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ON THE DOMAINS OF DEFINITION OF ANALYTIC CHARACTERISTIC FUNCTIONS[1]

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1. Introduction. It is known that analytic characteristic functions are always regular in a horizontal strip (the strip of regularity) which contains the real axis in its interior. It is, however, often possible to continue an analytic characteristic function beyond its strip of regularity. The problem arises whether an analytic characteristic function can have a natural domain of analyticity, that is, a domain beyond which it cannot be continued analytically. The accessible points of the boundary of this domain are then singular points. I. V. Ostrovskii [4] used a theorem of Cartan and Thullen (see Bochner–Martin [1] page 84) to show that any domain which is symmetric with respect to the imaginary axis and which contains a horizontal strip and the real axis can be the domain of analyticity of a characteristic function.

In the present note we construct by an elementary method characteristic functions which have a given domain of analyticity. The results are then extended to characteristic functions which are boundary functions of analytic functions.

2. Some lemmas.

LEMMA 1. *The function*

$$f(t) = \left[\left(1+\frac{it}{\alpha}\right)\left(1+\frac{it}{\beta+i\gamma}\right)\left(1+\frac{it}{\beta-i\gamma}\right)\right]^{-1}$$

where α, β and γ are real numbers is a characteristic function if $0<\alpha\leqq\beta$.

The lemma is proven by expanding $f(t)$ into partial fractions and applying the inversion formula.

LEMMA 2. *It is always possible to find a sequence $\{a_k\}$ of positive numbers such that $\sum_{k=1}^{\infty} a_k = 1$ while $a_n \geqq 2\sum_{k=n+1}^{\infty} a_k$. An example of such a sequence is* $a_k = (c-1)c^{-k}$ $(k = 1, 2, \cdots)$ *with* $c > 3$.

LEMMA 3. *Let G be an arbitrary domain; then there exists a denumerable set of points which is dense in the boundary ∂G of G.*

We consider the set of all points of G which have rational coordinates and arrange them in a sequence $\{\alpha_n\}$. For each point α_n let β_n be the point of ∂G which is nearest to α_n and which has the property that $\arg(\alpha_n-\beta_n)$ has the smallest possible value. In general the sequence $\{\beta_n\}$ contains repetitions. We omit these and obtain a sequence $\{b_n\}$ which is dense in ∂G.

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3. The case of analytic characteristic functions. We note that analytic characteristic functions have the Hermitian property; hence the boundary ∂G of G must be symmetric with respect to the imaginary axis. We consider first the case where ∂G lies in a half-plane which has no interior point in common with the real axis. Let α be the greatest positive number such that ∂G is contained in the half-plane $\operatorname{Im}(z) \geqq \alpha > 0$. Then the point $i\alpha$ is necessarily a singular point (possibly an isolated singularity) of the analytic characteristic function.

Let $\{b_n\}$ be the sequence of points on ∂G which was used in the proof of Lemma 3 and which is dense in ∂G. We suppose $b_n \neq i\alpha$ and introduce the function $f_0(z) = (1 - z/i\alpha)^{-1}$ and

$$f_n(z) = \left[\left(1-\frac{z}{i\alpha}\right)\left(1-\frac{z}{b_n}\right)\left(1+\frac{z}{\bar{b}_n}\right)\right]^{-1} \tag{3.1a}$$

or

$$f_n(z) = \frac{A_n}{z-i\alpha} + \frac{B_n}{z-b_n} + \frac{C_n}{z+\bar{b}_n} \tag{3.1b}$$

where

$$A_n = \frac{i\alpha b_n \bar{b}_n}{(i\alpha - b_n)(i\alpha + \bar{b}_n)} \tag{3.2}$$

$$B_n = \frac{i\alpha b_n \bar{b}_n}{(b_n - i\alpha)(b_n + \bar{b}_n)} = -\bar{C}_n .$$

It follows from Lemma 1 and the fact that $\operatorname{Im}(b_n) \geqq \alpha$ that $f_n(z)$ is an analytic characteristic function. We select a sequence $\{a_n\}$ of positive numbers which satisfies the conditions of Lemma 2. Then

$$f(z) = \textstyle\sum_{n=0}^{\infty} a_n f_n(z) \tag{3.3}$$

is an analytic characteristic function. Its strip of regularity is the half-plane $\operatorname{Im} z < \alpha$.

