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ASYMPTOTIC DISTRIBUTIONS OF SOME MULTIVARIATE TESTS[1]

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1. Introduction and summary. In this paper it is shown how the systems of partial differential equations developed by the author [15] for the hypergeometric functions of matrix argument may be used to obtain asymptotic expansions for some distributions occurring in multivariate analysis. In particular expansions are derived for the distributions of Hotelling's generalized $T_0{}^2$-statistic, Pillai's $V^{(s)}$ criterion, and the largest latent root of the sample covariance matrix.

2. Notation and preliminary results. It will be seen later that the functions occurring in the distribution problems mentioned above are matrix generalizations of the confluent hypergeometric functions (see Erdélyi et al., Chapter 6 [6]). As in the case $m = 1$, there are two types of confluent function. The first is the ${}_1F_1$ function which can be defined by a power series expansion (Constantine [2]), or by the integral representation (Herz [9])

$$(2.1)\quad {}_1F_1(a; c; R) = [\Gamma_m(c)/\Gamma_m(a)\Gamma_m(c-a)] \int_{S=0}^{I} \operatorname{etr}(RS)(\det S)^{a-p} \det(I-S)^{c-a-p}\, dS,$$

and the other confluent function we define by the integral

$$(2.2)\quad \Psi(a, c; R) = [1/\Gamma_m(a)] \int_{S>0} \operatorname{etr}(-RS)(\det S)^{a-p} \det(I+S)^{c-a-p}\, dS,$$

where R is an $m \times m$ symmetric matrix, $\operatorname{etr}(X) = \exp(\operatorname{tr} X)$, and, throughout this paper, $p = (m+1)/2$. (2.2) is valid for $Re(R) > 0$. In the case $m = 1$ these functions are both solutions of the confluent differential equation (see [6]). A system of partial differential equations satisfied by ${}_1F_1(a; c; R)$ has been given by Muirhead [15] (Equation (5.1)). We shall see shortly that $\Psi(a, c; R)$ also satisfies this system. Firstly we obtain Ψ as a limiting function from the matrix generalization of the Gaussian hypergeometric function, ${}_2F_1(a, b; c; R)$.

LEMMA.

$$(2.3)\quad \lim_{c\to\infty} {}_2F_1(a, b; c; I - cR^{-1}) = (\det R)^b \Psi(b, b-a+p; R)$$

This can easily be proved using the integral representation for ${}_2F_1$ given by Herz [9]. Now, from the system of partial differential equations satisfied by

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${}_2F_1(a, b; c; R)$ (see Muirhead [15]) it is readily verified that $(\det R)^{-b}{}_2F_1(a, b; c; I-cR^{-1})$ satisfies the system

$$\begin{aligned}(2.4)\qquad & \frac{R_i^2}{c}\left(\frac{c}{R_i}-1\right)\frac{\partial^2 y}{\partial R_i^2}+\Big\{b-a+1+\tfrac{1}{2}(m-1)-R_i+\frac{R_i}{c}(a-b-1)\\ & -\frac{1}{2c}\sum_{j=1, j\neq i}^{m}\frac{R_iR_j(1-c/R_i)}{R_i-R_j}\Big\}\frac{\partial y}{\partial R_i}+\frac{1}{2c}\sum_{j=1, j\neq i}^{m}\frac{R_j^2(1-c/R_j)}{R_i-R_j}\frac{\partial y}{\partial R_j}\\ & =\left(b-\frac{ab}{c}\right)y \qquad (i = 1, 2, \cdots, m).\end{aligned}$$

