

First and second order analysis for periodic random arrays using block bootstrap methods*

Anna E. Dudek

*Institut de Recherche Mathématique de Rennes, Université Rennes 2, Rennes, France
and*

*AGH University of Science and Technology, al. Mickiewicza 30, 30-059 Krakow, Poland
e-mail: aedudek@agh.edu.pl*

Abstract: In the paper row-wise periodically correlated triangular arrays are considered. The period length is assumed to grow in time. The Fourier decomposition of the mean and autocovariance functions for each row of the matrix is presented. To construct bootstrap estimators of the Fourier coefficients two block bootstrap techniques are used. These are the circular version of the Generalized Seasonal Block Bootstrap and the Circular Block Bootstrap. Consistency results for both methods are presented. Bootstrap-t equal-tailed confidence intervals for parameters of interest are constructed. Results are illustrated by an example based on simulated data.

MSC 2010 subject classifications: 62G09, 62M10, 94A12, 62P30, 94A13.

Keywords and phrases: Block bootstrap, consistency, Fourier coefficients of mean and autocovariance functions, periodic triangular array.

Received January 2016.

1. Introduction

In the recent years the number of bootstrap applications for periodic processes has constantly grown. An important class of periodic processes are periodically correlated (PC) processes, which were introduced by Gladyshev (1961). They are widely used for modeling in many different settings such as climatology, hydrology, mechanics, vibroacoustics and economics. Many motivating examples can be found in Gardner et al. (2006); Hurd and Miamee (2007); Antoni (2009) and Napolitano (2012).

Time series X_t is called PC with period d if it has periodic mean and autocovariance functions, i.e.

$$E(X_{t+d}) = E(X_t) \quad \text{and} \quad B(t, \tau) = \text{Cov}(X_t, X_{t+\tau}) = \text{Cov}(X_{t+d}, X_{t+\tau+d}) \quad (1)$$

for each $t, \tau \in \mathcal{Z}$. The period d is taken as the smallest positive integer such that (1) holds.

*This project has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 655394.

There are two block bootstrap methods that can be applied for PC processes. These are the Moving Block Bootstrap (MBB) and the Generalized Seasonal Block Bootstrap (GSBB). The first method was introduced independently by Künsch (1989) and Liu and Singh (1992). It is very general and it does not preserve the periodic structure contained in the data. As a result the number of possible applications of this technique for PC time series is limited. On the other hand, the GSBB of Dudek et al. (2014a) was designed for periodic processes. It keeps the periodicity but to apply it one needs to know the period length. The GSBB is the generalization of two block bootstrap methods: the Seasonal Block Bootstrap (SBB) (Politis (2001)) and the Periodic Block Bootstrap (Chan et al. (2004)).

The first and the second order characteristics of PC time series that are often considered can be split into two groups depending on the domain in which the analysis is performed. In the time domain one can be interested, for example, in the overall mean and the seasonal means. In the frequency domain these are Fourier coefficients of the mean and the autocovariance functions. For all mentioned characteristics consistency of the MBB and the GSBB has been already shown (see Synowiecki (2007), Dudek et al. (2014a), Dudek et al. (2014b), Dudek (2015)).

PC processes can be applied when the period length is constant. In this paper we focus on a different case, i.e. when the period length is changing over time. A very important example illustrating this phenomena is a chirp signal. It is a signal in which frequency increases or decreases over time. It can be described by the following equation

$$\chi(u) = S(u) \exp(i\phi(u)),$$

where $S(u)$ is a positive, low-pass, smooth amplitude function whose evolution is slow when compared to the oscillations of the phase $\phi(u)$. Chirps are commonly met in nature. For example in audio signals (animal communication, echolocation), radar and sonar systems, astrophysics (gravitational waves radiated by coalescing binaries), mechanics and vibrations (e.g. car engines), medicine (EEG data – epileptic seizure) and seismography. For more details and examples we refer the reader to Flandrin (2001) and the references therein.

In this paper we consider a case, when period length is growing in time. This corresponds to the so-called down-chirp signal. An example of such signal is presented in Figure 1. To model such phenomena periodic random arrays can be used. Bootstrap for triangular arrays with the period growing in time was first considered by Leśkow and Synowiecki (2010) and later by Dudek et al. (2014a). In both papers only the problem of the estimation of the overall mean was discussed. Leśkow and Synowiecki showed the consistency of the Periodic Block Bootstrap and Dudek et al. of the Generalized Seasonal Block Bootstrap. In this paper we would like to extend the area of possible applications of bootstrap methods for the periodic random arrays to the Fourier coefficients of the mean and autocovariance functions.

The paper is organized as follows. Section 2 contains the formulation of the problem. In Section 3 the algorithms of the circular version of the GSBB and the

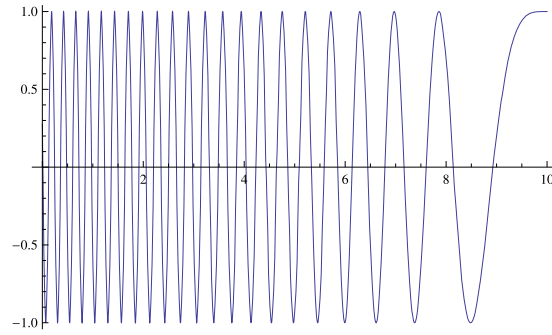


FIG 1. Example of down-chirp signal.

MBB are recalled. The bootstrap consistency results for the coefficients of the mean and the autocovariance functions for triangular arrays with growing period are presented in Section 4. In Section 5 the bootstrap-t confidence intervals are constructed and a simulated data example is presented. Finally, in Section 6 we discuss possible future extensions of our work. All proofs can be found in the [Appendix](#).

2. Problem formulation

Let $\{X_{n,t} : t = 1, \dots, m_n\}$ be an array of real valued random variables. We assume that it is row-wise periodically correlated (PC), i.e. in each row the mean function $\mu_n(t) = E(X_{n,t})$ and the autocovariance function $B_n(t, \tau) = \text{Cov}(X_{n,t}, X_{n,t+\tau})$ are periodic in variable t with period d_n .

Below we present the Fourier decomposition of the mean and the autocovariance functions in the n -th row of the triangular array. We have

$$\begin{aligned}\mu_n(t) &= \sum_{\gamma_n \in \Gamma_n} b_n(\gamma_n) \exp(i\gamma_n t), \\ B_n(t, \tau) &= \sum_{\lambda_n \in \Lambda_{n,\tau}} a_n(\lambda_n, \tau) \exp(i\lambda_n t),\end{aligned}$$

where the sets

$$\Gamma_n = \{\gamma_n : b_n(\gamma_n) \neq 0\}, \quad \Lambda_{n,\tau} = \{\lambda_n : a_n(\lambda_n, \tau) \neq 0\}$$

are finite and $\Gamma_n, \Lambda_{n,\tau} \subseteq \{2k\pi/d_n, k = 0, \dots, d_n - 1\}$.

Moreover, the coefficients $b_n(\gamma_n)$ and $a_n(\lambda_n, \tau)$ can be calculated using the following formulas

$$b_n(\gamma_n) = \frac{1}{d_n} \sum_{t=1}^{d_n} \mu_n(t) \exp(-i\gamma_n t), \quad a_n(\lambda_n, \tau) = \frac{1}{d_n} \sum_{t=1}^{d_n} B_n(t, \tau) \exp(-i\lambda_n t).$$

and their estimators are of the form

$$\hat{b}_n(\gamma_n) = \frac{1}{m_n} \sum_{t=1}^{m_n} X_{n,t} \exp(-i\gamma_n t), \quad (2)$$

$$\hat{a}_n(\lambda_n, \tau) = \frac{1}{m_n} \sum_{t=1-\min\{\tau, 0\}}^{m_n - \max\{\tau, 0\}} (X_{n,t+\tau} - \hat{\mu}_n(t+\tau)) (X_{n,t} - \hat{\mu}_n(t)) \exp(-i\lambda_n t), \quad (3)$$

where $\hat{\mu}_n(t) = \sum_{\gamma_n \in \Gamma_n} \hat{b}_n(\gamma_n) \exp(i\gamma_n t)$.

In the following we use the simplified version of (3). Without loss of generality we can assume that $E(X_{n,t}) \equiv 0$ and taking $\tau \geq 0$ we get

$$\hat{a}_n(\lambda_n, \tau) = \frac{1}{m_n} \sum_{t=1}^{m_n - \tau} X_{n,t} X_{n,t+\tau} \exp(-i\lambda_n t). \quad (4)$$

Detailed information concerning the coefficients of the mean and autocovariance functions and their estimators can be found in the papers of Hurd (1989), (1991) and Hurd and Leśkow (1992a), (1992b).

3. Bootstrap algorithms

In this paper we apply two block bootstrap methods. However, these will not be the MBB and the GSBB in their standard form, but their circular versions. In this approach the data are treated as wrapped on the circle. This allows us to reduce the edge effects caused by the fact that in the MBB and the GSBB pseudo-samples the observations from the beginning and the end of the original sample appear less often than the other observations.

To simplify notation in this section we assume that Y_t is a PC process with the known period d . Let (Y_1, \dots, Y_n) be the observed sample. Moreover, let the sample length n be an integer multiple of the period length d ($n = wd, w \in \mathcal{N}$).

