

## THE JAIN–MONRAD CRITERION FOR ROUGH PATHS AND APPLICATIONS TO RANDOM FOURIER SERIES AND NON-MARKOVIAN HÖRMANDER THEORY

BY PETER K. FRIZ<sup>1</sup>, BENJAMIN GESS<sup>1</sup>,  
ARCHIL GULISASHVILI AND SEBASTIAN RIEDEL<sup>1,2</sup>

*Technische Universität Berlin, Humboldt-Universität zu Berlin,  
Ohio University and Technische Universität Berlin*

We discuss stochastic calculus for large classes of Gaussian processes, based on rough path analysis. Our key condition is a covariance measure structure combined with a classical criterion due to Jain and Monrad [*Ann. Probab.* **11** (1983) 46–57]. This condition is verified in many examples, even in absence of explicit expressions for the covariance or Volterra kernels. Of special interest are random Fourier series, with covariance given as Fourier series itself, and we formulate conditions directly in terms of the Fourier coefficients. We also establish convergence and rates of convergence in rough path metrics of approximations to such random Fourier series. An application to SPDE is given. Our criterion also leads to an embedding result for Cameron–Martin paths and complementary Young regularity (CYR) of the Cameron–Martin space and Gaussian sample paths. CYR is known to imply Malliavin regularity and also Itô-like probabilistic estimates for stochastic integrals (resp., stochastic differential equations) despite their (rough) pathwise construction. At last, we give an application in the context of non-Markovian Hörmander theory.

**Introduction.** There is a lot of interest, from financial mathematics to nonlinear SPDE theory, in having a stochastic calculus for nonsemimartingales. In the past, much emphasis was laid upon stochastic integration (resp., stochastic differential equations) driven by fractional Brownian motion (fBm), and then general Volterra processes; cf., for example [42], Section 5, [9]. More recently, an effort was made to dispense with the Volterra structure (cf. [35, 36]) leading to a key condition of finite planar (or 2D) variation of the covariance. A completely different approach was started by Lyons [39]; cf. also [19, 20, 38, 40]. In essence, it suffices to have a.s. enough  $p$ -variation regularity of sample paths  $X(\omega)$  and existence of stochastic area(s), also subject to some variation-type regularity. The problem is

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then shifted away from developing a general stochastic integration theory to the (arguably) much simpler task of constructing the first few iterated (stochastic) integrals; the rest then follows from deterministic rough path integration theory.

In the case of Gaussian sample paths, a general sufficient condition for the existence of stochastic areas was introduced in [20]. Namely, it was shown that if the covariance of the underlying process is sufficiently regular in terms of finite two-dimensional  $\rho$ -variation, the process can be enhanced with stochastic areas in a canonical way. The point is that uniform  $L^2$ -estimates on the stochastic areas (more precisely, smooth approximations thereof) are possible, thanks to two-dimensional Young estimates, as long as  $\rho < \rho^* = 2$ . It is then fairly straightforward and carried out in detail in [20], Chapter 15 (cf. also [19]) to construct a (random) rough path  $\mathbf{X}$  associated to  $X$ . This setup has proven rather useful, applications include non-Markovian Hörmander theory ([3, 5], more below) and Hairer’s construction [22, 23] of a spatial rough path associated to the stochastic heat equation (in one space dimension) which laid the foundation to prove well-posedness of certain nonlinear SPDEs. However, finding bounds for the  $\rho$ -variation of the covariance of a stochastic process in concrete examples is not an easy task, and checkable conditions have been dearly missing in the literature.<sup>2</sup> Providing such conditions is the first main contribution of the present work.

These conditions immediately apply to known examples such as fractional Brownian motion with Hurst parameter  $H$ . In this case, it is known that  $\rho = 1/(2H) \vee 1$  and the critical  $\rho < 2$  corresponds to  $H > 1/4$ ; sharpness of this condition follows from the well-documented divergence of the Lévy area for  $H^* = 1/4$ .

Knowing the precise parameter  $\rho$  also has other benefits: it was shown (cf. [17]) that finite  $\rho$ -variation of the covariance of a Gaussian process implies that the Cameron–Martin space  $\mathcal{H}$  can be continuously embedded in the space of paths with finite  $\rho$ -variation; in other words,

$$\mathcal{H} \hookrightarrow C^{\rho\text{-var}}$$

holds. In the case  $\rho < 3/2$ , this embedding assures that the mixed iterated integral between a Gaussian sample path and a Cameron–Martin path can be defined via Young’s integration theory, and we thus speak of “complementary Young regularity” (CYR) here. CYR has many consequences: for instance, it allows for a Malliavin calculus [3, 4], [20], Chapters 15, 20, w.r.t. Gaussian rough paths. In fact, SDE solutions—by which we mean solutions to rough differential equations driven by  $\mathbf{X}(\omega)$  for a.e.  $\omega$ —will a.s. be Fréchet-smooth in Cameron–Martin directions as long as CYR holds. This led to the development of non-Markovian Hörmander theory [3, 5], a significant extension of previous work [1] specific to fBm with  $H > 1/2$ . CYR is important also for other reasons. It is the condition under which

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<sup>2</sup>The situation is easier when  $\rho = 1$ . In this case, the covariance has finite 1-variation if and only if its mixed distributional derivative is a finite signed measure. In the fBm case this means precisely  $H \geq 1/2$ .

one has Stroock–Varadhan-type support theorems (see [20], Chapter 19, and the references therein). It is also the key to good probabilistic estimates in (Gaussian) rough path theory. To appreciate this, note that the available pathwise estimates in rough path theory are ill-suited to see the probabilistic cancellations which are the heart of the Itô theory. It was only recently understood that Gaussian isoperimetry (in the form of the Borell–Sudakov–Tsirelson inequality) can bridge this gap (cf. [6] and also [13]): in the generic setting of  $\rho = 1$ , if applied to stochastic integrals (cf. [14]) of Lip 1-forms (as it is typical in rough path integration theory), one obtains identical (Gaussian) moment estimates as in the Itô theory. This deteriorates as  $\rho$  increases, but exponential integrability—and even better, depending on  $\rho$ —remains true.<sup>3</sup> A natural question is whether one can extend CYR to processes which have finite  $\rho$ -variation for  $\rho \geq 3/2$ . In the case of fractional Brownian motion, a direct analysis of its Cameron–Martin paths (using the Volterra structure of fBm) reveals that in this special case the stronger embedding

$$\mathcal{H}^H \hookrightarrow C^{q\text{-var}} \quad \text{for any } q > \frac{1}{H + (1/2)}$$

holds (cf. [16]) which implies CYR for all  $H > 1/4$ . Another contribution of the present work is to show that this stronger embedding holds in much greater generality and, in particular, even in absence of a Volterra structure of the process under consideration, which readily implies CYR for all  $\rho < 2$  and thus closes this gap.