We show next that $f(z)$ is analytic in the domain G. This is done by proving that the series (3.3) is uniformly convergent in every domain $G_{\varepsilon, R}$ of the form

$$G_{\varepsilon, R} = \{z \mid z \in G, |z| < R, d(z, \partial G) > \varepsilon\}$$

where $d(z, \partial G)$ is the distance of the point z to the boundary ∂G and where ε and R are arbitrary positive constants.

We see from (3.2) that the relations

$$\frac{A_n}{z - i\alpha} = O(1)$$

$$\frac{B_n}{z - b_n} = o(1)$$

$$\frac{C_n}{z + \bar{b}_n} = o(1)$$

hold as $|b_n| \to \infty$. Therefore we conclude from (3.1b) that

$$f_n(z) = O(1) \quad \text{as} \quad |b_n| \to \infty.$$

Hence there exist constants K and M such that

$$|f_n(z)| < K \quad \text{if} \quad |b_n| > M. \tag{3.4}$$

Let n be such that $|b_n| \leqq M$, then one sees from (3.1a) that

$$|f_n(z)| = \left| \frac{\alpha b_n \bar{b}_n}{(z-i\alpha)(z-b_n)(z+\bar{b}_n)} \right| < \frac{\alpha M^2}{\varepsilon^3}. \tag{3.5}$$

The uniform convergence of the series (3.3) follows from (3.4) and (3.5) and from the choice of the a_n.

We show next that each b_n is a singular point of $f(z)$.

We consider, in the following, values of z for which

$$|z-b_n| < |z-b_k| \tag{3.6}$$

for all $k \neq n$. The existence of such z is assured by the construction of the sequence $\{b_k\}$.

It follows from (3.1b) and (3.6) that

$$|f_k(z)| \leqq \left| \frac{A_k}{z-i\alpha} + \frac{C_k}{z+\bar{b}_k} \right| + \left| \frac{B_k}{z-b_n} \right|.$$

Therefore

$$\left| \frac{f_k(z)}{\dfrac{B_n}{z-b_n}} \right| \leqq \frac{\left| \dfrac{A_k}{z-i\alpha} + \dfrac{C_k}{z+\bar{b}_k} \right|}{\left| \dfrac{B_n}{z-b_n} \right|} + \left| \frac{B_k}{B_n} \right|.$$

It follows from this inequality that it is possible to find positive constants δ and η_1 so small that

$$\left| \frac{f_k(z)}{\dfrac{B_n}{z-b_n}} \right| < \frac{3}{2} \tag{3.7}$$

provided $0 < |z-b_n| < \eta_1$ and $|b_k - b_n| < \delta$. We can also select a number $\eta_2 > 0$ so small that

$$\left| \frac{f_n(z)}{\dfrac{B_n}{z-b_n}} \right| > \frac{2}{3} \tag{3.8}$$

for $|z-b_n| < \eta_2$. We see from (3.7) and (3.8) that

$$\left| \frac{f_k(z)}{f_n(z)} \right| < 1 \tag{3.9}$$

for $|z-b_n| < \eta < \min(\eta_1, \eta_2)$ and $|b_k-b_n| < \delta$. Moreover, the inequality (3.9) is even valid if $|b_k-b_n| \geqq \delta$ and $|z-b_n| < \eta$ provided η is chosen sufficiently small. We see from (3.3) that

$$\frac{f(z)}{f_n(z)} = \sum_{k=0}^{n-1} a_k \frac{f_k(z)}{f_n(z)} + a_n + \sum_{k=n+1}^{\infty} a_k \frac{f_k(z)}{f_n(z)}. \tag{3.10}$$

The first sum on the right-hand side of (3.10) tends to zero as z approaches b_n while one sees from (3.9) and Lemma 2 that

$$\left| \sum_{k=n+1}^{\infty} a_k \frac{f_k(z)}{f_n(z)} \right| < \frac{1}{2} a_n.$$

It follows then from (3.10) that $|f(z)/f_n(z)| > a_n/4$ if $z \to b_n$. Therefore b_n is a singular point of $f(z)$.

