A RELATIONSHIP BETWEEN HODGES' BIVARIATE SIGN TEST AND A NON-PARAMETRIC TEST OF DANIELS[1]

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The null distribution of the statistic K of Hodges' bivariate sign test ([2] and [3]) is the same as the null distribution of the statistic m proposed by Daniels [1] to test the hypothesis: Median $\{Y \mid x\} = \alpha_o + \beta_o x$, where α_0 and β_0 are given. For suppose we consider a sequence $S_1, S_2, \cdots, S_n$, where each S_k is either $+1$ or -1. We say that two such sequences agree i times if there exist exactly i places at which the sequences agree. Let t_k, $k = 0, 1, \cdots, n-1$, be the number of agreements of the sequence $S_1, \cdots, S_n$ with the sequence whose first $n-k$ values are $+1$, and whose last k values are -1 (called the kth sequence). Let t_{n+i}, $i = 0, \cdots, n-1$, be the number of agreements of $S_1, \cdots, S_n$ with the sequence obtained by changing each sign of the ith sequence. Clearly $t_{n+i} = n - t_i$, $i = 0, \cdots, n-1$, and

$$t_{k+1} = \begin{cases} 1 + t_k \text{ if } S_{n-k} = -1 \\ -1 + t_k \text{ if } S_{n-k} = +1 \end{cases}, \qquad k = 0, \cdots, n-1.$$

Now envisage the sequence $S_1, \cdots, S_n$ placed in order reading from left to right at equal intervals on the upper half of a circle (S_1 and S_n being above the horizontal diameter), and the value $-S_k$ placed at the point on the circle diametrically opposed to S_k, $k = 1, \cdots, n$. Such an arrangement is sketched below for the case $n = 5$.

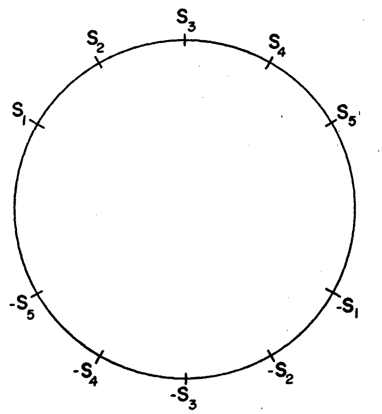
Let P_k, $k = 0, \cdots, 2n-1$, be the number of positive S_i lying on the upper semi-circle after k steps of a clockwise rotation have been taken (at which time S_1 will occupy the position formerly occupied by S_{k+1}, and S_{n-k} will occupy the position formerly occupied by S_n). Then clearly $P_0 = t_0$,

$$P_{k+1} = \begin{cases} P_k + 1 \quad \text{if} \quad S_{n-k} = -1 \\ P_k - 1 \quad \text{if} \quad S_{n-k} = +1 \end{cases}, \qquad k = 0, \cdots, n-1,$$

and $P_{n+i} = n - P_i$, $i = 0, \cdots, n-1$. Since the t_k also satisfy these last two relationships, it follows that $t_k = P_k$, $k = 0, \cdots, 2n-1$. Hence $m = n - \text{Max } t_i = n - \text{Max } P_i = K$, where both maxima are over $i = 0, 1, \cdots, 2n-1$. Since each sequence $S_1, \cdots, S_n$ has probability $1/2^n$ under the null hypothesis of both Daniels' and Hodges' tests, and since m and K depend only on the observed sequence $S_1, \cdots, S_n$, it follows that the null distributions of m and K are identical.

Received February 8, 1960; revised July 25, 1960.

[1] This report was supported in part by a Public Health Service Training Grant in Biometry.



Results for Daniels' m may thus be applied to Hodges' K, and vice versa. For example, we can apply Daniels' approximation (under the null hypothesis),

$$\Pr\{m > m_o\} \backsim 4(n - 2m_o)n^{-\frac{1}{2}}\sum_{j=0}^{\infty} (2\pi)^{-\frac{1}{2}} \exp\left[-\tfrac{1}{2}(2j + 1)^2(n - 2m_o)^2/n\right],$$

to Hodges' K. Also we have $K = m \leqq [(n - 1)/2]$, whereas Hodges only shows $K \leqq n/2$.

Hodges' restriction to $K < n/3$ [2] seems to have no relevance to Daniels' problem, and Klotz [3] has already obtained the null distribution of Hodges' test with the restriction removed. Each of the three authors includes a null distribution table in his paper, and these tables agree, that of Klotz being most complete.