The structure of our article is as follows. In Section 1 we answer in the affirmative the following question: given a multidimensional Gaussian process with covariance of finite  $\rho$ -variation,  $\rho < 2$ , does CYR hold? The caveat here is that the  $\rho$  is not related anymore to the  $\rho$ -variation of the covariance but instead to finite *mixed*  $(1, \rho)$ -variation, a mild strengthening that we prove not to be restrictive at all in applications. The usefulness of such a result stands and falls with one’s ability to verify this condition in concrete cases. The situation is aggravated by the examples from random Fourier series (rFs) where the covariance itself is not known explicitly, but only given as a Fourier series in its own right. A general and checkable condition for finite mixed  $(1, \rho)$ -variation is the main result of Section 2; see Theorem 2.2. Loosely speaking, our condition is a combination of a classical criterion for Gaussian processes to have  $p$ -variation sample paths due to Jain–Monrad, with a covariance measure structure condition (the distributional mixed derivative is assumed to be Radon away from the diagonal). We then run through a (long) list of examples (see Examples 2.4–2.16) which illustrate the wide applicability of our criterion. (This way, we also recover from general principles previously-known results on fBM, such as [16].) In Section 3 we apply the results of Section 2 to study rFs in greater depth. In particular, once we have established finite  $\rho$ -variation for the covariance of rFs and therefore the existence of

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<sup>3</sup>Such integrability properties can be crucial in SPDE theory [10, 14, 22] and in robust filtering theory [7, 11].

associated (random) rough paths, we ask for convergence (with rates in rough path metrics) of natural approximations given in terms of Fourier multipliers.<sup>4</sup> The best rates are obtained by considering the rough paths under consideration as  $p$ -rough paths with large  $p$ , which also means that one has to go beyond level 2, 3 considerations. Thankfully, we can rely here on general results for Gaussian rough paths established in [15]. The main results in Section 3 are Theorems 3.2 and 3.17. In Sections 4 and 5, we discuss some concrete rFs (resp., random Fourier transforms) arising from (fractional) stochastic heat equations in the study of the stochastic Burgers’s [22] and the KPZ [23] equation. Namely, we show how to regard a [fractional, with dissipative term  $-(-\partial_{xx})^\alpha, \alpha \leq 1$ ] heat equation with space–time white noise, on bounded intervals subject to various boundary conditions (resp., the entire real line) as an evolution in rough path space. The key here is spatial covariance of finite  $\rho$ -variation, where  $2\alpha = 1 + 1/\rho$ . Note  $\rho = 1$  if and only if  $\alpha = 1$  and that  $\alpha > 3/4$  is handled by our theory.<sup>5</sup> This type of spatial rough path was first used by Hairer (with  $\alpha = 1$ , and periodic boundary conditions) to analyze the stochastic Burgers equation [22]; a similar construction with other boundary conditions (incl. those we handle here) was left as open (technical) problem in [22]. In a recent preprint, Gubinelli et al. [21] consider the fractional stochastic Burgers equation, also with periodic boundary conditions, when  $\alpha > 5/6$  based on a direct spatial rough path construction.<sup>6</sup> Finally, in Section 6 we illustrate (by the example of a driving rFs) how our results can be used to check the technical conditions put forward in [5] (cf. also [3, 24]), under which differential equations driven by such Gaussian signals and along Hörmander vector fields possess a smooth density at positive times.

*Notation.* Let  $I = [S, T] \subset \mathbb{R}$  be a closed interval. We define the simplex by  $\Delta_I := \{(s, t) | s \leq t \in I\}$ . A dissection  $D$  of an interval  $I = [S, T]$  is of the form

$$D = (S = t_0 \leq t_1 \leq \dots \leq t_n = T),$$

and we write  $\mathcal{D}(I)$  for the family of all such dissections.

We will now very briefly recall the elements of rough paths theory used in this paper. For more details we refer to [20]. Let  $T^N(\mathbb{R}^d) = \mathbb{R} \oplus \mathbb{R}^d \oplus (\mathbb{R}^d \otimes \mathbb{R}^d) \oplus \dots \oplus (\mathbb{R}^d)^{\otimes N}$  be the truncated step- $N$  tensor algebra. For paths in  $T^N(\mathbb{R}^d)$  starting at the fixed point  $e := 1 + 0 + \dots + 0$ , one may define  $\beta$ -Hölder and  $p$ -variation

<sup>4</sup>Rough path convergence of piecewise linear-, mollifier, Karhunen–Loeve approximation follows from general Gaussian rough path theory [20] and requires no further discussion.

<sup>5</sup>The covariance structure, including local decorrelation as measured by mixed variational regularity, of the fractional SHE in the space variables is similar to fBm with  $H = \alpha - 1/2$ .

<sup>6</sup>In absence of  $\rho$ -variation estimates, no conclusions toward CYR and its numerous consequences are drawn in [21], nor do the results allow one to use the general body of Gaussian rough path approximation theory [15, 17, 20] based on uniform  $\rho$ -variation estimates. That said, the overall aim of [21] was quite different.

metrics, extending the usual metrics for paths in  $\mathbb{R}^d$  starting at zero: the *homogeneous*  $\beta$ -Hölder and  $p$ -variation metrics will be denoted by  $d_{\beta\text{-Hö}}l$  and  $d_{p\text{-var}}$ , the *inhomogeneous* ones by  $\rho_{\beta\text{-Hö}}l$  and  $\rho_{p\text{-var}}$ , respectively. Note that both  $\beta$ -Hölder and  $p$ -variation metrics induce the same topology on the path spaces. Corresponding norms are defined by  $\|\cdot\|_{\beta\text{-Hö}}l = d_{\beta\text{-Hö}}l(\cdot, 0)$  and  $\|\cdot\|_{p\text{-var}} = d_{p\text{-var}}(\cdot, 0)$  where 0 denotes the constant  $e$ -valued path.

A geometric  $\beta$ -Hölder rough path  $\mathbf{x}$  is a path in  $T^{\lfloor 1/\beta \rfloor}(\mathbb{R}^d)$  which can be approximated by lifts of smooth paths in the  $d_{\beta\text{-Hö}}l$  metric; geometric  $p$ -rough paths are defined similarly. Given a rough path  $\mathbf{x}$ , the projection on the first level is an  $\mathbb{R}^d$ -valued path and will be denoted by  $\pi_1(\mathbf{x})$ . It can be seen that rough paths actually take values in the smaller set  $G^N(\mathbb{R}^d) \subset T^N(\mathbb{R}^d)$ , where  $G^N(\mathbb{R}^d)$  denotes the free step- $N$  nilpotent Lie group with  $d$  generators. The Carnot–Carathéodory metric turns  $(G^N(\mathbb{R}^d), d)$  into a metric space. Consequently, we denote by

$$C_0^{0,\beta\text{-Hö}}l(I, G^{\lfloor 1/\beta \rfloor}(\mathbb{R}^d)) \quad \text{and} \quad C_0^{0,p\text{-var}}(I, G^{\lfloor p \rfloor}(\mathbb{R}^d))$$

the rough paths spaces where  $\beta \in (0, 1]$  and  $p \in [1, \infty)$ . Note that both spaces are Polish spaces.