As first we present the circular version of the GSBB (cGSBB).

cGSBB algorithm

1. Choose a (positive) integer block size $b(< n)$. Then, sample size n can be split into l blocks of the length b and a shorter block of the length r i.e., $n = lb + r, l \in \mathcal{N}$ and $r \in \{0, \dots, b-1\}$.
2. For $t = 1, b+1, 2b+1, \dots, lb+1$, let

$$(Y_t^*, Y_{t+1}^*, \dots, Y_{t+b-1}^*) = (Y_{k_t}, Y_{k_t+1}, \dots, Y_{k_t+b-1})$$

where k_t is iid from a discrete uniform distribution

$$P(k_t = t + vd) = \frac{1}{w} \quad \text{for } v = 0, 1, \dots, w-1.$$

Since we consider the circular version of GSBB, when $t + vd > n$ we take the shifted observations $t + vd - n$.

3. Join the l blocks $(Y_{k_t}, Y_{k_t+1}, \dots, Y_{k_t+b-1})$ and take the first n observations (Y_1^*, \dots, Y_n^*) to obtain a bootstrap sample of the same length as original one.

The second bootstrap method that we consider is the Circular Block Bootstrap of Politis and Romano (1992), which is a circular version of the Moving Block Bootstrap. Comparing with the cGSBB the only difference is in the second step of the algorithm. We do not keep the periodic structure of the data, so we do not need to restrict the set of blocks which are selected.

CBB algorithm

1. Choose a (positive) integer block size $b(< n)$. Then, sample size n can be split into l blocks of length b and a shorter block of length r i.e., $n = lb + r, l \in \mathcal{N}$ and $r \in \{0, \dots, b-1\}$.
2. For $t = 1, b+1, 2b+1, \dots, lb+1$, let

$$(Y_t^*, Y_{t+1}^*, \dots, Y_{t+b-1}^*) = (Y_{k_t}, Y_{k_t+1}, \dots, Y_{k_t+b-1})$$

where k_t is iid from a discrete uniform distribution

$$P(k_t = j) = \frac{1}{n} \quad \text{for } j = 0, 1, \dots, n-1.$$

If $j > n$ we take $j - n$ instead.

3. Join the l blocks $(Y_{k_t}, Y_{k_t+1}, \dots, Y_{k_t+b-1})$ and take the first n observations (Y_1^*, \dots, Y_n^*) to obtain a bootstrap sample of the same length as original one.

4. Main results

In the first part of this section we use the cGSBB method to construct consistent estimators of the coefficients of the autocovariance function defined by formula (4). Then, the corresponding result for the coefficients of the mean function is presented. The second part is dedicated to the CBB method.

Let $m_n = l_n b_n + r_n$, where $r_n \in \{0, \dots, b_n - 1\}$ i.e., the observations in the n -th row of the considered triangular array $\{X_{n,t} : t = 1, \dots, m_n\}$ can be split into l_n blocks of the length b_n and some remaining part of length r_n . For the sake of simplicity we use notation r, l, b instead of r_n, l_n, b_n , whenever it is possible. Moreover, without loss of generality we assume that $m_n = w_n d_n, w_n \in \mathcal{N}$, i.e. each row of the considered array contains w_n periods.

To obtain our results additionally we need to assume the following conditions:

- A1** let $\{X_{n,t} : t = 1, \dots, m_n\}$ be a WP(4) row-wise periodically correlated array of real valued random variables with period d_n ; WP(4) denotes the weakly periodic process of order 4. This means that $E|X_{n,t}|^4 < \infty$ and for any $t, \tau_1, \tau_2, \tau_3 \in \mathbb{Z}$

$$E(X_{n,t} X_{n,t+\tau_1} X_{n,t+\tau_2} X_{n,t+\tau_3})$$

$$= \mathbb{E}(X_{n,t+d_n} X_{n,t+\tau_1+d_n} X_{n,t+\tau_2+d_n} X_{n,t+\tau_3+d_n});$$

A2 $m_n = w_n d_n$, $d_n \rightarrow \infty$ and $w_n \rightarrow \infty$ as $n \rightarrow \infty$;

A3 let $\{\alpha_n(k), k = 1, 2, \dots\}$ be α -mixing coefficients of the n th row of the array $\{X_{n,t}\}$, i.e.

$$\alpha_n(k) = \sup_t \sup_{\substack{A \in \mathcal{F}_n(1,t) \\ B \in \mathcal{F}_n(t+k, m_n)}} |P(A \cap B) - P(A)P(B)|,$$

where $\mathcal{F}_n(t_1, t_2)$ is a σ -field generated by observations $\{X_{n,t_1}, X_{n,t_1+1}, \dots, X_{n,t_2}\}$;

A4(q) there exists $\delta > 0$ such that

$$\sup_{n,t} \mathbb{E} |X_{n,t}|^{q(4+\delta)} < \infty \quad \text{and} \quad \sup_n \sum_{k=1}^{m_n} k \alpha_n^{\delta/(4+\delta)}(k) = K < \infty;$$

A5

$$\sup_{n,k} \left\{ k^{1+\zeta} \alpha_n^{1/2}(k) \right\} < \infty \text{ for some } \zeta > 0.$$

Condition **A2** denotes that the period length d_n and the number of observed periods w_n are growing to infinity together with the sample size. Constant q in **A4** will take value 1 or 2 depending on the considered case. α -mixing assumption **A4** and **A5** are technical and also appear in Synowiecki (2007). These kind of conditions are also considered for usual PC time series (see e.g. Synowiecki (2007), Dudek et al. (2014a)). For more details on mixing we refer the reader to Doukhan (1994).

As first we define three different bootstrap versions of $\hat{a}_n(\lambda_n, \tau)$.

$$\hat{a}_n^*(\lambda_n, \tau) = \frac{1}{m_n} \sum_{t=1}^{m_n-\tau} X_{n,t}^* X_{n,t+\tau}^* \exp(-i\lambda_n t), \quad (5)$$

$$\bar{a}_n^*(\lambda_n, \tau) = \frac{1}{m_n} \sum_{t=1}^{m_n-\tau} X_{n,t}^* X_{n,t+\tau}^* \exp(-i\lambda_n t^*), \quad (6)$$

$$\begin{aligned} \tilde{a}_n^*(\lambda, \tau) &= \frac{1}{m_n} \sum_{k=0}^{l-1} \sum_{j=0}^{b-\tau-1} X_{n,1+kb+j}^* X_{n,1+kb+j+\tau}^* \exp(-i\lambda_n(1+kb+j)^*) \\ &\quad + \frac{1}{m_n} \sum_{j=0}^{\max\{0, r-\tau-1\}} X_{n,1+kb+j}^* X_{n,1+kb+j+\tau}^* \exp(-i\lambda_n(1+kb+j)^*), \end{aligned} \quad (7)$$

where $X_{n,t}^*$ for $t = 1, \dots, m_n$ is the cGSBB version of the n -th row of $X_{n,t}$. Note that in the formulas (6) and (7) the symbol * is present additionally in the argument of the exponential function. In this case the time argument no longer varies from 1 to $m_n - \tau$, but corresponds to the time indices of observations chosen in the block selection process. To be more precise let us assume that the chosen block $(X_{n,t}^*, \dots, X_{n,t+b-1}^*)$ is of the form $(X_{n,k_t}^*, \dots, X_{n,k_t+b-1}^*)$. Then

the values $(t^*, \dots, (t+b-1)^*)$ correspond to selected time indices, i.e. $(t^*, \dots, (t+b-1)^*) = (k_t, \dots, k_t + b - 1)$.

Estimator (5) is the most natural. It is constructed by replacing in formula (4) the original sample by its bootstrap counterpart. As we show later this estimator is consistent, because the GSBB preserves the periodic structure in the bootstrap sample. This form of estimator was also applied in Dudek et al. (2014b) for the autocovariance function of PC time series. The second estimator uses ideas presented in Dudek (2015). In this paper results from Dudek et al. (2014b) are extended to the wider class of processes called almost periodically correlated. Since in this case the period usually does not exist, the GSBB cannot be used and hence Dudek considered the CBB. One may notice that (5) and (6) are equal for $\lambda_n \in \Lambda_{n,\tau}$. Finally estimator (7) was obtained by removing some summands from (6). To be more precise (7) has been got from (6) after subtraction of those summands for which $X_{n,t}^*$ and $X_{n,t+\tau}^*$ belong to two different blocks selected in the second step of the cGSBB algorithm. Let B_{1+kb}^* for $k = 0, \dots, l-1$ be the block of length b of the form $(X_{n,1+kb}^*, \dots, X_{n,(k+1)b}^*)$. The last block of length r is denoted by $B_{1+lb}^* = (X_{n,lb+1}^*, \dots, X_{n,m_n}^*)$. Since $(X_{n,1}^*, \dots, X_{n,m_n}^*) = (B_1^*, \dots, B_{1+lb}^*)$, we rewrite formula (6) using the new notation.

$$\begin{aligned} \bar{a}_n^*(\lambda_n, \tau) &= \frac{1}{m_n} \sum_{t \notin C_{b,\tau}^*} X_{n,t}^* X_{n,t+\tau}^* \exp(-i\lambda_n t^*) \\ &+ \frac{1}{m_n} \sum_{t \in C_{b,\tau}^*} X_{n,t}^* X_{n,t+\tau}^* \exp(-i\lambda_n t^*) \\ &= \tilde{a}_n^*(\lambda, \tau) + \frac{1}{m_n} \sum_{t \in C_{b,\tau}^*} X_{n,t}^* X_{n,t+\tau}^* \exp(-i\lambda_n t^*) \exp(-i\lambda_n t^*), \end{aligned}$$

where the set $C_{b,\tau}^*$ contains those time indices for which $X_{n,t}^*$ and $X_{n,t+\tau}^*$ belong to different blocks

$$\begin{aligned} C_{b,\tau}^* &= \left\{ t : X_{n,t}^* \in B_{1+kb}^*, X_{n,t+\tau}^* \in B_{1+(k+1)b}^*, \right. \\ &\quad \left. k = 0, \dots, l-1 \text{ and } 1 \leq t, t+\tau \leq m_n \right\}. \end{aligned}$$