Letting $c \to \infty$ the system (2.4) tends to the system

$$(2.5)\quad R_i\frac{\partial^2 y}{\partial R_i^2}+\Big\{b-a+1-R_i+\tfrac{1}{2}\sum_{j=1, j\neq i}^{m}\frac{R_i}{R_i-R_j}\Big\}\frac{\partial y}{\partial R_i}-\tfrac{1}{2}\sum_{j=1, j\neq i}^{m}\frac{R_j}{R_i-R_j}\frac{\partial y}{\partial R_j}=by \qquad (i = 1, 2, \cdots, m)$$

which, by (2.3), must be satisfied by $\Psi(b, b-a+p; R)$. But (2.5) is exactly the system satisfied by ${}_1F_1(b; b-a+p; R)$, (see [15]). Hence we conclude that

THEOREM. *${}_1F_1(a; c; R)$ and $\Psi(a, c; R)$ both satisfy the same system of partial differential equations.*

3. Hotelling's generalized T_0^2. Hotelling's generalized T_0^2-statistic is used as a test of significance in the multivariate analysis of variance and is defined as

$$(3.1)\qquad T = T_0^2/n_2 = \operatorname{tr} S_1 S_2^{-1},$$

where the $m \times m$ matrices S_1 and S_2 are independently distributed on n_1 and n_2 degrees of freedom respectively, estimating the same covariance matrix, with S_2 having the Wishart distribution and S_1 having the (possibly) non-central Wishart distribution. Constantine [3] has obtained the exact distribution of T_0^2 over the range $|T| < 1$ as a power series involving generalized Laguerre polynomials. The distribution of T_0^2 tends to that of χ^2 on mn_1 degrees of freedom for large n_2, and Ito [10] has derived asymptotic expansions both for the percentage points and for the cumulative distribution function (cdf) of T_0^2. This latter expansion has also been obtained independently by Davis [4].

We shall consider the case when S_1 and S_2 have central Wishart distributions and estimate different covariance matrices Σ_1 and Σ_2 respectively. In order to test the null hypothesis $H_0: \Sigma_1 = \Sigma_2$, Pillai [16], Khatri [12] studied

$$(3.2)\qquad T' = n_2 \operatorname{tr} S_1 S_2^{-1}.$$

When H_0 is true, $T' = T_0^2$. Starting with the joint distribution of the latent roots of $S_1 S_2^{-1}$ (see James [11]) we may readily obtain the moment generating function (mgf) of T', as

$$(3.3)\qquad g(t, \Omega) = [\Gamma_m(\tfrac{1}{2}(n_1+n_2))/\Gamma_m(\tfrac{1}{2}n_2)]\Psi(\tfrac{1}{2}n_1, p-\tfrac{1}{2}n_2; n_2 t\Omega)$$

where $\Omega = \operatorname{diag}(\omega_1, \omega_2, \cdots, \omega_m)$ and $\omega_1, \cdots, \omega_m$ are the latent roots of $\Sigma_1 \Sigma_2^{-1}$. Since $g(t, \Omega)$ is an mgf the boundary condition $g(0, \Omega) = 1$ must be satisfied. Let us put

$$g_1(R) = [\Gamma_m(\tfrac{1}{2}(n_1+n_2))/\Gamma_m(\tfrac{1}{2}n_2)]\Psi(\tfrac{1}{2}n_1, p-\tfrac{1}{2}n_2; \tfrac{1}{2}n_2 R) \tag{3.4}$$

where $g_1(0) = 1$ and $g_1(2t\Omega) = g(t, \Omega)$. $g_1(2tI)$ is the mgf of $T_0{}^2$. We know from Section 2 the differential equations satisfied by $g_1(R)$ and these will be used later to derive an asymptotic expansion for $g_1(R)$. Using (2.2) it is easily seen that $g_1(R) \to \det(I+R)^{-\frac{1}{2}n_1}$ as $n_2 \to \infty$, and this limit will be helpful later.