**1. Complementary Young regularity under mixed  $(1, \rho)$ -variation assumption.** Let  $X : [0, T] \rightarrow \mathbb{R}$  be a real-valued, centered, continuous Gaussian process with covariance

$$R_X(s, t) = \mathbb{E}X_s X_t.$$

We will denote the associated Cameron–Martin space by  $\mathcal{H}$ . It is well known that  $\mathcal{H} \subset C([0, T], \mathbb{R})$  and each  $h \in \mathcal{H}$  is of the form  $h_t = \mathbb{E}Z X_t$  with  $Z$  being an element of the  $L^2$ -closure of  $\text{span}\{X_t | t \in [0, T]\}$ , a Gaussian random variable. If  $h_t = \mathbb{E}Z X_t$ ,  $h'_t = \mathbb{E}Z' X_t$ ,  $\langle h, h' \rangle_{\mathcal{H}} = \mathbb{E}Z Z'$ .

For any function  $h : [0, T] \rightarrow \mathbb{R}$  we define  $h_{s,t} := h_t - h_s$  for all  $s, t \in [0, T]$ . We recall the definition of mixed right  $(\gamma, \rho)$ -variation given in [46]: for  $\gamma, \rho \geq 1$  let

$$(1.1) \quad \begin{aligned} &V_{\gamma,\rho}(R_X; [s, t] \times [u, v]) \\ &:= \sup_{\substack{(t_i) \in \mathcal{D}([s, t]) \\ (t'_j) \in \mathcal{D}([u, v])}} \left( \sum_{t'_j} \left( \sum_{t_i} \left| R_X \left( \begin{matrix} t_i, t_{i+1} \\ t'_j, t'_{j+1} \end{matrix} \right) \right|^\gamma \right)^{\rho/\gamma} \right)^{1/\rho}, \end{aligned}$$

where  $\mathcal{D}([s, t])$  denotes the set of all dissections of  $[s, t]$  and

$$R_X \left( \begin{matrix} t_i, t_{i+1} \\ t'_j, t'_{j+1} \end{matrix} \right) = \mathbb{E}X_{t_i, t_{i+1}} X_{t'_j, t'_{j+1}}.$$

The notion of the 2D  $\rho$ -variation is recovered as  $V_\rho = V_{\rho,\rho}$ . Recall that  $V_\rho$ -regularity plays a key role in Gaussian rough path theory [17, 19, 20] and

in particular yields a stochastic integration theory for large classes of multidimensional Gaussian processes. Clearly,  $V_{\gamma \vee \rho}(R; A) \leq V_{\gamma, \rho}(R; A) \leq V_{\gamma \wedge \rho}(R; A)$  for all rectangles  $A \subseteq [0, T]^2$ . As the main result of this section, we present the following embedding theorem for the Cameron–Martin space.

**THEOREM 1.1.** *Assume that the covariance  $R_X$  has finite mixed  $(1, \rho)$ -variation in 2D sense. Then there is a continuous embedding*

$$\mathcal{H} \hookrightarrow C^{q\text{-var}} \quad \text{with } q = \frac{1}{1/(2\rho) + 1/2} < 2.$$

*More precisely,*

$$\|h\|_{q\text{-var}; [s, t]} \leq \|h\|_{\mathcal{H}} \sqrt{V_{1, \rho}(R_X; [s, t]^2)} \quad \forall [s, t] \subseteq [0, T].$$

The following is then immediate.

**COROLLARY 1.2.** *Assume  $\rho \in [1, 2)$ . Then complementary Young regularity holds, that is, we can choose  $p > 2\rho$  small enough such that  $X$  has a.s.  $p$ -variation sample paths,  $h \in \mathcal{H}$  has finite  $q$ -variation with  $1/p + 1/q > 1$ .*

We shall in see in Section 2 (as one of many examples) that the assumption of mixed  $(1, \rho)$ -variation is met in the case of fBm in the rough regime  $H \leq 1/2$  with  $\rho = 1/(2H)$ . (E.g., Example 2.9 applies with  $k = 0$  and in fact gives a neat criterion for processes with stationary increments.) It then follows that fractional Cameron–Martin paths enjoy finite  $q = \frac{1}{H+1/2}$ -variational regularity, which is consistent (and in fact a mild sharpening) of  $q > \frac{1}{H+1/2}$ , previously obtained in [16] with methods specific to fBm. Let us also note that, for the sole purpose of Theorem 1.1, it would have been enough to consider identical dissections  $(t_i) \equiv (t'_j)$  in the definition of mixed variation  $V_{\gamma, \rho}$  in (1.1). The criteria in Theorem 2.2 below would then allow for a mildly simplified proof. On the other hand, this criteria derived in Theorem 2.2 below are also sufficient (and interesting) for finite  $\rho$ -variation  $V_\rho = V_{\rho, \rho}$  which is the key condition for the construction of Gaussian rough paths needed later on, hence the additional generality of different vertical and horizontal dissections.

**REMARK 1.3.** Let  $X: [0, T] \rightarrow \mathbb{R}^d$  be a multidimensional centered Gaussian process. Then every path  $h$  in the associated Cameron–Martin space  $\mathcal{H}$  is of the form  $h_t = \mathbb{E}ZX_t$  with  $Z$  being an element of the  $L^2$ -closure of  $\text{span}\{X_t^i | t \in I, i = 1, \dots, d\}$  and  $\|h\|_{\mathcal{H}} = \|Z\|_{L^2}$ . The  $q$ -variation of  $h$  is finite if and only if the  $q$ -variation of every  $h^i = \mathbb{E}ZX^i$  is finite, and we obtain the bound

$$\|h\|_{q\text{-var}; [s, t]} \leq C \|h\|_{\mathcal{H}} \max_{i=1, \dots, d} \sqrt{V_{1, \rho}(R_{X^i}; [s, t]^2)},$$

where  $C$  is a constant depending only on the dimension  $d$ .

We now give the proof of Theorem 1.1. In fact, having identified the importance of mixed variation, the proof is pleasantly short.

**PROOF OF THEOREM 1.1.** Let  $h = \mathbb{E}ZX \in \mathcal{H}$ . Fix a dissection  $D = (t_j) \subset [s, t]$ , write  $h_j \equiv h_{t_j, t_{j+1}}$ ,  $X_j = X_{t_j, t_{j+1}}$  and also  $\|h\|_q^q := \sum_j |h_j|^q$ . Let  $q'$  and  $\rho'$  be the conjugate exponents of  $q$  and  $\rho$ . An easy calculation shows that  $\rho' = q'/2$ . By duality,

$$\|h\|_q = \sup_{\beta: \|\beta\|_{q'} \leq 1} \sum \beta_j h_j = \sup_{\beta: \|\beta\|_{q'} \leq 1} \mathbb{E} \left( Z \sum_j \beta_j X_j \right),$$

and so by Cauchy–Schwarz

$$\|h\|_q^2 \leq \|h\|_{\mathcal{H}}^2 \sup_{\beta: \|\beta\|_{q'} \leq 1} \sum_{j,k} \beta_j \beta_k \mathbb{E} X_j X_k.$$

