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THE NONEXISTENCE OF LINKED BLOCK DESIGNS WITH LATIN SQUARE ASSOCIATION SCHEMES

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0. Summary. The listing of partially balanced linked block designs by Roy and Laha (1957) contains no Latin square designs. The listing of designs with Latin square association schemes by Clatworthy (1956), which includes those given by Bose, Clatworthy and Shrikhande (1954) and by Bose and Shimamoto (1952), and the later listings by Chang and Liu (1964) and by Clatworthy (1967) contain no linked block designs. The question then arises whether any linked block designs exist having the Latin square association scheme. In this note a partial answer to the question is given. It is shown that there do not exist any linked block designs which are partially balanced with two associate classes and have the L_i association scheme for $i = 2$, 3 or 4.

1. Introduction. In the L_2 association scheme with $v = n^2$ varieties, the varieties are arranged in a square: two varieties are said to be first associates if they lie in the same row or in the same column; otherwise they are second associates. In the L_i scheme for $i > 2$, $i-2$ mutually orthogonal Latin squares are superimposed on the square array; two varieties are then said to be first associates if they lie in the same row or column, or if they correspond to the same letter in one of the Latin squares; otherwise they are second associates. The number of varieties in each associate class are

$$n_1 = i(n-1), \qquad n_2 = (n-1)(n-i+1),$$

and the equation $r(k-1) = \lambda_1 n_1 + \lambda_2 n_2$ becomes

$$r(k-1) - i(n-1)\lambda_1 - (n-1)(n-i+1)\lambda_2 = 0. \tag{1}$$

For any design with the L_i scheme, the latent roots, θ_i, of $\mathbf{NN}'$ and their multiplicities, α_i, were shown by Connor and Clatworthy (1954) to be

$$\theta_0 = rk, \qquad \alpha_0 = 1;$$

$$\theta_1 = r+(n-i)\lambda_1 - (n-i+1)\lambda_2, \qquad \alpha_1 = i(n-1);$$

$$\theta_2 = r - i\lambda_1 + (i-1)\lambda_2, \qquad \alpha_2 = (n-1)(n-i+1).$$

Since linked block designs are the duals of balanced incomplete block designs, there are two possibilities, either $\theta_1 = 0$ and $b = \alpha_2 + 1$, or $\theta_2 = 0$ and $b = \alpha_1 + 1$. We shall call these Case 1 and Case 2, respectively.

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In Case 1 we solve the equations $\theta_1 = 0$ and (1) and get

$$\lambda_1 = \frac{r(k-n)}{n(n-1)}, \qquad \lambda_2 = \frac{r(ni-n+(n-i)k)}{n(n-1)(n-i+1)}; \tag{2}$$

$$r = \frac{bk}{v} = \frac{(n^2-(n-1)i)k}{n^2} = k - \frac{(n-1)ik}{n^2}.$$

But r is an integer and n^2, $n-1$ are relatively prime so that ik/n^2 must be an integer; k must be of the form n^2t/i where t and k are integers.

In case 2 we solve $\theta_2 = 0$ and (1) obtaining

$$\lambda_1 = \frac{r(n^2-ni+k(i-1))}{in(n-1)}, \qquad \lambda_2 = \frac{r(k-n)}{n(n-1)}; \qquad r = (i(n-1)+1)k/n^2. \tag{3}$$

Since the complement of a linked block design is itself a linked block design (their duals being complementary BIB designs), it is enough to consider only designs with $2k \leqq v$, i.e., $2k \leqq n^2$.

For each scheme our technique will be to take the possible values (if any) of $k \leqq v/2$ which make r an integer and substitute in (2) to obtain λ_1 or in (3) to obtain λ_2. In either case we shall then show that the fraction obtained for λ_i cannot be reduced to an integer. Hence no designs exist.

2. The L_2 scheme. *Case* 1. We have $k = n^2t/2$. If n is odd, $k \geqq n^2 = v$ and there are no designs. Suppose n is even. Then there is the possibility $k=n^2/2$, in which case $r = (n^2-2n+2)/2$ and

$$4\lambda_1 = \frac{(n^2-2n+2)(n^2-2n)}{n(n-1)} = (n-1)(n-2)+\frac{n-2}{n-1}.$$

But $n-1$, $n-2$ are relatively prime and so this expression for $4\lambda_1$ cannot be an integer unless $n = 2$. If $n = 2$ we can have a design with $b = k = 2$, $v = 4$, $r = 1$, $\lambda_1 = 0$, $\lambda_2 = 1$, but it is not a linked block design since the two blocks are disjoint.

Case 2. We have $r = (2n-1)k/n^2$. But $2n-1$ and n^2 are relatively prime and so k/n^2 must be an integer; hence $k \geqq n^2$, and there are no designs.

3. The L_3 scheme. *Case* 1. We must have $3k/n^2$ an integer, and this implies $k \geqq n^2$ unless n is divisible by 3. Let $n = 3s$, where s is an integer. The only possibility with $2k \leqq v$ is $k = 3s^2$ and $r = 3s^2-3s+1$. Then $\lambda_1 = (3s^2-3s+1)(s-1)/(3s-1)$ and $3\lambda_1 = 3s^2-5s+3-2s/(3s-1)$. However, except in the trivial case of $s = 1$, which gives three disjoint blocks, $0 < 2s/(3s-1) < 1$, and so $3\lambda_1$ cannot be an integer.

Case 2. We have $r = (3n-2)k/n^2$. If n is odd, $3n-2$ and n^2 are relatively prime, so that $k \geqq n^2$, which allows no designs. Suppose that $n = 2s$ where s is an integer, and $s > 1$; then $r = (3s-1)k/2s^2$. If s is even, $3s-1$ and $2s^2$ are relatively prime, and the only possibility is $k = 2s^2$, $r = 3s-1$, in which case $\lambda_2 = (3s-1)(s-1)/(2s-1)$, and $2\lambda_2 = 3s-2-s/(2s-1)$, which cannot be an integer. If s is odd, there is the possibility of $k = s^2$, $2r = 3s-1$. Then $4\lambda_2 = (3s-1)(s-2)/(2s-1)$ and $8\lambda_2 = 3s-4-3s/(2s-1)$, which cannot be an integer.

4. The L_4 scheme. It can be shown by the same methods that there are no linked block designs with the L_4 scheme. The proof is omitted.

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