2}e^{-\lambda+\lambda(1-2t)^{-1}} = \prod_{i=1}^{n} (1 - 2t\, d_{ii})^{-1/2} e^{-\lambda_i + \lambda_i(1-2d_{ii}t)^{-1}}.$$

It is clear that there exists a neighborhood of zero for t such that the quantities on the left- and right-hand sides of Eq. (2) exist and have derivatives of all orders.

If any of the d_{ii} were neither 0 nor 1, the right-hand side of this identity would be an analytic function of t with different singularities than the left-hand side. By this same argument it follows that exactly k of the d_{ii} are one and the others vanish. It also follows that $\lambda = \sum \lambda_i$.

Thus we have shown that if $Y'AY$ is distributed as $\chi'^2(k, \lambda)$, then k of the d_{ii} are equal to unity, and $n - k$ of the d_{ii} are equal to zero. But the d_{ii} are the characteristic roots of A, and hence A is an idempotent matrix of rank k and the theorem is established.

It might be pointed out that $\lambda = 0$, and $\chi'^2(k, \lambda)$ degenerates to $\chi^2(k)$ if and only if $A\mu = \varphi$.

Using Theorem 1 and Theorem 2 of this section and Theorem H of the preceding section, we can state the following Theorems:

THEOREM 3. *If Y is distributed as $N_n(\mu, I)$ and if $Y'AY = \sum_{i=1}^k Y'A_iY$ where the rank of A equals p and the rank of A_i equals p_i, then*

(1) *any two of the three conditions* C_1, C_2, C_3 *are necessary and sufficient for all the remaining conditions* $C_1, \cdots, E_1$;
(2) *any two of the three conditions* D_1, D_2, D_3 *are necessary and sufficient for all the remaining conditions,* $C_1, \cdots, E_1$.
(3) *any two conditions* C_i *and* D_j $i \neq j$ *are necessary and sufficient for all the remaining conditions*;
(4) E_1 *and* C_3 *are necessary and sufficient for all the remaining conditions*;
(5) E_1 *and* D_3 *are necessary and sufficient for all the remaining conditions.*

C_1 : $Y'A_iY$ is distributed as $\chi'^2(p_i, \lambda_i)$ where $\lambda_i = (\mu'A_i\mu)/2$ for $i = 1, 2, \cdots, k$.
C_2 : *$Y'A_iY$ and $Y'A_jY$ are independent for all $i \neq j$.*
C_3 : *$Y'AY$ is distributed as $\chi'^2(p, \lambda)$ where $\lambda = (\mu'A\mu)/2$.*
D_1 : *Each A_i is an idempotent matrix.*
D_2 : *$A_iA_j = \varphi$ for all $i \neq j$.*
D_3 : *A is an idempotent matrix.*
E_1 : $\sum_{i=1}^k p_i = p$.

THEOREM 4. *In Theorems 2 and 3 if Y is distributed as $N_n(\mu, \sigma^2 I)$ then all the results follow except each quadratic form and each λ and λ_i must be divided by σ^2.*

Cochran's theorem states: *if Y is distributed as $N_n(\varphi, I)$, and if*

$$Y'Y = \sum_{i=1}^k Y'A_iY$$

(where the rank of A_i is n_i), then a necessary and sufficient condition that $Y'A_iY$ ($i = 1, 2, \cdots, k$) are independently distributed respectively as $\chi^2(n_i)$($i = 1, 2, \cdots, k$) is that $\sum_{i=1}^k n_i = n$. Madow extended this to (Madow, 1940): *if Y is distributed as $N_n(\mu, I)$ and if $Y'Y = \sum_{i=1}^k Y'A_iY$ (where the rank of A_i is n_i), then a necessary and sufficient condition that $Y'A_iY$ ($i = 1, 2, \cdots, k$) are independently distributed as $\chi'^2(n_i, \lambda_i)$ is that $\sum_{i=1}^k n_i = n$.*

We will now extend these theorems.