Set  $R_{j,k} = \mathbb{E} X_j X_k$ . Then, using the symmetry of  $R$  and Hölder’s inequality,

$$\begin{aligned} \sum_{k,j} \beta_j \beta_k R_{k,j} &\leq \frac{1}{2} \sum_{j,k} \beta_j^2 |R_{j,k}| + \frac{1}{2} \sum_{j,k} \beta_k^2 |R_{j,k}| \\ &= \sum_j \beta_j^2 \sum_k |R_{k,j}| \\ &\leq \|\beta\|_{2\rho'}^2 \left( \sum_j \left( \sum_k |R_{i,k}| \right)^\rho \right)^{1/\rho} \\ &\leq V_{1,\rho}(R; [s, t]^2) \end{aligned}$$

when  $\|\beta\|_{2\rho'} = \|\beta\|_{q'} \leq 1$  which shows the claim.  $\square$

**2. Jain–Monrad revisited.**

2.1. *Preliminaries and motivation from fBm.* Let  $I \subset \mathbb{R}$  be a compact interval and  $R: I \times I \rightarrow \mathbb{R}$  be a symmetric, continuous function. We set  $T = |I|$ ,

$$(2.1) \quad D_h := \{(s, t) \in I^2 : |s - t| \leq h\}$$

and let  $D := D_0$  be the diagonal of  $I^2$ . In this section we will give conditions under which  $R$  has finite  $\rho$ -variation on  $I^2 = I \times I$ . For a rectangle  $[s, t] \times [u, v] \subseteq I^2$ , we define the rectangular increment by

$$R \begin{pmatrix} s, t \\ u, v \end{pmatrix} = R(s, u) - R(s, v) - R(t, u) + R(t, v),$$

and we set

$$(2.2) \quad \sigma^2(s, t) := R \begin{pmatrix} s, t \\ s, t \end{pmatrix} = R(s, s) + R(t, t) - 2R(s, t),$$

where symmetry of  $R$  was used in the last step. Note that

$$\partial_{s,t}\sigma^2 = -2\partial_{s,t}R$$

whenever these mixed derivatives make sense. In many applications  $R$  is the covariance function of a zero mean stochastic process  $X$ , that is,  $R(s, t) = EX_sX_t$ , and in this case  $\sigma^2(s, t) = \text{Var}(X_t - X_s) \geq 0$  is the variance of increments. However, it will be important to conduct the present discussion in a generality that goes beyond covariance functions.

Given a dissection  $(t_i)$  of  $I = [0, T]$ , the square  $[0, T]^2$  can be decomposed into little squares  $\cup_j [t_j, t_{j+1}]^2$  and off-diagonal rectangles, say  $\{Q_j\}$ . Then

$$\sum_i \sigma^2(t_i, t_{i+1}) + \sum_j R(Q_j) = R\left(\begin{matrix} 0, T \\ 0, T \end{matrix}\right) = \sigma^2(0, T) < \infty,$$

and the right-hand side is independent of the dissection. Depending on the behavior of  $\sigma^2(s, t)$ , we can or cannot ignore the on-diagonal contributions in the limit  $\text{mesh}(t_i) \rightarrow 0$ . For instance, if  $\sigma^2(s, t) = |t - s|^{2H}$  with  $H > 1/2$ , then

$$\lim_{\text{mesh}(t_i) \rightarrow 0} \sum_i \sigma^2(t_i, t_{i+1}) = 0$$

and with  $R(Q_j) \approx \partial_{s,t}R \cdot \Delta_j$  for small  $Q_j$ , or by direct calculus, we find

$$\begin{aligned} \sigma^2(0, T) &= T^{2H} = -\frac{1}{2} \int_0^T \int_0^T \partial_{s,t}|t - s|^{2H} ds dt \\ (2.3) \qquad &= H(2H - 1) \int_0^T \int_0^T |t - s|^{2H-2} ds dt, \end{aligned}$$

noting that  $|t - s|^{2H-2} = |t - s|^{-1+2(H-1/2)}$  is integrable at the diagonal (and then everywhere on  $[0, T]^2$ ) if and only if  $H > 1/2$ . When  $H = 1/2$  this computation fails. Indeed, the prefactor  $2H - 1 = 0$  combined with the diverging integral effectively leaves us with  $0 \cdot \infty$ . The reason of course is that  $R(Q_j) = 0$  in this case (Brownian increments are uncorrelated), and everything hinges on the (nonvanishing) on-diagonal contribution

$$\sum_i \sigma^2(t_i, t_{i+1}) = \sum_i (t_{i+1} - t_i) = T.$$

As a Schwartz distribution  $\partial_{s,t}R = \partial_{s,t} \min(s, t) = \delta_{\{s=t\}}$  is a ‘‘Dirac’’ on the diagonal and indeed with this interpretation as a measure,

$$\sigma^2(0, T) = R\left(\begin{matrix} 0, T \\ 0, T \end{matrix}\right) = \int_0^T \int_0^T \delta_{\{s=t\}} ds dt = T.$$

When  $H < 1/2$ ,  $\sigma^2(s, t) = |t - s|^{2H}$ , the on-diagonal contributions are not only nonvanishing but divergent [as the mesh of  $(t_i)$  goes to zero]. That is,

$$\sigma^2(0, T) = T^{2H} = \underbrace{\sum_i \sigma^2(t_i, t_{i+1})}_{\rightarrow \infty} + \sum_j R(Q_j)$$

and so, necessarily,  $\sum_j R(Q_j) \rightarrow -\infty$ . Translated to the calculus setting, this causes (2.3) to fail. Indeed, ignoring the infinite contribution from the diagonal leaves us with

$$T^{2H} \neq H(2H - 1) \underbrace{\int_0^T \int_0^T |t - s|^{2H-2} ds dt}_{=+\infty} = -\infty \quad \text{for } H < 1/2.$$

Let us remark that, with our standing assumption  $R \in C([0, T]^2)$  the (distributional) mixed derivative  $\partial_{s,t}R$  always exists, that is,

$$\langle \partial_{s,t}R, \varphi \rangle := \int_0^T \int_0^T R(s, t) \partial_{s,t}\varphi(s, t) ds dt \quad \forall \varphi \in C_c^\infty((0, T)^2).$$

One can ask if, or when,  $\partial_{s,t}R$  is given by a signed and finite (i.e., of finite total variation) Borel measure  $\mu$  on  $[0, T]^2$ , say

$$\langle \partial_{s,t}R, \varphi \rangle = \int_{[0,T]^2} \varphi d\mu,$$

with associated Hahn–Jordan decomposition  $\mu = \mu_+ - \mu_-$ . When  $H > 1/2$ , the answer is affirmative with  $\mu = \mu_+ = H(2H - 1)|t - s|^{2H-2} ds dt$ . For  $H = 1/2$ , the answer is also affirmative with  $\mu = \mu_+ = \delta_{\{s=t\}}$ . For  $H < 1/2$ , the answer is negative.

However, for all values of  $H \in (0, 1)$  it is possible to define a (signed)  $\sigma$ -finite measure by

$$\mu(A) := \int_A H(2H - 1)|t - s|^{2H-2} ds dt$$

which we shall regard as a signed Radon measure on  $(0, T)^2 \setminus D$ . Note

$$\begin{aligned} \mu &\equiv \mu_+, & \mu &\equiv 0, & \mu &\equiv -\mu_- \\ & & & & \text{for } H > 1/2, H = 0, H < 1/2, \text{ respectively.} \end{aligned}$$

In general, as seen when  $H < 1/2$ ,  $\mu$  does not need to be a finite measure on  $(0, T)^2 \setminus D$ . On the other hand, its restriction to any compact in  $(0, T)^2 \setminus D$  is finite so that  $\mu$  defines a signed Radon measure on  $(0, T)^2 \setminus D$ . Hence, for all values of  $H \in (0, 1)$  the (distributional) mixed derivative  $\partial_{s,t}R$  on  $(0, T)^2 \setminus D$  is given by the Radon measure  $\mu$ . (This was certainly observed previously, e.g., in [35].)