Estimator (7) is the most convenient for practical applications among the proposed formulas. As we discuss in Section 5 construction of the bootstrap confidence intervals for $a_n(\lambda_n, \tau)$ is computationally very expensive. In our study we will generate 500 000 bootstrap samples. Estimator (7) allows us to reduce this cost substantially. To obtain (5) and (6) one needs to calculate the estimator value according to the appropriate formula for each generated bootstrap sample. Formula (7) allows for a different approach. Note that each summand in (7) is based on a different block B_{1+kb}^* , $k = 0, \dots, l$. Thus, instead of using cGSBB to get $(X_{n,1}^*, \dots, X_{n,m_n}^*)$ and then calculating $\tilde{a}_n^*(\lambda, \tau)$, one can create new random variables

$$Y_j^b = \sum_{t=j}^{j+b-\tau-1} X_{n,t} X_{n,t+\tau} \exp(-i\lambda_n t) \quad \text{and} \quad Y_j^r = \sum_{t=j}^{j+r-\tau-1} X_{n,t} X_{n,t+\tau} \exp(-i\lambda_n t)$$

for $j = 1, \dots, m_n$. Y_j^b and Y_j^r represent these parts of estimator $\hat{a}_n(\lambda_n, \tau)$ that are based on blocks starting with observation $X_{n,i}$ of length b and r , respectively. As in the cGSBB algorithm we treat data as wrapped on the circle, which means that if any time index t is bigger than m_n we take $t - m_n$ instead. Now $\tilde{a}_n^*(\lambda, \tau)$ can be rewritten as follows

$$\tilde{a}_n^*(\lambda, \tau) = \frac{1}{m_n} \left(\sum_{j=0}^{l-1} Y_{k_{1+jb}}^b + Y_{k_{1+lb}}^r \right),$$

where $k_{1+kb}, k = 0, \dots, l$ are iid random variables from step 2 of the cGSBB algorithm (see Section 3). Thus, when we calculate values of Y_j^b and Y_j^r , we can very easily obtain many values of $\tilde{a}_n^*(\lambda, \tau)$. It is enough to generate the appropriate number of strings $k_{1+kb}, k = 0, \dots, l$ and sum the corresponding $Y_{k_{1+kb}}^b$ and $Y_{k_{1+lb}}^r$ values, which is computationally very cheap.

Below we present the cGSBB consistency result. For the sake of simplicity we formulate theorem using notation $a_n^*(\lambda_n, \tau)$, which represents any of the estimators (5)–(7). That means that $a_n^*(\lambda_n, \tau)$ can be replaced by any of the estimators (5)–(7).

Theorem 4.1. *Let $\{X_{n,t} : t = 1, \dots, m_n\}$ be a triangular array such that $E(X_{n,t}) \equiv 0$. Assume that conditions **A1**–**A3**, **A4(2)**, **A5** hold.*

If $b_n \rightarrow \infty$ as $n \rightarrow \infty$ such that $b_n = o(m_n)$, then the cGSBB is consistent i.e.

$$\sup_{x \in \mathcal{R}} \left| P \left(\frac{\sqrt{m_n}}{\sigma_n^R} (\Re \hat{a}_n(\lambda_n, \tau) - \Re a_n(\lambda_n, \tau)) \leq x \right) - P^* \left(\frac{\sqrt{m_n}}{\sigma_n^{*R}} (\Re a_n^*(\lambda_n, \tau) - E^*(\Re a_n^*(\lambda_n, \tau))) \leq x \right) \right| \xrightarrow{p} 0, \quad (8)$$

$$\sup_{x \in \mathcal{R}} \left| P \left(\frac{\sqrt{m_n}}{\sigma_n^I} (\Im \hat{a}_n(\lambda_n, \tau) - \Im a_n(\lambda_n, \tau)) \leq x \right) - P^* \left(\frac{\sqrt{m_n}}{\sigma_n^{*I}} (\Im a_n^*(\lambda_n, \tau) - E^*(\Im a_n^*(\lambda_n, \tau))) \leq x \right) \right| \xrightarrow{p} 0, \quad (9)$$

where

$$\sigma_n^{2R} = \frac{1}{d_n} \sum_{t=1}^{d_n} \sum_{s=-\infty}^{\infty} B_n^{\tilde{X}}(t, s) \cos(\lambda_n t) \cos(\lambda_n(t+s)), \quad (10)$$

$$\sigma_n^{2I} = \frac{1}{d_n} \sum_{t=1}^{d_n} \sum_{s=-\infty}^{\infty} B_n^{\tilde{X}}(t, s) \sin(\lambda_n t) \sin(\lambda_n(t+s)), \quad (11)$$

and $\sigma_n^{2R}, \sigma_n^{2I}$ are assumed to be positive. Moreover, $\tilde{X}_{n,t} = X_{n,t} X_{n,t+\tau}$ ($t = 1, \dots, m_n - \tau$) and $B_n^{\tilde{X}}(t, \tau)$ is the autocovariance function in the n -th row. Additionally, $\sigma_n^{*R}, \sigma_n^{*I}$ are the bootstrap counterparts of σ_n^R, σ_n^I .

The analogous result can be also formulated in terms of the coefficients of the mean function. In this case we consider two bootstrap estimators of (2):

$$\widehat{b}_n^*(\gamma_n) = \frac{1}{m_n} \sum_{t=1}^{m_n} X_{n,t}^* \exp(-i\gamma_n t), \quad (12)$$

$$\overline{b}_n^*(\gamma_n) = \frac{1}{m_n} \sum_{t=1}^{m_n} X_{n,t}^* \exp(-i\gamma_n t^*). \quad (13)$$

One may note that the estimator of (2) corresponding to (7) is exactly equal to (13). Moreover, if $m_n = l_n b_n$ the estimator (13) obtained using the CBB method is unbiased, i.e. $E^*(\overline{b}_n^*(\gamma_n)) = \overline{b}_n(\gamma_n)$. If additionally $m_n = w_n d_n$, its cGSBB version is also unbiased. One should be aware that this property does not hold for any of the bootstrap estimators (5)–(7). Using the CBB or the cGSBB ensures that each observation appears in the same number of blocks, but this fact allows us to provide unbiased estimator only when $\tau = 0$. In this case estimator (6) is unbiased under the same conditions as just discussed for estimator (13). If $\tau > 0$, then the bias is caused by the fact that joining selected blocks to create a bootstrap pseudo-sample we introduce a dependence structure that was not present in the original sample. To calculate $X_t^* X_{t+\tau}^*$ for X_t^* and $X_{t+\tau}^*$ belonging to two consecutive bootstrap blocks, we often use observations that were not τ time units apart in the original data. This effect cannot be removed by using circular versions of block bootstrap methods.

In the theorem below we present bootstrap consistency for the coefficients of the mean function. For simplicity we use notation $b_n^*(\gamma_n)$ to avoid the formulation of the result for (12) and (13) separately.

Theorem 4.2. *Let $\{X_{n,t} : t = 1, \dots, m_n\}$ be a row-wise periodically correlated array of real valued random variables with period d_n . Assume that conditions **A2**, **A3**, **A4(1)**, **A5** hold.*

If $b_n \rightarrow \infty$ as $n \rightarrow \infty$ such that $b_n = o(m_n)$, then the cGSBB is consistent i.e.

$$\sup_{x \in \mathcal{R}} \left| P \left(\frac{\sqrt{m_n}}{\sigma_n^R} \left(\Re \widehat{b}_n(\gamma_n) - \Re b_n(\gamma_n) \right) \leq x \right) - P^* \left(\frac{\sqrt{m_n}}{\sigma_n^{*R}} \left(\Re b_n^*(\gamma_n) - E^*(\Re b_n^*(\gamma_n)) \right) \leq x \right) \right| \xrightarrow{p} 0, \quad (14)$$

$$\sup_{x \in \mathcal{R}} \left| P \left(\frac{\sqrt{m_n}}{\sigma_n^I} \left(\Im \widehat{b}_n(\gamma_n) - \Im b_n(\gamma_n) \right) \leq x \right) - P^* \left(\frac{\sqrt{m_n}}{\sigma_n^{*I}} \left(\Im b_n^*(\gamma_n) - E^*(\Im b_n^*(\gamma_n)) \right) \leq x \right) \right| \xrightarrow{p} 0, \quad (15)$$

where

$$\sigma_n^{2R} = \frac{1}{d_n} \sum_{t=1}^{d_n} \sum_{s=-\infty}^{\infty} B_n^X(t, s) \cos(\lambda_n t) \cos(\lambda_n(t+s)), \quad (16)$$

$$\sigma_n^{2I} = \frac{1}{d_n} \sum_{t=1}^{d_n} \sum_{s=-\infty}^{\infty} B_n^X(t, s) \sin(\lambda_n t) \sin(\lambda_n(t+s)), \quad (17)$$

and $\sigma_n^{2R}, \sigma_n^{2I}$ are assumed to be positive. Moreover, $B_n^X(t, \tau)$ is the autocovariance function in n -th row of the array $X_{n,t}$. Moreover, $\sigma_n^{*R}, \sigma_n^{*I}$ are the bootstrap counterparts of σ_n^R, σ_n^I .