THEOREM 5. *If Y is distributed as $N_n(\mu, V)$ where V is an $n \times n$ positive definite symmetric matrix, and if $Y'BY = \sum_{i=1}^k Y'B_iY$ where the rank of B_i is p_i and the rank of B is p, then any one of the six conditions,* C_1, C_2, C_3, C_4, C_5, C_6, *is necessary and sufficient that the $Y'B_iY$ be independently distributed as $\chi'^2(p_i, \lambda_i)$ where $\lambda_i = \frac{1}{2}\mu'A_i\mu$.*

C_1 : *BV be idempotent and $\sum_{i=1}^k p_i = p$.*
C_2 : *BV and each B_iV be idempotent.*
C_3 : *BV be idempotent and $B_iVB_j = \varphi$ for all $i \neq j$.*
C_4 : *$Y'BY$ be distributed as $\chi'^2(p, \lambda)$ and $p = \sum_{i=1}^k p_i$.* ($\lambda = \frac{1}{2}\mu'A\mu$).
C_5 : *$Y'BY$ be distributed as $\chi'^2(p, \lambda)$ and B_iV be idempotent* (where $\lambda = \frac{1}{2}\mu'B\mu$).

C_6 : *$Y'BY$ be distributed as $\chi'^2(p, \lambda)$ and $B_iVB_j = \varphi$ for $i \neq j$* (where $\lambda = \frac{1}{2}\mu'B\mu$).

Proof. Since V is positive definite, there exists a non-singular matrix P such that $P'VP = I_n$. Let $Z = P'Y$; then Z is distributed as $N_n(P'\mu, I_n)$. Also $Y'BY = Z'P^{-1}BP'^{-1}Z$, $Y'B_iY = Z'P^{-1}B_iP'^{-1}Z$, and

$$Z'(P^{-1}BP'^{-1})Z = \sum_{i=1}^{k} Z'(P^{-1}B_iP'^{-1})Z.$$

If we let $A = P^{-1}BP'^{-1}$ and $A_i = P^{-1}B_iP'^{-1}$, then we have $Z'AZ = \sum_{i=1}^{k} Z'A_iZ$, and the results follow immediately from Theorem 3 if we can show that: A being idempotent is equivalent to BV being idempotent; A_i being idempotent is equivalent to B_iV being idempotent; $B_iVB_j = \varphi$ for $i \neq j$ is equivalent to $A_iA_j = 0$ for $i \neq j$. To show these we proceed as follows: If A is idempotent then this means $(P^{-1}BP'^{-1})(P^{-1}BP'^{-1}) = P^{-1}BP'^{-1}$. Performing left multiplication by P and right multiplication by P' gives $BP'^{-1}P^{-1}B = B$. But $P'^{-1}P^{-1} = V$, hence $BVB = B$ or $(BV)(BV) = BV$. Thus A being idempotent implies that BV is idempotent. Starting with $(BV)(BV) = BV$ we will arrive at $AA = A$. Hence A being idempotent is equivalent to BV being idempotent. A similar procedure will work when applied to B_iV. To show that $B_iVB_j = \varphi$ for $i \neq j$ is equivalent to $A_iA_j = \varphi$ for $i \neq j$, proceed as follows: If $B_iVB_j = \varphi$ for $i \neq j$, then $\varphi = P^{-1}B_iP'^{-1}P'VPP^{-1}B_jP'^{-1} = A_iIA_j = A_iA_j$. The reverse procedure also follows, and hence the theorem is established.

(In this theorem, AV and the A_iV need not be symmetric. Also it should be remembered that AV being idempotent is equivalent to VA being idempotent and similarly for A_iV).