Daniels is able to obtain the power of his test only in the case where the alternative line is parallel to the null hypothesis line, while Hodges does not consider the power function. The alternatives for the Hodges test which correspond to the parallel line alternatives for the Daniels test satisfy (in Hodges' notation)

$$\Pr\left\{ y_i' - y_i > 0 \,\middle|\, \frac{y_i' - y_i}{x_i' - x_i} = \beta, (x_i' - x_i)^2 + (y_i' - y_i)^2 = d \right\} = P_i(\beta, d) = P,$$

where $-\infty \leqq \beta \neq 0 \leqq +\infty, 0 < d < +\infty, i = 1, \cdots, n$. Letting $g_i(\xi, \eta)$ be the density function of $(x_i' - x_i, y_i' - y_i)$, we must then have

$$g_i(\xi, \eta)/[g_i(\xi, \eta) + g_i(-\xi, -\eta)] = P,$$

or

$$(1 - P)g_i(\xi, \eta) = Pg_i(-\xi, -\eta), \quad \text{for } i = 1, \cdots, n, \quad \eta > 0, \quad \text{and all } \xi.$$

The conditional power of Hodges' test against those alternatives for which $(1 - P)g_i(\xi, \eta) = Pg_i(-\xi, -\eta), i = 1, \cdots, n, \eta > 0$, is then given by Daniels' approximation [1],

$$\begin{aligned}\Pr\{K \leqq m_o\} &= \Pr\{m \leqq m_o\} \\ &= 1 - \sum_{j=-\infty}^{\infty} e^{2j\mu z_o}[\Phi((2j+1)z_o + \mu) - \Phi((2j-1)z_o + \mu)] \\ &\qquad + 2e^{-\mu^2/2}[e^{\mu z_o} - e^{-\mu z_o}](2\mu^3)^{-\frac{1}{2}} \sum_{j=0}^{\infty} e^{-(2j+1)^2 z_o^2/2},\end{aligned}$$

where

$$z_o = (n - 2m_o)n^{-\frac{1}{2}},$$

$$\Pr\{\epsilon_i > \alpha_o - \alpha\} = P_i = P = (1 - \mu n^{-\frac{1}{2}})/2,$$

$$Q_i = Q = (1 + \mu n^{-\frac{1}{2}})/2.$$

Here $\Pr\{m > m_o\}$ is Daniels' probability of rejecting the null hypothesis that the true line is $\alpha_o + \beta_o x$ when in fact the true line is $\alpha + \beta_o x$, and the rejection criterion is $m > m_o$.

The above class of alternatives for the Hodges' test is rather restrictive ($g_i(\xi, \eta)$ must be discontinuous at the origin, $i = 1, \cdots, n$) and does not seem particularly interesting. In the general case, with

$$\frac{g_i(\xi, \eta)}{g_i(\xi, \eta) + g_i(-\xi, -\eta)} = P_i(\xi, \eta), \quad i = 1, \cdots, n, \eta > 0,$$

the conditional power of Hodges' test, given that $(x_i' - x_i, y_i' - y_i) = (\xi_i, \eta_i)$, (with $0 > \eta_1/\xi_1 > \eta_2/\xi_2 > \cdots > \eta_j/\xi_j, 0 < \eta_n/\xi_n < \eta_{n-1}/\xi_{n-1} < \cdots < \eta_{j+1}/\xi_{j+i}$ for some j and $i = 1, \cdots, n$), is then

$$\Pr\{K \leqq k_o\} = 1 - \sum_{t=k_o+1}^{n-k_o-1} P_n(t, k_o),$$

where

$$\begin{aligned}&P_n(t, k_o) \\ &\quad = \Pr\{k_o < t + \omega_i < n - k_o, i = 1, 2, \cdots, n-1; t + \omega_n = n - t\},\end{aligned}$$

and ω_i takes on the values $+1$ or -1 with probabilities

$$P_i(\xi_i, \eta_i) \quad \text{and} \quad 1 - P_i(\xi_i, \eta_i),$$

respectively.

I wish to express my thanks to Professor Lincoln E. Moses for suggesting the relationship between the two statistics.

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[2] J. L. Hodges, Jr., "A bivariate sign test," *Ann. Math. Stat.*, Vol. 26 (1955), pp. 523–527.
[3] Jerome Klotz, "Null distribution of the Hodges bivariate sign test," *Ann. Math. Stat.*, Vol. 30 (1959), pp. 1029–1033.