We still have to consider the point $i\alpha$. If $i\alpha$ is not an isolated point of ∂G then it is the limit of singular points b_j and is therefore also a singular point of $f(z)$. Suppose next that $i\alpha$ is an isolated point of ∂G. We note that $A_n = -im_n\,(m_n > 0)$ for all n and conclude from (3.1b) and (3.3) that $i\alpha$ is a singularity of $f(z)$. We have therefore obtained the following result.

THEOREM 1. *Let G be a domain which satisfies the following conditions*
(i) *The boundary of G is contained in the half-plane* $\operatorname{Im}(z) \geqq \alpha > 0$
(ii) *G is symmetric with respect to the imaginary axis*
(iii) *The point $i\alpha$ belongs to the boundary of G.*
Then there exists an analytic characteristic function whose natural domain of analyticity is G.

A similar statement is valid if the boundary of the region is contained in the half-plane $\operatorname{Im}(z) \leqq \alpha' < 0$. If we combine this with the preceding statement, we obtain the following statement.

COROLLARY TO THEOREM 1. *Let G be a domain which satisfies the following conditions*
(i) *G contains a strip* $-\alpha' < \operatorname{Im}(z) < \alpha$ $(\alpha' > 0, \alpha > 0)$
(ii) *G is symmetric with respect to the imaginary axis*
(iii) *The points $i\alpha$ and $-i\alpha'$ belong to the boundary of G. Then there exists an analytic characteristic function whose natural domain of analyticity is G.*

4. Generalizations. In this section we extend Theorem 1 to characteristic functions $f(t)$ which are boundary values of analytic functions. We say that a characteristic function $f(t)$ is the boundary value of an analytic function if there exists a complex valued function $A(z)$ of the complex variable $z = t+iy$ (t, y real) such that $A(z)$ is regular in a rectangle $R_1 = [|t| < \xi, -\eta < y < 0]$ (respectively in a rectangle $R_2 = [|t| < \xi, 0 < y < \eta]$) and which has the property that

$$f(t) = \lim_{y \uparrow 0} A(t+iy) \tag{4.1}$$

if $A(z)$ is regular in R_1 but

$$f(t) = \lim_{y \downarrow 0} A(t+iy) \tag{4.1a}$$

in case $A(z)$ is regular in R_2.

It follows then from a theorem due to Marcinkiewicz [3] that $A(z)$ is regular at least in the strip $-\eta < y < 0$ [respectively $0 < y < \eta$] and that the relation (4.1) holds for $-\eta < y < 0$ [respectively $0 < y < \eta$] and all real t.

We first prove the following theorem.

THEOREM 2. *Let G be a domain which satisfies the following conditions*[2]

(i) *G contains a strip $-\alpha < \operatorname{Im} z < 0$;*

(ii) *G is symmetric with respect to the imaginary axis;*

(iii) *The points 0 and $-i\alpha$ belong to the boundary ∂G of G;*

(iv) *If $b \in CG$ and $i \operatorname{Im} b > 0$ then $\operatorname{Im} b \in CG$;*

(v) *Every point x of the real axis which belongs to ∂G is the limit of a sequence of points $x_n + iy_n$ belonging to CG such that $y_n > 0$.*

Then there exists an analytic function $A(z)$ whose natural domain of analyticity is G and a characteristic function $f(t)$ such that (4.1) *holds.*

First, we suppose that G is not the strip $\{-\alpha < \operatorname{Im} z < 0\}$.