We also noted that putting $k = 1$ we get the

Corollary 5.1. *If Y is distributed as $N_n(\mu, V)$ where V is a positive definite matrix, then a necessary and sufficient condition that $Y'AY$ be distributed as $\chi'^2(p, \lambda)$ where p is the rank of A and where $\lambda = \frac{1}{2}\mu'A\mu$ is that AV be idempotent (not necessarily symmetric).*

4. Illustrations. Consider the linear hypothesis model $Y = X\beta + e$ defined in Sec. 3. If we partition the X matrix and β vector such that

$$X = (X_1, X_2) \quad \text{and} \quad \beta = \begin{pmatrix} \alpha \\ \gamma \end{pmatrix}$$

where X_1 is of order $n \times p_1$ and α is a $p_1 \times 1$ vector, then we can write $Y = X\beta + e$ as $Y = X_1\alpha + X_2\gamma + e$.

To test the hypothesis $H_0 : \alpha = \varphi$, we can form the ratio

$$u = \frac{Q_1}{Q} \cdot \frac{n - p}{p_1}, \tag{4.1}$$

where u is distributed as Snedecor's F with p_1 and $n - p$ degrees of freedom. The quantities Q_1 and Q can be derived (Kempthorne, 1952) by the following process:

Q is the minimum value of $e'e$ with respect to the parameters in the model $Y = X\beta + e = X_1\alpha + X_2\gamma + e$.

$Q_1 = Q - Q_2$ when Q_2 is the minimum value of $e'e$ with respect to the model $Y = X_2\gamma + e$ (the model restricted by H_0). By a straightforward application of a minimization procedure we see that

$$Q = Y'(I - XS^{-1}X')Y = Y'AY \quad \text{and} \quad Q_2 = Y'(\mathrm{I} - X_2S_2^{-1}X_2')Y = Y'BY$$

where $S = X'X$, $S_2 = X_2'X_2$, $I - XS^{-1}X' = A$, and $I - X_2S_2^{-1}X_2' = B$. To find the distribution of Q/σ^2 and Q_1/σ^2 the method sometimes employed (Kempthorne, 1952) is quite a complex procedure of finding the ranks of the corresponding matrices A and B and applying Cochran's theorem. An alternative method using theorems on idempotent matrices to obtain the distribution of u when H_0 is true and when $H_1 : \alpha \neq \varphi$ is true is as follows:

Obviously A and B are each idempotent. Since

$$X'(I - XS^{-1}X') = \varphi, \tag{4.2}$$

it is clear that $X_2'(I - XS^{-1}X') = \varphi$ and $X_1'(I - XS^{-1}X') = \varphi$. Let $C = B - A$; then by using 4.2,

$$C = (I - X_2S_2^{-1}X_2') - (I - XS^{-1}X')$$

is clearly idempotent and $AC = \varphi$. Hence by Theorem 3 we have

1. $Q/\sigma^2 = (Y'AY)/\sigma^2$ is distributed as $\chi'^2(n - p, \lambda_A)$.
2. $Q_1/\sigma^2 = (Y'CY)/\sigma^2$ is distributed as $\chi'^2(p_1, \lambda_C)$.
3. Q and Q_1 are independent.
4. $\lambda_A = 1/2\sigma^2\ (\beta'X'AX\beta) = 1/2\sigma^2\ [\beta'X'(I - XS^{-1}X')X\beta] = 0$, so Q/σ^2 is distributed as $\chi^2(n - p)$.
5. $\lambda_C = [1/(2\sigma^2)]\ [\beta'X'\{(I - X_2S_2^{-1}X_2') - (I - XS^{-1}X')\}X\beta]$
   $= [1/(2\sigma^2)]\ [\alpha'(X_1'X_1 - X_1'X_2S_2^{-1}X_2'X_1)\alpha]$,
   and since $X_1'X_1 - X_1'X_2S_2^{-1}X_2'X_1$ is positive definite, Q_1/σ^2 has the central Chi-square distribution if and only if $\alpha = \varphi$; i.e., if and only if H_0 is true.

Hence by Theorem M, $u = (Q_1/Q)\cdot[(n - p)/p_1]$ is distributed as $F'(p_1, n - p, \lambda_C)$ and reduces to the central F (Snedecor's F) if and only if H_0 is true.

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