The advantage of the GSBB method is that it preserves the periodic structure contained in the data. However, to use it we need to know the period length. In some applications (for more details we refer the reader to Napolitano (2012)) period length is not obvious. So far no research was done to prove consistency of the GSBB in such a case. Moreover, a study of the impact of the period length estimation error on the results should be performed. The alternative approach in such a situation is to apply the block bootstrap method that does not require knowledge of the period length such as the CBB. Please note that since the bootstrap sample obtained with the CBB is usually not periodic in moments, the estimator (5) in the general case is not consistent.

Theorem 4.3. *If in Theorems 4.1 and 4.2, instead of the cGSBB, the CBB is used and estimator (6) or (7) is considered, the assertions continue to hold.*

Remark. In this paper we consider the circular versions of the block bootstrap methods. The main reason for that is to reduce the edge effects. In the cGSBB and the CBB each observation is present in the same number of blocks. When the GSBB or the MBB is used, the observations from the beginning and the end of the sample appear in less blocks than the others. That can cause an additional bias. However, if in the theorems presented in this section the cGSBB and the CBB are replaced by the GSBB and the MBB respectively, all assertions will hold. The changes in proofs will be minor and hence we do not present technical details.

5. Confidence intervals

In this section we construct the bootstrap equal-tailed confidence intervals for the real and the imaginary parts of $a_n(\lambda_n, \tau)$ using the bootstrap-t method (see Efron and Tibshirani (1993)). So far in all applications for PC processes the percentile bootstrap confidence intervals were constructed (see e.g. Dudek et al. (2014a), Dudek (2015), Dudek et al. (2014b), Dehay and Dudek (2015)), because the bootstrap was applied to avoid variance estimation. In our simulation we substitute the variance estimator by its bootstrap counterpart (see chapter 6 in Efron and Tibshirani (1993)). Moreover, to calculate bootstrap quantiles we need to compute a bootstrap estimate of standard error for each bootstrap sample. Thus, we have two nested levels of bootstrap sampling (see p. 162 in Efron and Tibshirani (1993)). To be more precise for a given row of our triangular array $\{X_{n,t} : t = 1, \dots, m_n\}$ and chosen set of frequencies λ_n and shift τ we generate $B = 1000$ bootstrap samples. Then, for each sample we get $\Re \hat{a}_n^{*,i}(\lambda_n, \tau)$ and $\Im \hat{a}_n^{*,i}(\lambda_n, \tau)$, $i = 1, \dots, B$. To obtain the bootstrap esti-

mate of the standard error for each of the 1000 bootstrap samples, we generate $B_1 = 500$ bootstrap samples. For each new bootstrap sample we calculate again the bootstrap estimates $\Re \hat{a}_n^{*,i,j}(\lambda_n, \tau)$ and $\Im \hat{a}_n^{*,i,j}(\lambda_n, \tau)$, where $i = 1, \dots, B$ and $j = 1, \dots, B_1$. Finally,

$$\hat{\sigma}_n^{*R,i} = \left(\sum_{j=1}^{B_1} (\Re \hat{a}_n^{*,i,j}(\lambda_n, \tau) - \Re \hat{a}_n^{*,i,\cdot}(\lambda_n, \tau))^2 / (B_1 - 1) \right)^{1/2}, \quad (18)$$

$$\hat{\sigma}_n^{*I,i} = \left(\sum_{j=1}^{B_1} (\Im \hat{a}_n^{*,i,j}(\lambda_n, \tau) - \Im \hat{a}_n^{*,i,\cdot}(\lambda_n, \tau))^2 / (B_1 - 1) \right)^{1/2}, \quad (19)$$

where $i = 1, \dots, B$, $\Re \hat{a}_n^{*,i,\cdot}(\lambda_n, \tau) = 1/B_1 \sum_{j=1}^{B_1} \Re \hat{a}_n^{*,i,j}(\lambda_n, \tau)$ and $\Im \hat{a}_n^{*,i,\cdot}(\lambda_n, \tau) = 1/B_1 \sum_{j=1}^{B_1} \Im \hat{a}_n^{*,i,j}(\lambda_n, \tau)$. The quantiles are obtained using studentized statistics of the form

$$T^{*R}(i) = \frac{\Re \hat{a}_n^{*,i}(\lambda_n, \tau) - \Re \hat{a}_n^{*,i,\cdot}(\lambda_n, \tau)}{\hat{\sigma}_n^{*R,i}} \quad \text{and} \quad T^{*I}(i) = \frac{\Im \hat{a}_n^{*,i}(\lambda_n, \tau) - \Im \hat{a}_n^{*,i,\cdot}(\lambda_n, \tau)}{\hat{\sigma}_n^{*I,i}},$$

where $i = 1, \dots, B$.

Finally, for each point (λ_n, τ) the confidence intervals for $\Re a_n(\lambda_n, \tau)$ are of the form

$$(\hat{a}_n(\lambda_n, \tau) - t_{0.975}^{*R} \hat{\sigma}_n^{*R}, \hat{a}_n(\lambda_n, \tau) + t_{0.975}^{*R} \hat{\sigma}_n^{*R}),$$

where $t_{0.025}^{*R}, t_{0.975}^{*R}$ are the 2.5% and 97.5% quantiles, respectively. Moreover,

$$\hat{\sigma}_n^{*R} = \left(\sum_{i=1}^B (\Re \hat{a}_n^{*,i}(\lambda_n, \tau) - \Re \hat{a}_n^{*,\cdot}(\lambda_n, \tau))^2 / (B - 1) \right)^{1/2}, \quad (20)$$

where $\Re \hat{a}_n^{*,\cdot}(\lambda_n, \tau) = 1/B \sum_{i=1}^B \Re \hat{a}_n^{*,i}(\lambda_n, \tau)$.

The confidence interval for $\Im a_n(\lambda_n, \tau)$ is constructed correspondingly.

Below we provide an application of our results to the Doppler effect. The Doppler effect is a change in the frequency of the signal caused by contractions/dilations of time. It appears when the source (called the transmitter) emitting signal and observer (called the receiver) exhibit relative motion. It can be heard e.g. when a vehicle sounding a siren is approaching a person.

In our study we consider continuous time process of the form

$$\chi(u) = S(u) \cos \left(2\pi \left(f_0 + \frac{\gamma}{2} u \right) u \right) = S(u) \cos (2\pi f_0 u + \pi \gamma u^2),$$

where $u \in \mathcal{R}$ and γ is chirp rate, i.e. the instantaneous rate of change of the frequency of a waveform.

The chirp $\chi(u)$ describes the received signal in the case of the transmitted signal $S(u) \cos(2\pi f_0 u)$ and the relative motion between transmitter and receiver, with constant relative radial acceleration. A radial acceleration exists when an object moves on a curve. For more details we refer the reader

to Napolitano (2012) Secs. 2.2.6.1 and 7.8.1.1. If $S(u)$ is wide-sense stationary, then $S(u) \cos(2\pi f_0 u)$ is cyclostationary and $\chi(u)$ is generalized almost-cyclostationary (GACS). In the engineering language $\chi(u)$ is said to have a time-varying carrier frequency $f_0 + \frac{\gamma}{2}u$.

Let us consider the uniform sampling of $\chi(u)$ with sampling period $T_s = 1/f_s$ and let us assume that it is sampled in several time intervals labeled by $n = 1, \dots, N$. In a generic time interval $(u_{n,0}, u_{n,0} + g_n T_s)$ the time-varying carrier frequency of the chirp modulated process $\chi(u)$ ranges from $f_0 + \frac{\gamma}{2}u_{0,n}$ to $f_0 + \frac{\gamma}{2}(u_{0,n} + g_n T_s)$. Under the assumption that

$$\frac{|\gamma|}{2} g_n T_s \ll f_0 + \frac{|\gamma|}{2} u_{n,0} \quad (21)$$

the carrier frequency can be considered constant and equal to $f_0 + \frac{\gamma}{2}u_{n,0}$ in the whole interval. In such a case $\chi(u)$ for $u \in (u_{n,0}, u_{n,0} + g_n T_s)$ can be modeled as cyclostationary with period

$$T_{n,0} = \frac{1}{2(f_0 + \frac{\gamma}{2}u_{n,0})}.$$

If $T_{n,0}/T_s = q_n$ is rational and $q_n = c_n/d_n$ with c_n and d_n relatively prime then the discrete-time sampled signal

$$X_{n,t} = \chi(u)|_{u=tT_s} \quad \text{for } u \in (u_{n,0}, u_{n,0} + g_n T_s)$$

is cyclostationary with period d_n (see Izzo and Napolitano (1996) and (1997)). If $T_{n,0}$ and T_s are incommensurate, then $X_{n,t}$ is almost cyclostationary with cycle frequencies $kT_s/T_{n,0}$ modulo 1, where k is an integer number.

In addition, it should be noticed that in order to satisfy (21), the sample size of the block g_n cannot be too large. Consequently, g_n cannot be increased without a bound. Therefore, a limit exists to the best accuracy that can be achieved with the proposed technique in the considered example.

For our study the following values of parameters were set: $f_0 = -0.00005$, $\gamma = 0.2\sqrt{19}$. Moreover, $S(t)$ is a MA(2) process of the form

$$S(t) = 12 + 0.2\varepsilon_{t-1} + 0.3\varepsilon_{t-2} + \varepsilon_t,$$

where ε_t are iid random variables from the standard normal distribution.

