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NOTE ON A CHARACTERIZATION OF THE INVERSE GAUSSIAN DISTRIBUTION

BY J. K. WANI AND D. G. KABE

University of Calgary and St. Mary's University

1. Introduction and summary. A random variable x is said to have the inverse Gaussian [3], or Wald's ([4] pages 192–193) distribution function when its density function is given by

$$f(x; m, \lambda) = (\lambda/2\pi x^3)^{\frac{1}{2}} \exp\{-\lambda(x-m)^2/2m^2x\}, \qquad 0 < x < \infty, \tag{1}$$

where the constants λ and m are assumed to be positive. If $x_1, x_2, \cdots, x_N$ are N independent observations from (1), then it is proved by Tweedie [3] that the densities of the variates $y = \sum_{j=1}^N x_j$ and $z = \sum_{j=1}^N x_j^{-1} - N^2 y^{-1}$ are independent; the distribution of y is $f(y; Nm, N^2\lambda)$ and that of λz is χ^2 with $N-1$ degrees of freedom (df). By proving the converse of Tweedie's result, namely, if $x_1, \cdots, x_N$ are independently and identically distributed random variables and if y and z are independently distributed, then $x_1, x_2, \cdots, x_N$ each have inverse Gaussian distribution, Khatri [2] gave a characterization of the inverse Gaussian distribution. Now Wald ([4] pages 192–193) has proved that if x has the distribution (1), then $t^2 = (x-m)^2/m^2x$ has a χ^2 distribution with one df, or t has a normal distribution with mean zero and variance λ^{-1}. It thus appears that certain properties of the inverse Gaussian distribution may be studied via the known properties of the normal distribution. Khatri's result on the characterization of the inverse Gaussian distribution is one such result. The following known result for the normal distribution, namely, if $t_1, t_2, \cdots, t_N$ are independent and identically distributed random variables and if a linear form w in t variates and a quadratic form z of rank $N-1$ in t variates, where $\sum_1^N t_i^2 = w^2 + z$, are independently distributed, then the t variates are normally distributed. Now this result when translated for the inverse Gaussian distribution is the result stated by Khatri, who proves the result by an involved method of characteristic functions. Perhaps, the simpler proof of Khatri's result established in the present paper might be of pedagogical interest.

2. Proof. Now it is known that

$$\int_0^\infty f(a^2T^2 + b^2/T^2)\, dT = a^{-1} \int_0^\infty f(t^2 + 2ab)\, dt, \qquad a, b > 0, \tag{2}$$

see e.g., Edwards ([1] page 188 (1)). One of the substitutions used for proving the above result is

$$Q = (aT^2 - b)^2/T^2. \tag{3}$$

The transformation (3) is not one to one, for when $T = 0$, $Q = \infty$, when $T = b^{\frac{1}{2}}/a^{\frac{1}{2}}$, $Q = 0$, and when $T = \infty$, $Q = \infty$. Thus $Q = \infty$ for two values of T. However, if we consider the transformation (3), separately in two parts, between $T = 0$ and

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$T = b^{\frac{1}{2}}/a^{\frac{1}{2}}$, and $T = b^{\frac{1}{2}}/a^{\frac{1}{2}}$ and $T = \infty$, then the transformation is one to one in each part. Now T may be easily calculated in terms of Q, and we find that

$$\begin{aligned} T^2 &= [2ab+Q \pm (Q(Q+4ab))^{\frac{1}{2}}]/2a^2 \\ &= [4ab+Q+Q \pm 2(Q(Q+4ab))^{\frac{1}{2}}]/4a^2 \qquad (4) \\ &= [(Q+4ab)^{\frac{1}{2}} \pm Q^{\frac{1}{2}}]^2/(2a)^2. \end{aligned}$$

Since by assumption T is positive, we have from (4) that

$$(5) \qquad T = [(Q+4ab)^{\frac{1}{2}} + Q^{\frac{1}{2}}]/2a, \qquad \text{if} \quad T \geqq b^{\frac{1}{2}}/a^{\frac{1}{2}},$$

or that

$$(6) \qquad T = [(Q+4ab)^{\frac{1}{2}} - Q^{\frac{1}{2}}]/2a, \qquad \text{if} \quad T \leqq b^{\frac{1}{2}}/a^{\frac{1}{2}}.$$