4. Pillai's $V^{(m)}$.

Pillai's $V^{(m)}$ criterion is also used as a test of significance in multivariate analysis of variance and is defined as

$$V^{(m)} = \operatorname{tr}[S_1(S_1+S_2)^{-1}] \tag{4.1}$$

where the $m \times m$ matrices S_1 and S_2 are independently distributed on n_1 and n_2 degrees of freedom respectively, estimating the same covariance matrix, with S_2 having the Wishart distribution and S_1 having the (possibly) non-central Wishart distribution. Distribution problems associated with $V^{(m)}$ have been studied by Pillai (e.g. see [16], [17], [18]) and Khatri and Pillai [13]. The properties of the test have also been discussed by Giri [7] and Kiefer and Schwartz [14].

We shall consider the case when S_1 has the central Wishart distribution. Then James [11] (Equation 146) has shown that the mgf of

$$V = n_2 V^{(m)} \tag{4.2}$$

is given by

$$g(t) = {}_1F_1(\tfrac{1}{2}n_1; \tfrac{1}{2}(n_1+n_2); -n_2 tI). \tag{4.3}$$

Clearly $g(t) \to (1+2t)^{-\frac{1}{2}mn_1}$ as $n_2 \to \infty$ showing that the distribution of V tends to that of χ^2 on mn_1 degrees of freedom. Let us put

$$g_2(R) = {}_1F_1(\tfrac{1}{2}n_1; \tfrac{1}{2}(n_1+n_2); \tfrac{1}{2}n_2 R), \tag{4.4}$$

then $g_2(0) = 1$, $g_2(-2tI) = g(t)$, and $g_2(R) \to \det(I-R)^{-\frac{1}{2}n_1}$ as $n_2 \to \infty$.

An asymptotic expansion for $g_2(R)$ will be derived later.

5. Largest latent root of the covariance matrix. Suppose X is an $m \times n$ matrix variate whose columns are independently distributed as $N(0, \Sigma)$. Then, putting $S = n^{-1}XX'$ (the sample covariance matrix) we have, from Constantine [2]

$$\begin{aligned} &\Pr(S < L) \\ &\quad = [\Gamma_m(p)(\tfrac{1}{2}n)^{\frac{1}{2}nm}/\Gamma_m(\tfrac{1}{2}n+p)](\det \Sigma^{-1}L)^{\frac{1}{2}n} {}_1F_1(\tfrac{1}{2}n; \tfrac{1}{2}n+p; -\tfrac{1}{2}n\Sigma^{-1}L). \end{aligned} \tag{5.1}$$

When $L = \ell I$ this just reduces to the cdf of the largest latent root $\ell_{\max}$ of the covariance matrix, i.e. we have

$$\begin{aligned}(5.2)\quad \Pr(\ell_{\max} < \ell) &= [\Gamma_m(p)(\tfrac{1}{2}n\ell)^{\frac{1}{2}nm}/\Gamma_m(\tfrac{1}{2}n+p)](\det\Sigma)^{-\frac{1}{2}n}\,{}_1F_1(\tfrac{1}{2}n;\tfrac{1}{2}n+p;-\tfrac{1}{2}n\ell\Sigma^{-1})\\ &= [\Gamma_m(p)(\tfrac{1}{2}n\ell)^{\frac{1}{2}nm}/\Gamma_m(\tfrac{1}{2}n+p)](\det\Sigma)^{-\frac{1}{2}n}\operatorname{etr}(-\tfrac{1}{2}n\ell\Sigma^{-1})\\ &\qquad\cdot {}_1F_1(p;\tfrac{1}{2}n+p;\tfrac{1}{2}n\ell\Sigma^{-1})\end{aligned}$$

using the Kummer transformation (Herz [9]) $\operatorname{etr}(-R)_1F_1(a;b;R) = {}_1F_1(b-a;b;-R)$. Hanumara and Thompson [8] have tabulated approximate percentage points of the largest latent root, and the density function of $\ell_{\max}$ has been obtained by Sugiyama [19]. Let us put $g_3(R) = {}_1F_1(p;\tfrac{1}{2}n+p;\tfrac{1}{2}nR)$. Then $g_3(0) = 1$, and $g_3(R) \to \det(I-R)^{-p}$ as $n \to \infty$. An asymptotic expansion for $g_3(R)$ will be given later.