It is no restriction to assume that $\alpha = -\infty$. Let $\{b_k\}$ be the sequence of points on ∂G constructed according to Lemma 3 and let $\{\varepsilon_k\}$ be a decreasing sequence of positive numbers such that $\varepsilon_0 = +\infty$, $\lim_{k \to \infty} \varepsilon_k = 0$ and $\varepsilon_n \neq \operatorname{Im} b_k$ for all k and n. We define the sets

$$G_n' = \{z \mid z \in CG,\ \varepsilon_n \leqq \operatorname{Im} z \leqq \varepsilon_{n-1}\}$$

and

$$G_n = CG_n'.$$

If ε_1 is chosen small enough G_n is connected and G_n satisfies the conditions of Theorem 1, and we write $f_n(t)$ for the characteristic function constructed according to Theorem 1. The natural domain of $f_n(t)$ is G_n.

Let

$$f(t) = \sum_{j=1}^{\infty} a_j f_j(t)$$

where $\{a_j\}$ is the sequence defined by Lemma 2. Clearly, $f(t)$ is a characteristic function. We consider the region

$$H_{\varepsilon, R} = \{z \mid z \in G,\ |z| < R,\ d(z, \partial G) > \varepsilon\}.$$

It follows from the definition of the functions $f_j(z)$ and from (3.4) and (3.5) that there exists a constant M, depending on ε and R but not on j and z such that

$$|f_j(z)| < M$$

[2] CG denotes the complement of G.

if $z \in H_{\varepsilon, R}$. Therefore

$$f(z) = \sum_{j=1}^{\infty} a_j f_j(z)$$

is regular in G.

We show next that the points b_k are singular points of $f(z)$. We distinguish two cases. If $\operatorname{Im} b_k > 0$, then b_k belongs to one, and only one, of the sets G_n' and is therefore a singular point of the function $f_n(z)$. Moreover, there exists a constant M' such that $|f_j(z)| < M'$ for $j \neq n$ and $|z - b_k|$ sufficiently small. Therefore the sum $\sum_{j \neq n} a_j f_j(z)$ is finite so that b_k is a singular point of $f(z)$. We consider next the case where $\operatorname{Im} b_k = 0$. Then there exists, by assumption (v) of the theorem, a sequence $\{\beta_m\}$ such that $\beta_m \in CG$, $\operatorname{Im} \beta_m > 0$ and $\lim_{m \to \infty} \beta_m = b_k$. Since $f(z)$ is not regular in the points β_m we see that b_k is a singular point of $f(z)$. The statement of the theorem follows immediately in case G is not the strip $\{-\alpha < \operatorname{Im} z < 0\}$.

For the case when G is the strip $\{-\alpha < \operatorname{Im} z < 0\}$, it is sufficient to find a characteristic function the domain of analyticity of which is exactly $\{\operatorname{Im} z < 0\}$. Such an example can be given by means of the well-known Weierstrass function

$$f(t) = \sum_{k=0}^{\infty} 2^{-(k+1)} \exp(i5^k t).$$

The function $f(-t)$ belongs to a probability distribution which is bounded to the right and is, therefore, boundary of a function which is analytic in $\{\operatorname{Im} z < 0\}$ (cf. [2], page 309). Moreover, f is nowhere differentiable on the real axis. Therefore, any point of the real axis is a singular point of f.

As a consequence one obtains the following results.

COROLLARY TO THEOREM 2. *Let G be a domain which satisfies the following conditions:*

(i) *The domain G contains at least one interval of the real axis*;

(ii) *If G contains a point of the imaginary axis then G is symmetric with respect to the imaginary axis*;

(iii) *The origin does not belong to G*;

(iv) *If $b \in CG$ and $\operatorname{Im} b > 0$ then $i \operatorname{Im} b \in CG$*;

(v) *Every point x of the real axis which belongs to ∂G is the limit of a sequence of points $x_n + iy_n$ belonging to CG such that $y_n > 0$.*

Then there exists a regular function $A(z)$ whose natural domain of analyticity is G and a characteristic function $f(t)$ such that $f(t) = A(t)$ for any real t belonging to G.

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