We took values of $\chi(u)$ for $u = 995.8, 996.8, \dots, 1048.8$ and hence $m_n = 54$. In this time interval period length $d_n = 3$. Moreover, the shift parameter $\tau = 0$ and frequencies $\lambda_n \in \{0, 0.01, \dots, 3.14\}$. To construct bootstrap-t pointwise confidence intervals for $\Re a_n(\lambda_n, 0)$ and $\Im a_n(\lambda_n, 0)$ we used the cGSBB and the CBB methods. Four block lengths b were considered $\{3, 6, 9, 18\}$. Finally, since for each frequency we generate 0.5 mln samples, we use only estimator (7).

Let us recall that the set of true frequencies $\Lambda_{n,0} \subseteq \{2k\pi/3, k = 0, 1, 2\} = \{0, 2/3\pi, 4/3\pi\}$. For the sake of clarity in the figures we present results in Hz, i.e. for $\lambda_n/(2\pi)$. Then, the just mentioned set is of the form $\{0, 1/3, 2/3\} = \{0, 0.(3), 0.(6)\}$. Note that none of the considered frequencies is exactly equal

TABLE 1

For each value of block length b the detected frequencies (Hz) using 95% bootstrap- t equal-tailed pointwise confidence intervals for $\Re a_n(\lambda_n, 0)$ (column 2 and 4) and $\Im a_n(\lambda_n, 0)$ (column 3 and 5) are presented. Columns 2–3 contain results for cGSBB, columns 4–5 results for CBB.

	cGSBB		CBB	
$b = 3$	0.00	0.01	0.00	0.01
	0.34	0.33	0.32	0.33
	0.66	0.67	0.34	0.34
			0.66	0.66
			0.68	0.67
$b = 6$	0.00	0.01	0.00	0.01
	0.34	0.33	0.32	0.33
	0.66	0.67	0.34	0.67
			0.50	
			0.66	
$b = 9$	0.00	0.01	0.00	0.01
	0.34	0.33	0.32	0.33
	0.66	0.67	0.34	0.67
			0.66	
			0.68	
$b = 18$	0.00	0.01	0.00	0.01
	0.34	0.33	0.34	0.33
	0.66	0.67	0.39	0.67
			0.61	
			0.66	

to 0.(3) or 0.(6). Thus, we expect to detect frequencies that are close to these values.

For all considered block lengths b results are very similar and hence we restrict the presentation only to $b = 3$ (see Figure 2). Moreover, Table 1 contains the detected frequencies, i.e. those for which appropriate confidence intervals do not contain 0. One may notice that for the cGSBB we detect the same frequencies independently of the value of b . What is the most important is that these are exactly frequencies that are closest to the true ones. The situation is similar for the CBB used for $\Im a_n(\lambda_n, 0)$. Even if the number of detected frequencies is not constant (for $b = 3$ we have 5 frequencies and for $b \in \{6, 9, 18\}$ 3 frequencies), pointed frequencies are again the closest to the true ones. Additionally, for $b \in \{6, 9, 18\}$ these are exactly the same frequencies as were detected using cGSBB. The situation is slightly different in the case of $\Re a_n(\lambda_n, 0)$. We have 5 significant frequencies for $b \in \{3, 9, 18\}$ and 6 for $b = 6$. We always detect $0Hz$, $0.34Hz$ and $0.66Hz$. These are frequencies that were also detected using cGSBB for $\Re a_n(\lambda_n, 0)$. But among other frequencies we have definitely false detections: $\lambda_n = 0.5Hz$ (for $b = 6$), $\lambda_n = 0.39Hz$ and $\lambda_n = 0.61Hz$ (for $b = 18$). It seems that the CBB is more sensitive to the block length choice. However, one should remember that the CBB does not require knowledge of the period length. In the case when d_n needs to be estimated, the estimation error can affect the performance of the GSBB significantly. Since the CBB does not use d_n , it can be the best choice in many practical applications.

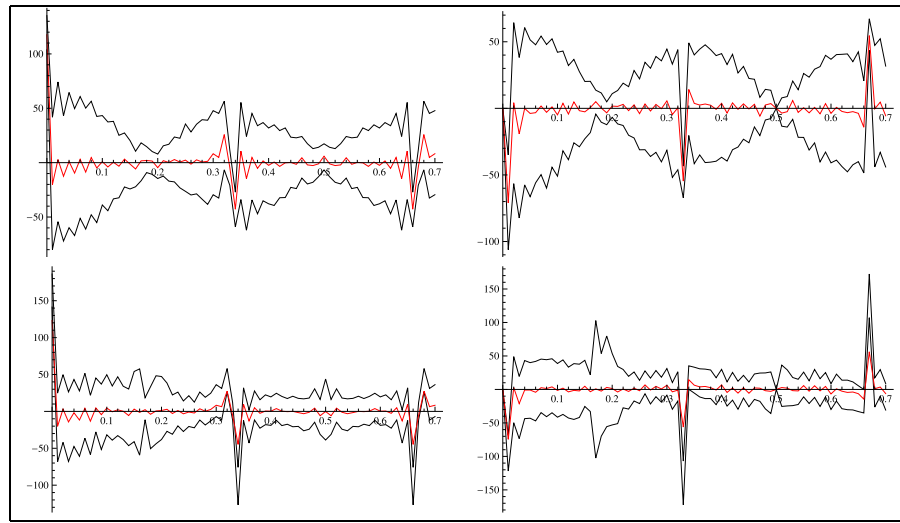


FIG 2. Estimators (red) of $\Re a_n(\lambda_n, 0)$ (left column) and $\Im a_n(\lambda_n, 0)$ (right column) together with bootstrap-t pointwise equal-tailed confidence intervals (black). First and second row results are for cGSBB and CBB, respectively. Nominal coverage probability is 95%, block length $b = 3$. On the x-axis frequencies $\lambda_n/(2\pi)$.

6. Conclusions and open questions

In this paper we consider time series with growing period. Under some moments and mixing assumptions we provide consistency results for two block bootstrap approaches, namely the Circular Block Bootstrap and the circular version of the Generalized Seasonal Block Bootstrap. The usual MBB and GSBB can also be applied for our problem. The consistent bootstrap-t equal-tailed confidence intervals for the coefficient of the mean and the autocovariance function are constructed. However, the considered model is very important and can be applied in many different fields e.g. for chirp signals, there is a further need to continue research in this topic. In Section 5 we considered the simple chirp signal with stationary component. However, in some applications this part of the signal is assumed to be heavy-tailed (see Sahmoudi et al. (2004)). Recently appeared some results for heavy tailed PC processes (Gajeczka-Mirek (2014), Drake et al. (2015), Gajeczka-Mirek (2015)) and hence our future research will be dedicated to the extension of our results to these kind of models.

Appendix

Proof of Theorem 4.1

Without loss of generality we assume that m_n is an integer multiple of the block length b_n ($m_n = l_n b_n$). To simplify notation we denote $b_n = b$.

To show asymptotic normality of the considered estimator we use Lemma 4 from Leśkow and Synowiecki (2010). We show details of the proof only for the real part of the estimator $\hat{a}_n(\lambda_n, \tau)$. The reasoning for the imaginary part is the same. Denote by $\{W_{n,t} : t = 1, \dots, m_n - \tau\}$ the array with elements are of the form $W_{n,t} = X_{n,t}X_{n,t+\tau} \cos(\lambda_n t)$. Since $X_{n,t}$ is row-wise PC and X_t is WP(4) we have that the array $\tilde{X}_{n,t} = X_{n,t}X_{n,t+\tau}$ is row-wise PC with period d_n . Moreover, $\cos(\lambda_n t)$ is a periodic function with period d_n . Thus, the array $W_{n,t}$ is also row-wise PC with period d_n . Moreover, we show that

$$\left| \text{Var} \left(\frac{1}{\sqrt{m_n}} \sum_{t=1}^{m_n-\tau} X_{n,t} X_{n,t+\tau} \cos(\lambda_n t) \right) - \sigma_n^{2R} \right| \rightarrow 0. \quad (22)$$

Note that

$$\begin{aligned} \text{Var} \left(\frac{1}{\sqrt{m_n}} \sum_{t=1}^{m_n-\tau} X_{n,t} X_{n,t+\tau} \cos(\lambda_n t) \right) &= \\ &= \frac{1}{m_n} \sum_{t=1}^{m_n-\tau} \sum_{s=1}^{m_n-\tau} \text{Cov} \left(\tilde{X}_{n,t}, \tilde{X}_{n,s} \right) \cos(\lambda_n t) \cos(\lambda_n s). \end{aligned}$$

Let $m_n - \tau = pd_n + r$, where $p \in \mathcal{N}$ and $r \in \{0, \dots, d_n - 1\}$. The right-hand side of the last equality we can rewrite as

$$\begin{aligned} &\frac{1}{m_n} \sum_{t=1}^{pd_n} \sum_{s=1}^{pd_n} \text{Cov} \left(\tilde{X}_{n,t}, \tilde{X}_{n,s} \right) \cos(\lambda_n t) \cos(\lambda_n s) \\ &+ \frac{1}{m_n} \sum_{t=pd_n+1}^{pd_n+r} \sum_{s=1}^{pd_n+r} \text{Cov} \left(\tilde{X}_{n,t}, \tilde{X}_{n,s} \right) \cos(\lambda_n t) \cos(\lambda_n s) \\ &+ \frac{1}{m_n} \sum_{t=1}^{pd_n+r} \sum_{s=pd_n+1}^{pd_n+r} \text{Cov} \left(\tilde{X}_{n,t}, \tilde{X}_{n,s} \right) \cos(\lambda_n t) \cos(\lambda_n s). \end{aligned} \quad (23)$$