6. A general asymptotic expansion. From Section 2 it is clear that $g_1(R)$, $g_2(R)$, and $g_3(R)$ all satisfy similar systems of partial differential equations and have similar limits as n_2 or $n \to \infty$. We consider now a function F which satisfies a system of partial differential equations with arbitrary parameters and we derive a general asymptotic expansion for F which can then be particularized to give asymptotic expansions for $g_1(R)$, $g_2(R)$, and $g_3(R)$. The system satisfied by F is

$$\begin{aligned}(6.1)\qquad & R_i\frac{\partial^2 F}{\partial R_i^2}+\Big\{\beta-\tfrac{1}{2}\varepsilon n-\tfrac{1}{2}(m-1)-\tfrac{1}{2}nR_i+\tfrac{1}{2}\sum_{j=1,j\neq i}^{m}\frac{R_i}{R_i-R_j}\Big\}\frac{\partial F}{\partial R_i}\\ &-\tfrac{1}{2}\sum_{j=1,j\neq i}^{m}\frac{R_j}{R_i-R_j}\frac{\partial F}{\partial R_j}=\tfrac{1}{2}\alpha nF \qquad (i=1,2,\cdots,m)\end{aligned}$$

where ε can be 1 or -1. We need only work with the first equation in the system (6.1) (i.e. $i = 1$) remembering that F is symmetric in $R_1, R_2, \cdots, R_m$. Clearly both the functions ${}_1F_1(\alpha;\beta-\tfrac{1}{2}\varepsilon n;\tfrac{1}{2}nR)$ and $\Psi(\alpha,\beta-\tfrac{1}{2}\varepsilon n;\tfrac{1}{2}nR)$ satisfy (6.1). Since we require F to be 1 at $R = 0$ we make the convenient change of variables $W = I-(I+\varepsilon R)^{-1}$ so that F is 1 at $W = 0$. Then from (6.1) with $i = 1$, we have that F satisfies the differential equation

$$\begin{aligned}(6.2)\quad & w_1(1-w_1)^2\frac{\partial^2 F}{\partial w_1^2}+\Big\{\beta-\tfrac{1}{2}(m-1)-2w_1-\tfrac{1}{2}n\varepsilon(1-w_1)^{-1}+\tfrac{1}{2}\sum_{j=2}^{m}\frac{w_1(1-w_j)}{w_1-w_j}\Big\}\\ &\cdot(1-w_1)\frac{\partial F}{\partial w_1}-\tfrac{1}{2}\sum_{j=2}^{m}\frac{w_j(1-w_j)^2}{w_1-w_j}\frac{\partial F}{\partial w_j}=\tfrac{1}{2}\varepsilon\alpha n(1-w_1)^{-1}F\end{aligned}$$

where $w_1, w_2, \cdots, w_m$ are the latent roots of W. The limits of $g_1(R)$, $g_2(R)$, $g_3(R)$ all suggest that our general F should have the limit $\det(I-W)^{\alpha}$ as $n \to \infty$, so putting

$$(6.3)\qquad F = \det(I-W)^{\alpha}\cdot G(W)$$

it is easily verified that G satisfies the differential equation

$$
\begin{aligned}
& w_1(1-w_1)^2 \frac{\partial^2 G}{\partial w_1{}^2} + \Big\{\beta - \tfrac{1}{2}(m-1) - \tfrac{1}{2}n\varepsilon - w_1(\beta - \tfrac{1}{2}(m-5)+2\alpha) + w_1{}^2(2\alpha+2) \\
(6.4) \qquad & \qquad + \tfrac{1}{2}\sum_{j=2}^{m} \frac{w_1(1-w_1)(1-w_j)}{w_1-w_j}\Big\} \frac{\partial G}{\partial w_1} - \tfrac{1}{2}\sum_{j=2}^{m} \frac{w_j(1-w_j)^2}{w_1-w_j}\frac{\partial G}{\partial w_j} \\
& = \Big[\alpha\beta - w_1(\alpha^2+\alpha) - \tfrac{1}{2}\alpha\sum_{j=2}^{m} w_j\Big] G.
\end{aligned}
$$