It thus follows that

$$\begin{aligned} &\int_0^\infty f(a^2T^2+b^2/T^2)\,dT \\ &\quad = \int_0^{b^{1/2}/a^{1/2}} f(a^2T^2+b^2/T^2)\,dT + \int_{b^{1/2}/a^{1/2}}^\infty f(a^2T^2+b^2/T^2)\,dT \qquad (7) \\ &\quad = \tfrac{1}{4}a^{-1}\int_0^\infty f(Q+2ab)Q^{-\frac{1}{2}}[1-(Q/(Q+4ab))^{\frac{1}{2}}]\,dQ \\ &\qquad + \tfrac{1}{4}a^{-1}\int_0^\infty f(Q+2ab)Q^{-\frac{1}{2}}[1+(Q/(Q+4ab))^{\frac{1}{2}}]\,dQ. \end{aligned}$$

By the substitution $Q = t^2$, $t > 0$ we may write (7) as

$$\begin{aligned} &\int_0^{b^{1/2}/a^{1/2}} f(a^2T^2+b^2/T^2)\,dT + \int_{b^{1/2}/a^{1/2}}^\infty f(a^2T^2+b^2/T^2)\,dT \\ &\quad = \tfrac{1}{2}a^{-1}\int_0^\infty f(t^2+2ab)[1-t/(t^2+4ab)^{\frac{1}{2}}]\,dt \qquad (8) \\ &\qquad + \tfrac{1}{2}a^{-1}\int_0^\infty f(t^2+2ab)[1+t/(t^2+4ab)^{\frac{1}{2}}]\,dt. \end{aligned}$$

Now in (1) we take $\lambda = 1$. $m = 1$, (there is no loss of generality), and set $x = T^{-2}$, and we find that the density of variate T is

$$(9) \qquad f(T; 1, 1) = (2/\pi)^{\frac{1}{2}} \exp\{-\tfrac{1}{2}(T^2-1)^2/T^2\}, \qquad 0 < T < \infty.$$

Further, by setting

$$(10) \qquad t^2 = (T^2-1)^2/T^2, \qquad t > 0,$$

and using (3) and (8), we find the density of t to be

$$\begin{aligned} (11) \qquad f(t) &= \tfrac{1}{2}(2/\pi)^{\frac{1}{2}} \exp\{-\tfrac{1}{2}t^2\}[1-t/(t^2+4)^{\frac{1}{2}}], \qquad 0 < t < \infty, \quad T \leqq 1 \\ &= \tfrac{1}{2}(2/\pi)^{\frac{1}{2}} \exp\{-\tfrac{1}{2}t^2\}[1+t/(t^2+4)^{\frac{1}{2}}], \qquad 0 < t < \infty, \quad T \geqq 1. \end{aligned}$$

However, if we are not given any information as to whether $T \leqq 1$ or $T \geqq 1$, then from (11) the density of $t = |T - T^{-1}|$ may be taken as

$$(12) \qquad f(t) = (2/\pi)^{\frac{1}{2}} \exp\{-\tfrac{1}{2}t^2\}, \qquad 0 < t < \infty.$$

In case t also assumes negative values then $t = T - T^{-1}$ has a unit normal distribution. Thus from (10) and (12) it follows that if $T_1, T_2, \cdots, T_N$ are independent identically distributed random variables with the same distribution function (9), then

$$\textstyle\sum t^2 = \sum T^2 - 2N + \sum T^{-2} = (\sum T^2 - N^2/\sum T^{-2}) \\ + (\sum T^{-2} - 2N + N^2/\sum T^{-2}) = z + y$$

has a χ^2 distribution with N df. However, Tweedie has proved that z has a χ^2 distribution with $N-1$ df. Now from the distribution theory of quadratic forms in normal independent variates it follows that y has a χ^2 distribution with one df and that y and z are independently distributed. This result shows that the transformation (10) implicitly transforms z to a quadratic form, of rank $N-1$, in t variates and y to (the square of) a linear form in t variates. Since the explicit forms of these quadratic forms in t variates are not necessarily required for the characterization, we have not obtained explicit expressions for these forms. The converse is now obvious. We are given that $T_1 - T_1{}^{-1} = t_1$, $T_2 - T_2{}^{-1} = t_2, \cdots, T_N - T_N{}^{-1} = t_N$, t variates being defined by (10), are independently and identically distributed random variables and that $z = \sum T^2 - N^2/\sum T^{-2}$ = a quadratic form of rank $N-1$ in t variates, and $y = \sum T^{-2} - 2N + N^2/\sum T^{-2}$ = a (square of) a linear form in t variates, are independently distributed.

We are asked to prove that t variates are normally distributed, i.e., T variates have inverse Gaussian distributions. However, from the given data and the distribution theory of quadratic forms in independent normal variates we know that t variates are normally distributed. Hence the converse is proved.

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