The array $\tilde{X}_{n,t}$ is row-wise α -mixing with $\alpha_n^{\tilde{X}}(k) = \alpha_n(k - \tau)$ for $k > \tau$ and $\alpha_n^{\tilde{X}}(k) = \alpha_n(0)$ otherwise. As first we show that the third summand is $O(1/m_n)$. For the second summand the reasoning is the same so this case will be omitted. Note that

$$\begin{aligned} &\frac{1}{m_n} \sum_{t=1}^{pd_n+r} \sum_{s=pd_n+1}^{pd_n+r} \text{Cov} \left(\tilde{X}_{n,t}, \tilde{X}_{n,s} \right) \cos(\lambda_n t) \cos(\lambda_n s) \\ &= \frac{1}{m_n} \sum_{t=1}^{pd_n+1-\tau} \sum_{s=pd_n+1}^{pd_n+r} \text{Cov} \left(\tilde{X}_{n,t}, \tilde{X}_{n,s} \right) \cos(\lambda_n t) \cos(\lambda_n s) \\ &+ \frac{1}{m_n} \sum_{t=pd_n+2-\tau}^{pd_n+r} \sum_{s=pd_n+1}^{pd_n+r} \text{Cov} \left(\tilde{X}_{n,t}, \tilde{X}_{n,s} \right) \cos(\lambda_n t) \cos(\lambda_n s) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{m_n} \sum_{t=1}^{pd_n+1-\tau} \sum_{s=pd_n+1}^{pd_n+r} \text{Cov} \left(\tilde{X}_{n,t}, \tilde{X}_{n,s} \right) \cos(\lambda_n t) \cos(\lambda_n s) + O \left(\frac{1}{m_n} \right) \\
&= \frac{1}{m_n} \sum_{s=pd_n+1}^{pd_n+r} \sum_{s_1=s-pd_n-1+\tau}^{s-1} \text{Cov} \left(\tilde{X}_{n,s-s_1}, \tilde{X}_{n,s} \right) \cos(\lambda_n t) \cos(\lambda_n s) \\
&\quad + O \left(\frac{1}{m_n} \right).
\end{aligned}$$

Using Lemma A.0.1. from Politis et al. (1999) we get

$$\left| \text{Cov} \left(\tilde{X}_{n,s-s_1}, \tilde{X}_{n,s} \right) \right| \leq 8 \sup_s \left(\mathbb{E} (X_{n,s})^{4+2\delta} \right)^{1/(2+\delta)} \alpha_n (s_1 - \tau)^{\delta/(2+\delta)} \quad (24)$$

and

$$\begin{aligned}
&\left| \frac{1}{m_n} \sum_{s=pd_n+1}^{pd_n+r} \sum_{s_1=s-pd_n-1+\tau}^{s-1} \text{Cov} \left(\tilde{X}_{n,s-s_1}, \tilde{X}_{n,s} \right) \cos(\lambda_n t) \cos(\lambda_n s) \right| \\
&\leq \frac{Cr}{m_n} \sum_{s_1=\tau}^{\infty} \alpha_n (s_1 - \tau)^{\delta/(2+\delta)} = O \left(\frac{1}{m_n} \right),
\end{aligned}$$

where C is a positive constant independent of n .

Moreover the first summand in (23) can be written as

$$\begin{aligned}
&\frac{1}{m_n} \sum_{t=1}^{pd_n} \sum_{s=1}^{pd_n} \text{Cov} \left(\tilde{X}_{n,t}, \tilde{X}_{n,s} \right) \cos(\lambda_n t) \cos(\lambda_n s) \\
&= \frac{1}{m_n} \sum_{t=1}^{d_n} \sum_{s=-(p-1)d_n+1}^{pd_n} \left(p - \left\lfloor \frac{s}{d_n} \right\rfloor \right) \text{Cov} \left(\tilde{X}_{n,t}, \tilde{X}_{n,s} \right) \cos(\lambda_n t) \cos(\lambda_n s) \\
&= \frac{p}{m_n} \sum_{t=1}^{d_n} \sum_{s=-(p-1)d_n+1}^{pd_n} \text{Cov} \left(\tilde{X}_{n,t}, \tilde{X}_{n,s} \right) \cos(\lambda_n t) \cos(\lambda_n s) \\
&\quad - \frac{1}{m_n} \sum_{t=1}^{d_n} \sum_{s=-(p-1)d_n+1}^{pd_n} \left\lfloor \frac{s}{d_n} \right\rfloor \text{Cov} \left(\tilde{X}_{n,t}, \tilde{X}_{n,s} \right) \cos(\lambda_n t) \cos(\lambda_n s).
\end{aligned}$$

The absolute value of the last term on the right-hand side can be bounded from above by

$$\begin{aligned}
&\frac{1}{d_n} \sum_{t=1}^{d_n} \frac{1}{m_n} \sum_{s=-m_n}^{m_n} \left| s \text{Cov} \left(\tilde{X}_{n,t}, \tilde{X}_{n,s} \right) \right| \\
&= \frac{1}{d_n} \sum_{t=1}^{d_n} \frac{1}{m_n} \sum_{s_1=-m_n-t}^{m_n-t} \left| (s_1 + t) \text{Cov} \left(\tilde{X}_{n,t}, \tilde{X}_{n,s_1+t} \right) \right|
\end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{d_n} \sum_{t=1}^{d_n} \frac{1}{m_n} \sum_{s_1=-m_n-t}^{t-\tau} \left| (s_1+t) \text{Cov} \left(\tilde{X}_{n,t}, \tilde{X}_{n,s_1+t} \right) \right| \\
 &+ \frac{1}{d_n} \sum_{t=1}^{d_n} \frac{1}{m_n} \sum_{s_1=t-\tau+1}^{t+\tau} \left| (s_1+t) \text{Cov} \left(\tilde{X}_{n,t}, \tilde{X}_{n,s_1+t} \right) \right| \\
 &+ \frac{1}{d_n} \sum_{t=1}^{d_n} \frac{1}{m_n} \sum_{s_1=t+\tau+1}^{m_n-t} \left| (s_1+t) \text{Cov} \left(\tilde{X}_{n,t}, \tilde{X}_{n,s_1+t} \right) \right|.
 \end{aligned}$$

The summands of the right-hand side we denote by I , II and III , respectively.

Using (24) and assumption (ii) we have that

$$\begin{aligned}
 III &\leq \frac{1}{d_n} \sum_{t=1}^{d_n} \frac{1}{m_n} \sum_{s_1=t+\tau+1}^{m_n-t} 2s_1 \left| \text{Cov} \left(\tilde{X}_{n,t}, \tilde{X}_{n,s_1+t} \right) \right| \\
 &\leq C_1 \frac{1}{d_n} \sum_{t=1}^{d_n} \frac{1}{m_n} \sum_{s_1=t+\tau+1}^{m_n-t} s_1 \alpha_n (s_1 - \tau)^{\delta/(2+\delta)} \\
 &\leq C_1 \frac{1}{d_n} \sum_{t=1}^{d_n} \frac{1}{m_n} \sum_{s_1=1}^{\infty} s_1 \alpha_n (s_1)^{\delta/(2+\delta)} = O \left(\frac{1}{m_n} \right),
 \end{aligned}$$

where C_1 is a positive constant independent of n . Analogously one can show that $I = O(1/m_n)$. Additionally, using assumption (ii) we get that

$$\begin{aligned}
 II &\leq C_2 \frac{1}{d_n} \sum_{t=1}^{d_n} \frac{1}{m_n} \sum_{s_1=t-\tau+1}^{t+\tau} |s_1+t| \leq C_2 \frac{1}{d_n} \sum_{t=1}^{d_n} \frac{1}{m_n} \sum_{s_1=t-\tau+1}^{t+\tau} (2t+\tau) \\
 &\leq C_2 \frac{1}{d_n} \sum_{t=1}^{d_n} \frac{1}{m_n} \sum_{s_1=t-\tau+1}^{t+\tau} (2d_n+\tau) \\
 &= C_2 \frac{1}{m_n} (2d_n+\tau) (2\tau-1) = O \left(\frac{1}{w_n} \right),
 \end{aligned}$$

where C_2 is a positive constant independent of n .

To get (22) we need to show

$$\left| \frac{p}{m_n} \sum_{t=1}^{d_n} \sum_{s=-(p-1)d_n+1}^{pd_n} \text{Cov} \left(\tilde{X}_{n,t}, \tilde{X}_{n,s} \right) \cos(\lambda_n t) \cos(\lambda_n s) - \sigma_n^{2R} \right| \rightarrow 0.$$

Since $|p/m_n - 1/d_n| \rightarrow 0$ as $n \rightarrow \infty$, the expression on the left-hand side can be rewritten equivalently as

$$\left| \frac{1}{d_n} \sum_{t=1}^{d_n} \left(\sum_{s_1=-(p-1)d_n+1-t}^{pd_n-t} \text{Cov} \left(\tilde{X}_{n,t}, \tilde{X}_{n,s_1+t} \right) \cos(\lambda_n (s_1+t)) \right) \right|$$

$$\begin{aligned}
& - \sum_{s=-\infty}^{\infty} \text{Cov} \left(\tilde{X}_{n,t}, \tilde{X}_{n,t+s} \cos(\lambda_n(t+s)) \right) \cos(\lambda_n t) \Bigg| \\
& \leq \left| \frac{1}{d_n} \sum_{t=1}^{d_n} \sum_{s=-\infty}^{-(p-1)d_n-t} \text{Cov} \left(\tilde{X}_{n,t}, \tilde{X}_{n,t+s} \right) \right| \\
& \quad + \left| \frac{1}{d_n} \sum_{t=1}^{d_n} \sum_{s=pd_n-t+1}^{\infty} \text{Cov} \left(\tilde{X}_{n,t}, \tilde{X}_{n,t+s} \right) \right|.
\end{aligned}$$

Using (24) the right-hand side of the above inequality can be bounded by

$$\frac{1}{d_n} \sum_{t=1}^{d_n} \sum_{s=-\infty}^{-(p-1)d_n-t} \alpha_n(s-\tau)^{\delta/(2+\delta)} + \frac{1}{d_n} \sum_{t=1}^{d_n} \sum_{s=pd_n-t+1}^{\infty} \alpha_n(s-\tau)^{\delta/(2+\delta)}.$$

Form the assumption (ii) and Toeplitz Lemma the last expression tends to 0 as $n \rightarrow \infty$ and simultaneously we get (22). As a result the $W_{n,t}/\sigma_n^R$ fulfills the assumptions of Lemma 4 from Leśkow and Synowiecki (2010) and

$$\frac{1}{\sqrt{m_n} \sigma_n^R} \sum_{t=1}^{m_n-\tau} W_{n,t} \xrightarrow{d} N(0, 1).$$

To prove the bootstrap consistency we use techniques developed in Dudek (2015) and Dudek et al. (2014b). We start with the estimator $\bar{a}_n^*(\lambda_n, \tau)$ and $\tilde{a}_n^*(\lambda_n, \tau)$. We show only (8) because proof of (9) is analogous.