Care is necessary, for important information has been lost by the restriction to  $(0, T)^2 \setminus D$ . For instance, nothing was left of Brownian motion ( $\mu = 0$ ). It follows that when  $H \leq 1/2$ , and in particular in the case  $H < 1/2$  where  $|\mu| = \mu_-$  has infinite mass on  $(0, T)^2 \setminus D$ , the on-diagonal information must be captured differently. We shall achieve this by a somewhat classical condition due to Jain–Monrad [12, 30] which imposes “on-diagonal”  $\rho$ -variation of  $\sigma^2$  by

$$v_\rho(\sigma^2; [s, t]) := \sup_{D=(t_i) \in \mathcal{D}([s,t])} \left( \sum_i |\sigma^2(t_i, t_{i+1})|^\rho \right)^{1/\rho} < \infty.$$

Clearly  $\rho = 1/2H \geq 1$  in the fBm example with  $H \leq 1/2$ , but the concept is much more general.

2.2. *Main result of the section.* Throughout we work on some closed interval  $I \subset \mathbb{R}$  with length  $T = |I|$ .

CONDITION 2.1 (Jain–Monrad). *Let  $\rho \geq 1$  and  $\omega: \Delta_I \rightarrow \mathbb{R}_+$  be a super additive function [i.e.,  $w(s, r) + w(r, t) \leq w(s, t)$  for all  $s \leq r \leq t$ ]. We say that  $(JM)_{\rho, \omega}$  holds if*

$$|\sigma^2(s, t)| \leq \omega(s, t)^{1/\rho}$$

holds for all  $s < t$ .

If  $v_\rho(\sigma^2; I) < \infty$ , we can always set  $\omega(s, t) = v_\rho(\sigma^2; [s, t])^\rho$ . Conversely, if  $(JM)_{\rho, \omega}$  holds, we have  $v_\rho(\sigma^2; [s, t]) \leq \omega(s, t)^{1/\rho}$  for all  $[s, t] \subseteq I$ .

Recall the definition of mixed right  $(\gamma, \rho)$ -variation given in (1.1), noting in particular the triangle inequality: for all rectangles  $A \subseteq I^2$ ,

$$(2.4) \quad V_{\gamma, \rho}(R_1 + R_2; A) \leq V_{\gamma, \rho}(R_1; A) + V_{\gamma, \rho}(R_2; A).$$

Recall that a signed Radon measure  $\mu$  is a locally finite signed Borel measure with decomposition  $\mu = \mu_+ - \mu_-$  where  $\mu_\pm$  are locally finite, nonnegative Borel measures, one of which has finite mass. For a finite measure  $\mu$  on  $(0, T)^2 \setminus D$  we will consider its extension to  $[0, T]^2$  by  $\mu(A) := \mu(A \cap ((0, T)^2 \setminus D))$  without further notice. We now give the main theorem of this section. For simplicity, we only formulate it for the case  $I = [0, T]$ .

THEOREM 2.2. *Let  $R: [0, T]^2 \rightarrow \mathbb{R}$  be a symmetric, continuous function and  $\sigma$  as in (2.2). Assume that the (Schwartz) distributional mixed derivative  $\mu := \frac{\partial^2 R}{\partial_t \partial_s} = -\frac{1}{2} \frac{\partial^2 \sigma^2}{\partial_t \partial_s}$  is a Radon measure on  $(0, T)^2 \setminus D$  with decomposition  $\mu = \mu_+ - \mu_-$ .*

Part A. *Assume that:*

- (A.i)  $\mu_-$  has finite mass and a continuous distribution function.
- (A.ii) There exists an  $h > 0$  such that  $\sigma^2(s, t) \geq 0$  whenever  $|t - s| \leq h$ .<sup>7</sup>

Then

$$V_1(R; [s, t] \times [u, v]) \leq R \begin{pmatrix} s, t \\ u, v \end{pmatrix} + 2\mu_-([s, t] \times [u, v])$$

$$\forall [s, t] \times [u, v] \subseteq [0, T]^2.$$

---

<sup>7</sup>Automatically true if  $R$  is a covariance function.

Part B. Assume that:

(B.i)  $\mu_+$  has finite mass and a continuous distribution function.

(B.ii) There exists an  $h > 0$  such that<sup>8</sup>

$$2R \begin{pmatrix} s, t \\ u, v \end{pmatrix} = \sigma^2(s, v) - \sigma^2(s, u) + \sigma^2(u, t) - \sigma^2(v, t) \geq 0$$

$$\forall [u, v] \subseteq [s, t] \subseteq I \text{ s.t. } |t - s| \leq h.$$

(B.iii)  $(JM)_{\rho, \omega}$  holds.

Then for all  $[s, t]^2 \subset D_h$ , as defined in (2.1), we have

$$(2.5) \quad V_{1, \rho}(R; [s, t]^2) \leq C(\omega^{1/\rho}(s, t) + \mu_+([s, t]^2)),$$

for some constant  $C = C(\rho)$ .

If, in addition,  $R: [0, T]^2 \rightarrow \mathbb{R}$  satisfies a Cauchy–Schwarz inequality<sup>9</sup> then, more generally, there is a constant  $C = C(\rho, h, T)$  such that

$$(2.6) \quad V_{1, \rho}(R; [s, t] \times [u, v])$$

$$\leq C(\omega^{1/(2\rho)}(s, t)\omega^{1/(2\rho)}(u, v) + \mu_+([s, t] \times [u, v])),$$

for all rectangles  $[s, t] \times [u, v] \subset [0, T]^2$ .

The interest in Theorem 2.2 is two-fold. First, it has far-reaching conclusions: mixed  $(1, \rho)$ -variation controls  $\rho$ -variation which, if applied (componentwise) to the covariance of a Gaussian process (multidimensional, with independent components), is the key quantity for the existence of associated rough paths; here one needs  $\rho < 2$  (which corresponds to  $H > 1/4$ ; cf. Example 2.8 below).

Let us state the consequence in terms of rough paths construction specifically as a corollary.

**COROLLARY 2.3.** Assume  $(X_t : 0 \leq t \leq T)$  is a  $d$ -dimensional, centered Gaussian process with independent components. For each component  $X^i$ , assume that either the assumptions of part A of Theorem 2.2 are satisfied, in which case we set  $\rho_i = 1$ , or those of part B for some  $\rho_i < 2$ . Set  $\rho := \max_{i=1, \dots, d} \rho_i < 2$ .

<sup>8</sup>With the exception of bi-fBm, Example 2.12, we typically check (B.ii) by simply showing that  $\tau \mapsto \sigma^2(\tau, t + \tau)$ , respectively,  $\sigma^2(t - \tau, t)$  are nondecreasing for all  $t$  and  $\tau < h$ . In particular, in stationary situations where  $\sigma^2(s, t) = F(t - s)$  this amounts for  $F$  to be nondecreasing on  $[0, h]$ ; conversely it is not hard to see (2.5) implies  $F$  nondecreasing on  $[0, h/2]$ .