We now look for a solution of (6.4) of the form

$$(6.5) \qquad G(W) = 1 + \textstyle\sum_{k=1}^{\infty} P_k(W) n^{-k}$$

where, for all k, $P_k(0) = 0$ so that $G(0) = 1$. We could now substitute the series (6.5) into (6.4) and equate coefficients of like powers of n^{-1} on both sides to yield the $P_k(W)$. However a significant reduction in work is obtained if instead of G, we consider the function

$$(6.6) \qquad H(W) = \ln G(W) = \textstyle\sum_{k=1}^{\infty} Q_k(W) n^{-k}$$

where $Q_k(0) = 0$ for all k. Then, using (6.4) we see that H satisfies the partial differential equation

$$
\begin{aligned}
& w_1(1-w_1)^2 \Big\{\frac{\partial^2 H}{\partial w_1{}^2} + \Big(\frac{\partial H}{\partial w_1}\Big)^2\Big\} \\
(6.7) \qquad & \qquad + \Big\{\beta - \tfrac{1}{2}(m-1) - \tfrac{1}{2}n\varepsilon - w_1(\beta - \tfrac{1}{2}(m-5)+2\alpha) + w_1{}^2(2\alpha+2) \\
& \qquad + \tfrac{1}{2}\sum_{j=2}^{m} \frac{w_1(1-w_1)(1-w_j)}{w_1-w_j}\Big\} \frac{\partial H}{\partial w_1} - \tfrac{1}{2}\sum_{j=2}^{m} \frac{w_j(1-w_j)^2}{w_1-w_j}\frac{\partial H}{\partial w_j} \\
& = \alpha\beta - w_1(\alpha^2+\alpha) - \tfrac{1}{2}\alpha\sum_{j=2}^{m} w_j .
\end{aligned}
$$

Now substitute the series (6.6) into (6.7). Equating constant terms gives $\partial Q_1/\partial w_1 = \varepsilon\alpha\{\sum_{j=2}^{m} w_j + w_1(2\alpha+2) - 2\beta\}$ so that

$$(6.8) \qquad Q_1 = \tfrac{1}{2}\varepsilon\alpha\{\sigma_1{}^2 + (2\alpha+1)\sigma_2 - 4\beta\sigma_1\}$$

where $\sigma_r = w_1{}^r + w_2{}^r + \cdots + w_m{}^r$ since the Q_k are symmetric.

Equating coefficients of n^{-1} in (6.7) gives

$$
\begin{aligned}
\frac{\partial Q_2}{\partial w_1} = 2\varepsilon\Big\{ & w_1(1-w_1)^2 \frac{\partial^2 Q_1}{\partial w_1{}^2} + \Big[\beta - \tfrac{1}{2}(m-1) - w_1(\beta - \tfrac{1}{2}(m-5)+2\alpha) \\
& + w_1{}^2(2\alpha+2) + \tfrac{1}{2}\sum_{j=2}^{m} \frac{w_1(1-w_1)(1-w_j)}{(w_1-w_j)}\Big]\frac{\partial Q_1}{\partial w_1} \\
& - \tfrac{1}{2}\sum_{j=2}^{m} \frac{w_j(1-w_j)^2}{w_1-w_j}\frac{\partial Q_1}{\partial w_j}\Big\}
\end{aligned}
$$

which gives, after substituting $\partial Q_1/\partial w_1$ and $\partial^2 Q_1/\partial w_1^2$ obtained from (6.8) and integrating,