The real part of $\Re \tilde{a}_n^*(\lambda_n, \tau)$ can be rewritten as following

$$\Re \tilde{a}_n^*(\lambda_n, \tau) = \frac{1}{n} \sum_{k=0}^{l-1} \tilde{Z}_{1+bk,b}^*,$$

where

$$\tilde{Z}_{t,b}^* = X_{n,t}^* X_{n,t+\tau}^* \cos(\lambda_n t^*) + \dots + X_{n,t+b-\tau-1}^* X_{n,t+b-1}^* \cos(\lambda_n(t+b-\tau-1)^*).$$

Thus, $\Re \tilde{a}_n^*(\lambda_n, \tau)$ is obtained from $\Re \bar{a}_n^*(\lambda_n, \tau)$ after subtraction of those summands for which $X_{n,t}^*$ and $X_{n,t+\tau}^*$ belong to two different blocks. Moreover, $\Re \tilde{a}_n^*(\lambda_n, \tau)$ and $\Re \bar{a}_n^*(\lambda_n, \tau)$ are asymptotically equivalent i.e.

$$\begin{aligned}
& \sqrt{m_n} |\Re(\bar{a}_n^*(\lambda_n, \tau)) - \Re(\tilde{a}_n^*(\lambda_n, \tau)) - E^*(\Re(\bar{a}_n^*(\lambda_n, \tau)))| \\
& \quad - E^*(\Re(\tilde{a}_n^*(\lambda_n, \tau)))| \xrightarrow{p^*} 0.
\end{aligned}$$

The proof follows the same steps as the one presented in Dudek et al. (2014b), so we omit technical details.

In the sequel, when it is not confusing, to simplify notation we omit sometimes subscript n . Now it is enough to show (8) for $\Re \tilde{a}_n^*(\lambda_n, \tau)$. Similarly to Dudek (2015) and Dudek et al. (2014b) we define

$$Z_{t,b} = \tilde{Z}_{t,b} - E \tilde{Z}_{t,b},$$

where

$$\tilde{Z}_{t,b} = X_{n,t}X_{n,t+\tau} \cos(\lambda t) + \cdots + X_{n,t+b-\tau-1}X_{n,t+b-1} \cos(\lambda(t+b-\tau-1)).$$

The bootstrap version of $Z_{t,b}$ is of the form

$$Z_{t,b}^* = \tilde{Z}_{t,b}^* - E^* \tilde{Z}_{t,b}^*.$$

For $t = 1, b+1, \dots, (l-1)b+1$ the random variables $Z_{t,b}^*$ are conditionally independent (given the sample (X_1, \dots, X_n) with common distribution

$$P^*(Z_{t,b}^* = Z_{t+kd,b}) = \frac{1}{w_n} \quad \text{for } k = 0, \dots, w_n - 1.$$

By Corollary 2.4.8 from Araujo and Giné (1980) to get condition (8) we need to show that for any $\nu > 0$

$$\sum_{k=0}^{l_n-1} P^*\left(\frac{1}{\sqrt{m_n}} |Z_{1+kb,b}^*| > \nu\right) \xrightarrow{p} 0, \quad (25)$$

$$\sum_{k=0}^{l_n-1} E^*\left(\frac{1}{\sqrt{m_n}} Z_{1+kb,b}^* \mathbf{1}_{|Z_{1+kb,b}^*| > \sqrt{m_n}\nu}\right) \xrightarrow{p} 0, \quad (26)$$

$$\left| \sum_{k=0}^{l_n-1} \text{Var}^*\left(\frac{1}{\sqrt{m_n}} Z_{1+kb,b}^* \mathbf{1}_{|Z_{1+kb,b}^*| \leq \sqrt{m_n}\nu}\right) - \sigma_n^{2R} \right| \xrightarrow{p} 0. \quad (27)$$

The proof of (25)–(27) is similar to the one presented in Dudek et al. (2014b).

Note that $E \left| 1/\sqrt{b} Z_{1+kb+sd,b} \right|^4$ is bounded (see Kim (1994) and Lemma A.5. from Synowiecki (2007)). Following the reasoning of Dudek et al. (2014a) we get that the absolute expected value of (25) and (26) can be bounded from above by

$$\sum_{k=0}^{l_n-1} \frac{1}{w_n} \sum_{s=0}^{w_n-1} \frac{E \left| 1/\sqrt{b} Z_{1+kb+sd,b} \right|^3}{l_n^{3/2} \nu^3} = O\left(\frac{1}{\sqrt{l_n}}\right)$$

and

$$\sum_{k=0}^{l_n-1} \frac{1}{w_n} \sum_{t=0}^{w_n-1} \frac{1}{\sqrt{l_n}} \sqrt{E \left| \frac{1}{\sqrt{b}} Z_{1+kb+td,b} \right|^2 \frac{E \left| \frac{1}{\sqrt{b}} Z_{1+kb+td,b} \right|^4}{l_n^2 \nu^4}} = O\left(\frac{1}{\sqrt{l_n}}\right),$$

respectively.

The condition (27) can be expressed as follows

$$\sum_{k=0}^{l_n-1} \text{Var}^*\left(\frac{1}{\sqrt{m_n}} Z_{1+kb,b}^* \mathbf{1}_{|Z_{1+kb,b}^*| \leq \sqrt{m_n}\nu}\right)$$

$$\begin{aligned}
&= \sum_{k=0}^{l_n-1} \frac{1}{w_n} \sum_{t=0}^{w_n-1} \frac{1}{n} Z_{1+kb+td_n, b}^2 \mathbf{1}_{|Z_{1+kb+td_n, b}| \leq \sqrt{m_n} \nu} \\
&\quad - \sum_{k=0}^{l_n-1} \left(\frac{1}{w_n} \sum_{t=0}^{w_n-1} \frac{1}{\sqrt{m_n}} Z_{1+kb+td_n, b} \mathbf{1}_{|Z_{1+kb+td_n, b}| \leq \sqrt{m_n} \nu} \right)^2.
\end{aligned}$$

The terms on the right-hand side we denote by IV and V , respectively. Following the reasoning of Dudek et al. (2014a) and Leśkow and Synowiecki (2010) we get that $|I - \sigma_n^{2R}|$ tends to 0 in probability as $n \rightarrow \infty$. The key step is to show that

$$\begin{aligned}
R_n = \left| \sum_{k=0}^{l_n-1} \frac{1}{w_n} \sum_{t=0}^{w_n-1} \frac{1}{l_n} \text{Var} \left(\frac{1}{\sqrt{b}} Z_{1+kb+td, b} \right) \right. \\
\left. - \text{Var} \left(\sum_{s=0}^{l_n-1} \frac{1}{\sqrt{l_n}} \frac{1}{\sqrt{b}} Z_{1+sb, b} \right) \right| \rightarrow 0.
\end{aligned}$$

Note that

$$\begin{aligned}
\text{Var} \left(\sum_{k=0}^{l_n-1} \frac{1}{\sqrt{b}} Z_{1+kb, b} \right) &= \sum_{k_1=0}^{l_n-1} \sum_{k_2=0}^{l_n-1} \text{Cov} \left(\frac{1}{\sqrt{b}} Z_{1+k_1b, b}, \frac{1}{\sqrt{b}} Z_{1+k_2b, b} \right) \\
&= \sum_{k=0}^{l_n-1} \text{Var} \left(\frac{1}{\sqrt{b}} Z_{1+kb, b} \right) + \sum_{k_1=0}^{l_n-1} \sum_{k_2 \neq k_1}^{l_n-1} \text{Cov} \left(\frac{1}{\sqrt{b}} Z_{1+k_1b, b}, \frac{1}{\sqrt{b}} Z_{1+k_2b, b} \right).
\end{aligned}$$

Additionally, since $\{W_{n,t}\}$ is row-wise periodically correlated we get

$$\sum_{k=0}^{l_n-1} \frac{1}{w_n} \sum_{t=0}^{w_n-1} \frac{1}{l_n} \text{Var} \left(\frac{1}{\sqrt{b}} Z_{1+kb+td, b} \right) = \sum_{k=0}^{l_n-1} \frac{1}{l_n} \text{Var} \left(\frac{1}{\sqrt{b}} Z_{1+kb, b} \right).$$

Thus, by Lemma A.0.1 of Politis et al. (1999) we have

$$\begin{aligned}
R_n &\leq \frac{1}{l_n} \sum_{k_1=0}^{l_n-1} \sum_{k_2 \neq k_1}^{l_n-1} \left| \text{Cov} \left(\frac{1}{\sqrt{b}} Z_{1+k_1b, b}, \frac{1}{\sqrt{b}} Z_{1+k_2b, b} \right) \right| \\
&\leq \frac{1}{l_n} \sum_{k_1=0}^{l_n-1} \sum_{k_2 \neq k_1}^{l_n-1} \left(\sup_k \mathbb{E} \left| \frac{1}{\sqrt{b}} Z_{1+kb, b} \right|^4 \right)^{1/2} \alpha_Y^{1/2} (|k_2 - k_1| b - b + 1) \\
&\leq C_3 \sum_{\tau=1}^{l_n-1} \alpha_n^{1/2} ((\tau - 1)b + 1),
\end{aligned}$$

where C_3 is some positive constant independent of n .