<sup>9</sup>That is,  $|R_{(u,v)}^{(s,t)}| \leq |R_{(s,t)}^{(s,t)}|^{1/2} |R_{(u,v)}^{(u,v)}|^{1/2}$ , for all  $[s, t] \times [u, v] \subseteq I^2$ , which is automatically true if  $R$  is a covariance function.

Then, for any  $p > 2\rho$ , it follows that  $X$  admits a “canonical” lift  $\mathbf{X} = \mathbf{X}(\omega)$  to a random geometric  $p$ -rough path.<sup>10</sup>

Moreover, mixed  $(1, \rho)$ -variation was seen in Section 1 to imply *complementary Young regularity*, an extremely important property leading to good probabilistic estimates of rough integrals, as explained in the [Introduction](#). It is also required for Stroock–Varadhan-type support theorems and is one of the key conditions for the applicability of Malliavin calculus and then non-Markovian Hörmander theory; cf. [3, 5].

Secondly, the theorem is practical because its conditions are easy to check and widely applicable. To illustrate this we now run through a list of examples. Roughly speaking, part A handles situations similar or nicer than Brownian motion, whereas part B handles situations similar or worse than Brownian motion. The finite measure  $m = \mu_-$  (resp.,  $\mu_+$ ) in part A (resp., B) should be considered as (harmless) perturbation which adds some extra flexibility. Typically  $m$  is given by a density, that is, by the (integrable) negative (resp., positive) part of some locally integrable function. Continuity of the distribution function is then trivial. In fact,  $m = 0$  in many interesting examples.

### 2.3. Examples.

#### 2.3.1. Examples handled by part A.

EXAMPLE 2.4 (Fractional Brownian motion  $H \geq 1/2$ ). Consider a (standard) fractional Brownian motion  $B^H$ , with  $\sigma^2(s, t) = |t - s|^{2H}$  in the regime  $H > 1/2$ . We have, as a measure on  $[0, T]^2 \setminus D$ ,

$$\begin{aligned} \mu = \mu_+ &= H(2H - 1)|t - s|^{2H-2} ds dt \geq 0 && \text{if } H > 1/2, \\ \mu &= 0 && \text{if } H = 1/2, \end{aligned}$$

which clearly yields a Radon measure on  $[0, T]^2 \setminus D$  (and even a finite Borel measure on  $[0, T]^2$ ). Note that  $\mu_- \equiv 0$  in the decomposition  $\mu = \mu_+ - \mu_-$ ; hence (A.i) holds trivially. Also, since  $R(s, t) = \frac{1}{2}(s^{2H} + t^{2H} - |t - s|^{2H})$  is a genuine covariance function, (A.ii) comes for free. It follows that  $R$  has finite “Hölder controlled” 1-variation, in the sense that

$$V_1(R; [s, t]^2) \leq R \begin{pmatrix} s, t \\ s, t \end{pmatrix} = |t - s|^{2H} = O(|t - s|).$$

---

<sup>10</sup>By “canonical” we mean that  $\mathbf{X}$  is the limit, in probability and  $p$ -variation rough path metric, of standard approximations procedures including piecewise linear, mollifications and of Karhunen–Loeve type. We also note that the estimates of Theorem 2.2 allow us to show, under natural assumptions on the quantities appearing on the right-hand side, that the covariances of  $X$  have finite “Hölder-controlled”  $\rho$ -variation, thereby allowing us to conclude that  $\mathbf{X}$  is a random geometric  $\alpha$ -Hölder rough path, for  $\alpha < \frac{1}{2\rho}$ . See [19, 20] for more details.

EXAMPLE 2.5 (Brownian bridge). Given a standard Brownian motion  $B$ , the *Brownian bridge* over  $[0, T]$  can be defined as

$$X_t = B_t - \frac{t}{T}B_T \implies R(s, t) = \min(s, t) - st/T.$$

It follows that  $\mu = \partial_{s,t}R$ , as a measure on  $[0, T]^2 \setminus D$ , decomposes into  $\mu_+ = 0$  and  $\mu_-$  with (constant) density  $1/T$ . Part A applies and immediately gives ‘‘Holder controlled’’ 1-variation, that is,  $V_1(R; [s, t]^2) = O(|t - s|)$ .

EXAMPLE 2.6 (Stationary increments I, Brownian and better regularity). Consider a process with stationary increments in the sense that the variance of its increments is given by

$$\sigma^2(s, t) = F(|t - s|) \geq 0,$$

for some  $F \in C^2([0, T])$ . A concrete (Gaussian) example is the *stationary Ornstein–Uhlenbeck process* with  $F(x) = 1 - e^{-x}$ . In any case, we may expand

$$F(h) = F'(0)h + F''(0)h^2/2 + o(h^2).$$

We compute

$$\partial_{s,t}\sigma^2(s, t) = -F''(|t - s|) + F'(0)2\delta(t - s)$$

so that

$$\frac{\partial^2 R}{\partial s \partial t} = -\frac{1}{2} \frac{\partial^2 \sigma^2}{\partial s \partial t} = \frac{1}{2} F''(|t - s|) \quad \text{on } (0, T)^2 \setminus D.$$

It then follows that (A.i) holds with

$$\mu(A) = \frac{1}{2} \int_A F''(|t - s|) ds dt,$$

and we immediately obtain finite (Holder controlled) 1-variation,

$$V_1(R; [s, t]^2) \leq \sigma^2(s, t) + |F''|_\infty |t - s|^2 = O(|t - s|).$$

For a concrete  $F$ , of course, one can compute  $\mu_-$  and obtain sharper conclusions. This may also be possible if we are in a ‘‘better than Brownian’’ setting, namely  $F'(0) = 0$ , in which case  $\sigma^2(s, t) = O(|t - s|^2)$ . Note that in this case  $F''(0) > 0$ , unless  $F$  is trivial.<sup>11</sup> It follows that, in a neighborhood of the diagonal,  $\mu > 0$ , and so  $\mu_- \equiv 0$ . We then have

$$V_1(R; [s, t]^2) \leq \sigma^2(s, t) = O(|t - s|^2),$$

for  $|t - s| \leq \sup\{h > 0 : F''(h) > 0\}$ .

<sup>11</sup>Indeed, if  $F'(0) = F''(0) = 0$ , then  $\|X_t - X_s\|_{L^2} = o(t - s)$  which is enough to conclude that  $X_t$  is a constant in  $L^2$ , but then  $\sigma^2(s, t) = \|X_t - X_s\|_{L^2}^2 = 0$ .