$$
\begin{aligned}
(6.9)\quad Q_2 = \tfrac{1}{12}\alpha\{&6\sigma_1^2\sigma_2+3(2\alpha+1)\sigma_2^2+12(2\alpha+1)\sigma_1\sigma_3+3(8\alpha^2+10\alpha+5)\sigma_4\\
&-8\sigma_1^3-24(2\alpha+2\beta+1)\sigma_1\sigma_2-8(4\alpha^2+6\alpha\beta+6\alpha+3\beta+4)\sigma_3\\
&+6(2\alpha+6\beta+1)\sigma_1^2+6(12\alpha\beta+4\beta^2+2\alpha+6\beta+3)\sigma_2-48\beta^2\sigma_1\}.
\end{aligned}
$$

Similarly equating coefficients of n^{-2} in (6.7) and integrating gives

$$
\begin{aligned}
(6.10)\quad Q_3 = \tfrac{1}{12}\varepsilon\alpha\{&6\sigma_1^2\sigma_2^2+4\sigma_1^3\sigma_3+2(2\alpha+1)\sigma_2^3+24(2\alpha+1)\sigma_1\sigma_2\sigma_3\\
&+18(2\alpha+1)\sigma_1^2\sigma_4+2(16\alpha^2+18\alpha+7)\sigma_3^2+6(8\alpha^2+10\alpha+5)\sigma_2\sigma_4\\
&+12(8\alpha^2+10\alpha+5)\sigma_1\sigma_5+2(40\alpha^3+88\alpha^2+104\alpha+41)\sigma_6\\
&-24\sigma_1^3\sigma_2-24(4\alpha+\beta+2)\sigma_1\sigma_2^2-24(6\alpha+\beta+3)\sigma_1^2\sigma_3\\
&-24(8\alpha^2+4\alpha\beta+10\alpha+2\beta+5)\sigma_2\sigma_3\\
&-24(12\alpha^2+6\alpha\beta+16\alpha+3\beta+9)\sigma_1\sigma_4\\
&-24(8\alpha^3+8\alpha^2\beta+20\alpha^2+10\alpha\beta+28\alpha+5\beta+12)\sigma_5+15\sigma_1^4\\
&+6(34\alpha+20\beta+17)\sigma_1^2\sigma_2+3(40\alpha^2+56\alpha\beta+8\beta^2+54\alpha+28\beta+31)\sigma_2^2\\
&+48(6\alpha^2+10\alpha\beta+\beta^2+9\alpha+5\beta+6)\sigma_1\sigma_3\\
&+6(20\alpha^3+80\alpha^2\beta+24\alpha\beta^2+68\alpha^2+108\alpha\beta+12\beta^2+127\alpha+62\beta+61)\sigma_4\\
&-8(6\alpha+10\beta+3)\sigma_1^3-24(4\alpha^2+22\alpha\beta+8\beta^2+8\alpha+11\beta+7)\sigma_1\sigma_2\\
&-8(40\alpha^2\beta+48\alpha\beta^2+4\beta^3+12\alpha^2+66\alpha\beta+24\beta^2+42\alpha+49\beta+24)\sigma_3\\
&+6(16\alpha\beta+24\beta^2+2\alpha+8\beta+3)\sigma_1^2\\
&+6(48\alpha\beta^2+16\beta^3+16\alpha\beta+24\beta^2+6\alpha+24\beta+5)\sigma_2-96\beta^3\sigma_1\}.
\end{aligned}
$$

Coefficients of higher powers of n^{-1} may be obtained in a similar manner.
Thus we have

$$
(6.11)\qquad \ln F = \alpha\ln[\det(I-W)]+n^{-1}Q_1+n^{-2}Q_2+n^{-3}Q_3+O(n^{-4})
$$

where Q_1, Q_2, Q_3 are given by (6.8), (6.9) and (6.10). Specifying the parameters α, β, ε will yield the expansions for $\ln g_1(R)$, $\ln g_2(R)$, and $\ln g_3(R)$ to be given in the next section.