Under assumption (iii) the sum on the right-hand side can be bounded from above by

$$C_4 \sum_{\tau=1}^{l_n-1} \frac{1}{((\tau - 1)b + 1)^{1+\zeta}} \leq C_4 + C_4 \sum_{\tau=2}^{l_n-1} \frac{1}{(\tau - 1)^{1+\zeta} b^{1+\zeta}} = O \left(\frac{1}{b^{1+\zeta}} \right),$$

where C_4 is some positive constant independent of n .

Note that V is tending to 0 in probability. The condition (ii) for α -mixing function is corresponding to one used by Dudek et al. (2014a). But since the period length $d_n \rightarrow \infty$ for some $\xi > 1/2$ we have

$$\begin{aligned} \frac{1}{l_n} \sum_{k=0}^{l_n-1} \mathbb{E} \left(\frac{1}{w_n} \sum_{t=0}^{w_n-1} \frac{1}{\sqrt{b}} Z_{1+kb+td_n, b} \right)^2 &\leq \frac{C}{w_n} \sum_{k=1}^{w_n-1} \frac{1}{k^\xi d_n^\xi} + O \left(\frac{1}{w_n} \right) \\ &= o \left(\frac{1}{d_n^\xi} \right) + O \left(\frac{1}{w_n} \right), \end{aligned}$$

which is the main difference compared to the arguments provided by Dudek et al. (2014a). This ends proof of theorem for $\mathfrak{R}\tilde{a}_n^*(\lambda_n, \tau)$ and $\mathfrak{R}\tilde{a}_n^*(\lambda_n, \tau)$. Finally, to get corresponding result for $\mathfrak{R}\hat{a}_n^*(\lambda_n, \tau)$ it is enough to follow the same reasoning as presented above with essential modifications of notation and using the fact that rows of the triangular array are $WP(4)$. \square

Proof of Theorem 4.2

Since the proof is the simplified version the reasoning presented in the proof of Theorem 4.1, it is omitted. \square

Proof of Theorem 4.3

The proof follows exactly the same steps as the proof of Theorem 4.1 and hence it is omitted. The main difference is the number of blocks that can be selected while the bootstrap sample is constructed. For the cGSBB this number is equal to w_n , while for the CBB it is equal to the length of the sample m_n . \square

References

- ANTONI, J., 2009. Cyclostationarity by examples, *Mech Syst Sig Process*, 23(4), 987–1036.
- ARAUJO, A. and GINÉ, E., 1980. The central limit theorem for real and Banach valued random variables. New York: Wiley. [MR0576407](#)
- CHAN, V., LAHIRI, S.N., MEEKER, W.Q., 2004. Block bootstrap estimation of the distribution of cumulative outdoor degradation, *Technometrics*, 46, 215–224. [MR2060017](#)
- DEHAY, D., DUDEK, A.E., 2015. Bootstrap method for Poisson sampling almost periodic process, *J. Time Ser. Anal.*, 36(3), 327–351. [MR3343005](#)
- DOUKHAN, P., 1994. Mixing. Properties and Examples, *Lecture Notes in Statistics*, 85. Springer-Verlag, New York. [MR1312160](#)

- DRAKE, C., LEŚKOW, J., GARAY, A.M., 2015. Imputation of missing observations for heavy tailed cyclostationary time series, in: F. CHAARI, J. LESKOW, A. NAPOLITANO, R. ZIMROZ, A. WYŁOMANSKA, A. DUDEK (eds), *Cyclostationarity: Theory and Methods – II*, vol. 3 of *Applied Condition Monitoring*, Springer International Publishing, pp. 179–187.
- DUDEK, A.E., 2015. Circular block bootstrap for coefficients of autocovariance function of almost periodically correlated time series, *Metrika*, 78(3), 313–335. [MR3320900](#)
- DUDEK, A.E., LEŚKOW, J., POLITIS, D., PAPARODITIS, E., 2014a. A generalized block bootstrap for seasonal time series, *J. Time Ser. Anal.*, 35, 89–114.
- DUDEK, A.E., MAIZ, S., ELBADAoui, M., 2014b. Generalized seasonal block bootstrap in frequency analysis of cyclostationary signals, *Signal Process.*, 104C, 358–368.
- EFRON, B., TIBSHIRANI, R.J., 1993. *An Introduction to the Bootstrap*, Chapman & Hall. [MR1270903](#)
- GAJECKA-MIREK, E., 2014. Subsampling for weakly dependent and periodically correlated sequences, in: F. CHAARI, J. LESKOW, A. NAPOLITANO, A. SANCHEZ-RAMIREZ (eds), *Cyclostationarity: Theory and Methods*, Lecture Notes in Mechanical Engineering, Springer International Publishing, pp. 37–51.
- GAJECKA-MIREK, E., 2015. Estimation of the parameters for non-stationary time series with long memory and heavy tails using weak dependence condition, PhD thesis, Institute of Mathematics, University of Silesia in Katowice.
- GARDNER, W.A., NAPOLITANO, A., PAURA, L., 2006. Cyclostationarity: half a century of research, *Signal Processing*, 86(4), 639–697.
- GLADYSHEV, E.G., 1961. Periodically correlated random sequences, *Soviet Mathematics*, 2, 385–388.
- HURD, H., 1989. Nonparametric time series analysis for periodically correlated processes, *IEEE Transactions on Information Theory*, 35, 350–359. [MR0999650](#)
- HURD, H., 1991. Correlation theory of almost periodically correlated processes, *Journal of Multivariate Analysis*, 37, 24–45. [MR1097303](#)
- HURD, H., LEŚKOW, J., 1992a. Estimation of the Fourier coefficient functions and their spectral densities for ϕ -mixing almost periodically correlated processes, *Statistics & Probability Letters*, 14(4), 299–306. [MR1179631](#)
- HURD, H., LEŚKOW, J., 1992b. Strongly consistent and asymptotically normal estimation of the covariance for almost periodically correlated processes, *Statistical Decisions*, 10(3), 201–25. [MR1183203](#)
- HURD, H.L., MIAMEE, A.G., 2007. *Periodically Correlated Random Sequences: Spectral. Theory and Practice*. Wiley. [MR2348769](#)
- IZZO, L., NAPOLITANO A., 1996. Higher-order cyclostationarity properties of sampled time-series, *Signal processing*, 54(3), 303–307.
- IZZO, L., NAPOLITANO A., 1997. Higher-order statistics for Rice's representation of cyclostationary signals, *Signal processing*, 56(3), 279–292.
- KIM, T., 1994. Moment bounds for non-stationary dependent sequences, *J. Appl. Probab.*, 31, 731–742. [MR1285511](#)

- KÜNSCH, H., 1989. The jackknife and the bootstrap for general stationary observations, *Ann. Statist.*, 17, 1217–1241. [MR1015147](#)
- LAHIRI, S.N., 2003. *Resampling Methods for Dependent Data*, Springer, New York. [MR2001447](#)
- LEŚKOW, J., SYNOWIECKI, R., 2010. On bootstrapping periodic random arrays with increasing period, *Metrika*, 71, 253–279. [MR2609997](#)
- LIU, R., SINGH, K., 1992. Moving blocks jackknife and bootstrap capture weak dependence. *Exploring the Limits of Bootstrap*, R. Lepage and L. Billard (eds), pp. 225–48. [MR1197787](#)
- NAPOLITANO, A., 2012. *Generalizations of Cyclostationary Signal Processing: Spectral Analysis and Applications*. Wiley-IEEE Press.
- POLITIS, D.N., 2001. Resampling time series with seasonal components, in: *Frontiers in Data Mining and Bioinformatics: Proceedings of the 33rd Symposium on the Interface of Computing Science and Statistics*, Orange County, California, June 13–17, 2001, pp. 619–621.
- POLITIS, D.N., ROMANO, J.P., 1992. A circular block-resampling procedure for stationary data. *Exploring the Limits of Bootstrap*, R. Lepage and L. Billard (eds), Wiley, New York, 263–270. [MR1197789](#)
- POLITIS, D.N., ROMANO, J.P., WOLF, M., 1999. *Subsampling*. Springer-Verlag, New York. [MR1707286](#)
- SAHMOUDI, M., ABED-MERAİM, K., BARKAT, B., 2004. IF estimation of multicomponent chirp signal in impulsive α -stable noise environment using parametric and non-parametric approaches, in: *Proc. of EU-SIPCO'2004*, Austria.
- SYNOWIECKI, R., 2007. Consistency and application of moving block bootstrap for nonstationary time series with periodic and almost periodic structure, *Bernoulli*, 13(4), 1151–1178. [MR2364230](#)