EXAMPLE 2.7 (Volterra processes I; Brownian and better regularity). Assume  $X_t = \int_0^t K(t, r) dB_r$  where  $K(t, \cdot)$  is assumed to be square-integrable. For  $s < t$ , we have

$$X_{s,t} = \int_0^t (K(t, r) - K(s, r)1_{\{r \leq s\}}) dB_r,$$

$$\sigma^2(s, t) = \mathbb{E}X_{s,t}^2 = \int_0^s (K(t, r) - K(s, r))^2 dr + \int_s^t K(t, r)^2 dr.$$

We assume a regular situation, by which we shall mean here that  $K$  is continuous on the simplex  $\{0 \leq s \leq t \leq T\}$ , and assuming suitable differentiability properties of  $K$ , one computes

$$\partial_{s,t}R = K(s, s) \partial_t K(t, s) + \int_0^s \partial_s K(s, r) \partial_t K(t, r) dr =: f(s, t).$$

If  $\mu := f(s, t) ds dt$  defines a Radon measure on  $[0, T]^2 \setminus D$ , with  $\mu_-$  having finite mass, part A is applicable. Rather than imposing technical conditions on  $K$ , we verify this in the model case of *Volterra fBm*,  $K(t, s) = (t - s)^{H-1/2}$ ,  $H > 1/2$  (As above, there is nothing to do in the Brownian case  $H = 1/2$  since then  $f \equiv 0$  and so  $\mu \equiv 0$ .) Specializing the above formula for  $\partial_{s,t}R$ , we have

$$\partial_{s,t}R = (H - 1/2)^2 \int_0^s (t - r)^{H-3/2} (s - r)^{H-3/2} dr =: f(s, t) \geq 0.$$

Since  $f$  remains bounded away from the diagonal, it clearly defines a (nonnegative!) Radon measure. Trivially,  $\mu_- \equiv 0$ , and so thanks to part A,

$$V_1(R; [s, t]^2) \leq \sigma^2(s, t) = O(|t - s|).$$

2.3.2. *Examples handled by part B.*

EXAMPLE 2.8 (Fractional Brownian motion  $H \leq 1/2$ ). Consider a (standard) fractional Brownian motion  $B^H$ , with  $\sigma^2(s, t) = |t - s|^{2H}$  in the regime  $H \leq 1/2$ . We compute  $\mu = \partial_{s,t}R = (-1/2) \partial_{s,t}\sigma^2$  away from the diagonal and find

$$\mu = -\mu_- = -H(1 - 2H)|t - s|^{2H-2} ds dt \leq 0$$

which clearly yields a Radon measure on  $[0, T]^2 \setminus D$ . Note that  $\mu_+ \equiv 0$  in the decomposition  $\mu = \mu_+ - \mu_-$ . Conditions (B.ii) and (B.iii) with  $\rho = 1/(2H)$ ,  $\omega(s, t) = t - s$  are clear. It follows that the fBm covariance function,  $R(s, t) = \frac{1}{2}(s^{2H} + t^{2H} - |t - s|^{2H})$ , has finite ‘‘Hölder controlled’’ mixed  $(1, \rho)$ -variation, in the sense that

$$V_{1,\rho}(R; [s, t]^2) \leq O(|t - s|^{1/\rho}).$$

EXAMPLE 2.9 (Stationary increments II, Brownian and worse regularity). Consider the case

$$\sigma^2(s, t) = F(|t - s|) \geq 0,$$

with  $F$  continuous, nonnegative and with  $F(0) = 0$ . A simple condition on  $F$  which generalizes at once the above fBm example and the previous Example 2.6 is *semi-concavity*, that is,

$$F'' \leq k \quad \text{in distributional sense on } (0, T) \text{ for some } k \in \mathbb{R},$$

which is tantamount to say that  $-F'' + k$  is a (nonnegative) Radon measure on  $(0, T)$ , which in turn induces a signed Radon measure on  $[0, T]^2 \setminus D$ , given by

$$A \mapsto \int_A (-F''(|t - s|) + k) ds dt - k\lambda(A),$$

where  $\lambda$  is the two-dimensional Lebesgue measure. Then  $\mu := \partial_{s,t} R = -\frac{1}{2} \partial_{s,t} \sigma^2$  is also a signed Radon measure, with  $\mu_+ \leq \frac{k}{2} \lambda$ . Clearly, there will always be some  $h > 0$  (depending on  $F$ ) such that (B.ii) holds. Under the additional assumption  $F(t) = O(t^{1/\rho})$  for some  $\rho \geq 1$ , we then have (B.iii), with  $\omega(s, t) = C(t - s)$  and conclude that, with changing constants,

$$V_{1,\rho}(R; [s, t]^2) \leq C \left( |t - s|^{1/\rho} + \frac{k}{2} |t - s|^2 \right) \leq O(|t - s|^{1/\rho}).$$

EXAMPLE 2.10 (Sums of fBm). In the previous example,  $F''$  was bounded, as a Schwartz distribution, by an  $L^\infty$ -function on  $[0, T]^2$ , namely by the constant  $k$ . But  $L^1$  would be enough. Consider  $X = B^{H_1} + B^{H_2}$ , a sum of two independent fBm with Hurst parameters  $H_1 \geq 1/2 \geq H_2$ . A look at our two previous fBm examples reveals that

$$\mu = \underbrace{H_1(2H_1 - 1)|t - s|^{2H_1-2} ds dt}_{=:\mu_+} - \underbrace{H_2(1 - 2H_2)|t - s|^{2H_2-2} ds dt}_{=:\mu_-}.$$

We easily check all conditions, in particular (B.iii) holds with  $\rho = 1/(2H_2) \geq 1$  and  $\omega(s, t) = t - s$ . As a consequence,

$$\begin{aligned} V_{1,\rho}(R; [s, t]^2) &\leq C \left( |t - s|^{1/\rho} + H_1(2H_1 - 1) \int_{[s,t]^2} |t' - s'|^{2H_1-2} ds' dt' \right) \\ &\leq C(|t - s|^{1/\rho} + |t - s|^{2H_1}) = O(|t - s|^{1/\rho}). \end{aligned}$$

(Of course, the same conclusion can be obtained from our previous fBm examples, using  $R_X = R_{B^{H_1}} + R_{B^{H_2}}$  and then the triangle inequality for the semi-norm  $V_{1,\rho}$ .)

EXAMPLE 2.11 (Volterra processes II). Volterra fBm with  $H < 1/2$ , that is, singular kernel  $K(t, s) = (t - s)^{H-1/2}$  is also covered by part B. More generally,

it is possible (thanks to the robustness of the conditions of part B), if tedious, to give technical assumptions on  $K$  which guarantee that (B.i)–(B.iii) are satisfied. We note that  $\rho \geq 1$  of condition (B.iii) is determined from the blow-up behavior of  $K$  near the diagonal.

2.3.3. *Further examples handled by part B.* (This section may be skipped at first reading. In particular, the reader may want to read Section 3 on random Fourier series before looking in detail at the “Fourier-based” examples below. Related applications to SPDEs are discussed in Section 4.)