7. The final asymptotic results. From the previous sections it is clear that putting $\alpha = \frac{1}{2}n_1$, $\beta = \frac{1}{2}n_1$, $\varepsilon = -1$, $n = n_2$, and $R = 2tI$ in (6.11) yields an asymptotic expansion for the cumulant generating function $\phi(t)$ of $V = n_2 V^{(m)}$, which may be expressed in the form

$$
(7.1)\qquad \phi(t) = \tfrac{1}{2}mn_1\ln(1+2t)+n_2^{-1}Q_1+n_2^{-2}Q_2+n_2^{-3}Q_3+O(n_2^{-4})
$$

where

$$Q_1 = \frac{A_1}{1+2t} + \frac{A_2}{(1+2t)^2} - (A_1+A_2),$$

$$Q_2 = \frac{B_2}{(1+2t)^2} + \frac{B_3}{(1+2t)^3} + \frac{B_4}{(1+2t)^4} - (B_2+B_3+B_4),$$

$$Q_3 = \frac{C_3}{(1+2t)^3} + \frac{C_4}{(1+2t)^4} + \frac{C_5}{(1+2t)^5} + \frac{C_6}{(1+2t)^6} - (C_3+C_4+C_5+C_6),$$

with

$$A_1 = \tfrac{1}{2}mn_1(m+1), \qquad A_2 = -\tfrac{1}{4}mn_1(m+n_1+1),$$

$$B_2 = \tfrac{1}{4}mn_1[2m^2+m(n_1+4)+(n_1+2)],$$

$$B_3 = -\tfrac{1}{6}mn_1[4m^2+3m(2n_1+3)+(n_1{}^2+6n_1+7)],$$

$$B_4 = \tfrac{1}{8}mn_1[2m^2+5m(n_1+1)+(2n_1{}^2+5n_1+5)],$$

$$C_3 = \tfrac{1}{6}mn_1[5m^3+2m^2(3n_1+8)+m(n_1{}^2+12n_1+19)+(n_1{}^2+6n_1+8)],$$

$$C_4 = -\tfrac{1}{8}mn_1[15m^3+2m^2(17n_1+27)+m(14n_1{}^2+73n_1+87) + (n_1{}^3+14n_1{}^2+49n_1+52)],$$

$$C_5 = \tfrac{1}{2}mn_1[3m^3+2m^2(5n_1+6)+m(7n_1{}^2+23n_1+24) + (n_1{}^3+7n_1{}^2+19n_1+17)],$$

$$C_6 = -\tfrac{1}{12}mn_1[5m^3+22m^2(n_1+1)+2m(11n_1{}^2+27n_1+26) + (5n_1{}^3+22n_1{}^2+52n_1+41)]\big|.$$

(7.1) is similar to the generalized form for the asymptotic expansion of the cumulant generating function obtained by Box [1]. Using (7.1) Davis [5] has derived the percentile expansion of V to order $n_2{}^{-3}$. Exponentiation of (7.1) yields the expansion of the mgf of V which may then be inverted, using the fact that $(1+2t)^{-\frac{1}{2}r}$ is the mgf of χ^2 on r degrees of freedom, to yield the expansion of the cdf of V. To order $n_2{}^{-2}$ this is

$$\text{(7.2)} \qquad \Pr(V > x) = \Pr(\chi^2_{mn_1} > x) + \frac{mn_1}{4n_2}\sum_{j=0}^{2} a_j \Pr(\chi^2_{mn_1+2j} > x) + \frac{mn_1}{96n_2{}^2}\sum_{j=0}^{4} b_j \Pr(\chi^2_{mn_1+2j} > x) + O(n_2{}^{-3})$$

where

$$a_0 = n_1-m-1, \qquad a_1 = 2(m+1), \qquad a_2 = -(m+n_1+1),$$

$$b_0 = 3m^3n_1 - 2m^2(3n_1{}^2-3n_1+4) + 3m(n_1{}^3-2n_1{}^2+5n_1-4) - 4(2n_1{}^2-3n_1-1),$$

$$b_1 = -12mn_1[m^2-m(n_1-2)-(n_1-1)],$$

$$b_2 = 6[3m^3n_1+2m^2(3n_1+4)-m(n_1{}^3-7n_1-16)+4(n_1+2)],$$

$$b_3 = -4[3m^3n_1+m^2(3n_1{}^2+6n_1+16)+3m(n_1{}^2+9n_1+12)+4(n_1{}^2+6n_1+7)],$$

$$b_4 = 3[m^3n_1+2m^2(n_1{}^2+n_1+4)+m(n_1{}^3+2n_1{}^2+21n_1+20)+4(2n_1{}^2+5n_1+5)].$$

The coefficient of n_2^{-3} in (7.2) can also be obtained from (7.1) but because of its complexity it is not given here. We can check the expansion (7.2) numerically when $m = 2$ and $n_1 = 3$. In this case the exact probability is easily shown to be (in the range $0 < y < 1$)

$$\Pr(V^{(2)} < y) = (n_2+1)[1-(1-\tfrac{1}{2}y)^{n_2}] - n_2[1-(1-y)^{\frac{1}{2}(n_2+1)}].$$

Taking $n_2 = 50$ this gives $\Pr(V > 12.592) = 0.0304$ while (7.2) gives $\Pr(V > 12.592) = 0.0302$.

Similarly we may obtain the expansion of the cumulant generating function of T_0^2, up to order n_2^{-3}, by putting $\alpha = \frac{1}{2}n_1$, $\beta = p$, $\varepsilon = 1$, $n = n_2$, and $R = 2tI$ in (6.11). Davis [5] has also obtained the resulting expansion by another method, and has used it to extend Ito's percentile expansion to order n_2^{-3}.

Finally, putting $\alpha = p$, $\beta = p$, $\varepsilon = -1$, and $R = \ell\Sigma^{-1}$ in (6.11) and exponentiating we obtain the expansion of $g_3(\ell\Sigma^{-1})$ which occurs in the expression (5.2) for $\Pr(\ell_{\max} < \ell)$. The final result, to order n_2^{-2} is

$$(7.3)\quad \Pr(\ell_{\max} < \ell) = [\Gamma_m(p)(\tfrac{1}{2}n\ell)^{\frac{1}{2}nm}/\Gamma_m(\tfrac{1}{2}n+p)](\det\Sigma)^{-\frac{1}{2}n}\operatorname{etr}(-\tfrac{1}{2}n\ell\Sigma^{-1})\det(I-\ell\Sigma^{-1})^{-p}F$$

where

$$(7.4)\qquad F = 1+n^{-1}P_1+n^{-2}P_2+O(n^{-3}) \qquad \text{with}$$

$$P_1 = \tfrac{1}{2}p\{-\sigma_1^2-(2p+1)\sigma_2+4p\sigma_1\} \qquad \text{and}$$

$$\begin{aligned} P_2 = \tfrac{1}{24}p\{&3p\sigma_1^4+6[2p^2+p+2]\sigma_1^2\sigma_2+3[4p^3+4p^2+5p+2]\sigma_2^2 \\ &+24[2p+1]\sigma_1\sigma_3+6[8p^2+10p+5]\sigma_4-8[3p^2+2]\sigma_1^3 \\ &-24[2p^3+p^2+6p+2]\sigma_1\sigma_2-16[10p^2+9p+4]\sigma_3 \\ &+12[4p^3+8p+1]\sigma_1^2+12[16p^2+8p+3]\sigma_2-96p^2\sigma_1\} \end{aligned}$$

where the σ_r are the power sums of the latent roots of the matrix $I-(I-\ell\Sigma^{-1})^{-1}$. The expansion (7.3) is valid over the range $0 \leqq \ell < \min(s_i)$ where the s_i are the latent roots of Σ.

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