EXAMPLE 2.12 (Bifractional Brownian motion). Consider a *bifractional Brownian motion* (cf., e.g., [29, 36, 44]), that is, a centered Gaussian process  $B^{H,K}$  on  $[0, T]$  with covariance function given by<sup>12</sup>

$$R(s, t) = \frac{1}{2^K} ((s^{2H} + t^{2H})^K - |t - s|^{2HK}),$$

for some  $H \in (0, 1)$  and  $K \in (0, 1]$ . It is known (cf. [29], Proposition 3.1) that whenever  $s < t$ ,

$$(2.7) \quad 2^{-K} |t - s|^{2HK} \leq \sigma^2(s, t) \leq 2^{1-K} |t - s|^{2HK}.$$

We claim that the case  $HK \geq \frac{1}{2}$  (resp.,  $\leq \frac{1}{2}$ ) is handled by part A (resp., B) of Theorem 2.2. To this end, first note that

$$\begin{aligned} \partial_{s,t} R(s, t) &= \frac{(2H)^2 K(K - 1)}{2^K} \frac{s^{2H-1} t^{2H-1}}{(s^{2H} + t^{2H})^{2-K}} \\ &\quad + \frac{2HK(2HK - 1)}{2^K} |t - s|^{2HK-2}. \end{aligned}$$

The measure

$$\nu := - \frac{(2H)^2 K(K - 1)}{2^K} \frac{s^{2H-1} t^{2H-1}}{(s^{2H} + t^{2H})^{2-K}} ds dt$$

has finite mass. Indeed, it is enough to show that

$$\int_{B_\delta(0)} \frac{|st|^{2H-1}}{(|s|^{2H} + |t|^{2H})^{2-K}} ds dt$$

is finite for some  $\delta > 0$ , where  $B_\delta(0)$  denotes the closed ball around 0 with radius  $\delta$ . Introducing polar coordinates, this integral equals

$$\begin{aligned} (2.8) \quad &\int_0^\delta \int_0^{2\pi} r^{2HK-1} \frac{|\sin(\theta) \cos(\theta)|^{2H-1}}{(|\sin(\theta)|^{2H} + |\cos(\theta)|^{2H})^{2-K}} d\theta dr \\ &\leq 2^{1-2H} \int_0^{2\pi} |\sin(2\theta)|^{2H-1} d\theta \int_0^\delta r^{2HK-1} dr \end{aligned}$$

<sup>12</sup>As pointed out, for example, in [36] this process does not fit in the Volterra framework.

and both integrals are finite for  $H, K > 0$ . Note that estimate (2.8) also implies that  $v([s, t]^2) \leq C|t - s|^{2HK}$  for some constant  $C$  depending of  $H, K$  and  $T$ .

Hence we obtain that  $\partial_{s,t}R(s, t) := \mu$  is a Radon measure on  $(0, T)^2 \setminus D$ . If  $HK \geq \frac{1}{2}$ , we have the decomposition  $\mu = \mu_+ - \mu_-$  with  $\mu_- = v$ , and we have already seen that (A.i) holds. (A.ii) is trivially satisfied, and with (2.7) we may conclude that

$$V_1(R; [s, t]^2) \leq 2^{1-K}|t - s|^{2HK} + 2v([s, t]^2) \leq C|t - s|^{2HK}$$

for all  $[s, t] \subseteq [0, T]$ .

If  $HK \leq \frac{1}{2}$ ,  $\mu_+ \equiv 0$  on  $(0, T)^2 \setminus D$ , thus (B.i) is satisfied in both cases. (B.ii) is also easy to see. Indeed, since  $B^{H,K}$  is a self-similar process with index  $HK$ , one can use scaling to see that it is enough to show that for all  $t_0 \in \mathbb{R}_+$  and  $h_0 \in [0, 1]$ , the function

$$h \mapsto R \left( \begin{matrix} t_0, t_0 + 1 \\ t_0 + h_0, t_0 + h_0 + h \end{matrix} \right) =: \phi(h)$$

is nonnegative on  $[0, 1 - h_0]$ . Since  $\phi(0) = 0$ , it is enough to show that  $\phi' \geq 0$  on  $(0, 1 - h_0)$  which follows by a simple calculation. Finally, from (2.7) we see that (B.iii) holds with  $\rho = \frac{1}{2HK}$  and  $\omega(s, t) = |t - s|$ , therefore

$$V_{1,\rho}(R; [s, t]^2) = O(|t - s|^{1/\rho}).$$

**EXAMPLE 2.13 (Random Fourier series I: stationary).** Consider a stationary random Fourier series<sup>13</sup>

$$\Psi(t) = \sum_{k=1}^{\infty} \alpha_k Y^k \sin(kt) + \alpha_{-k} Y^{-k} \cos(kt), \quad t \in [0, 2\pi],$$

with zero-mean, independent Gaussians  $\{Y^k | k \in \mathbb{Z}\}$  with unit variance. We compute

$$\begin{aligned} R(s, t) &= \sum \alpha_k^2 \sin(ks) \sin(kt) + \alpha_{-k}^2 \cos(ks) \cos(kt) \\ &= \frac{1}{2} \sum (\alpha_k^2 + \alpha_{-k}^2) \cos(k(t - s)) + (\alpha_k^2 - \alpha_{-k}^2) \cos(k(t + s)) \end{aligned}$$

and note that  $\alpha_k^2 \equiv \alpha_{-k}^2$  due to the assumed stationarity of  $\Psi$ . This leaves us with

$$\begin{aligned} R(s, t) &= K(|t - s|), \\ \sigma^2(s, t) &= 2(K(0) - K(|t - s|)) =: F(|t - s|), \end{aligned}$$

---

<sup>13</sup>We may ignore the (constant, random) zero-mode in the series since we are only interested in properties of the increments of the process.

where

$$K(t) := \sum_{k=1}^{\infty} \alpha_k^2 \cos(kt).$$

In special situations, for example, when  $\alpha_k^2 = 1/k^2$ , one can find  $K \in C^2([0, 2\pi])$  in closed form, which brings us back to Example 2.6. This is not possible in general, but in view of Example 2.9 above, it would suffice to know that  $K$  is convex and  $1/\rho$ -Hölder. Conditions on the Fourier-coefficients for this to hold true are known from Fourier analysis (recalled in detail in Section 3 below). For instance, given (eventually) decreasing  $(\alpha_k^2)$ ,  $K$  is  $1/\rho$ -Hölder if and only if  $\alpha_k^2 = O(k^{-(1+1/\rho)})$ . In particular, in the model case

$$\alpha_k^2 = \frac{1}{k^{2\alpha}},$$

the desired decay holds true if and only if

$$2\alpha = 1 + 1/\rho \Leftrightarrow \rho = \frac{1}{2\alpha - 1} \geq 1 \quad (\text{for } \alpha \leq 1).$$

Convexity also holds true here and we conclude that for all  $[s, t] \subset [0, 2\pi]$ ,

$$V_{1,\rho}(R; [s, t]^2) = O(|t - s|^{1/\rho}).$$

EXAMPLE 2.14 (Random Fourier series II: nonstationary, general case). As seen in the previous example, the covariance may be written as

$$(2.9) \quad R(s, t) = K(|t - s|) + K(|t + s|) + \tilde{K}(|t - s|) - \tilde{K}(|t + s|)$$

$$(2.10) \quad =: R^-(s, t) + R^+(s, t) + \tilde{R}^-(s, t) - \tilde{R}^+(s, t),$$

where  $R^\pm$  and  $\tilde{K}$  are as before and

$$\tilde{K}(t) := \sum_{k=1}^{\infty} \alpha_{-k}^2 \cos(kt).$$

Under the assumption that  $K, \tilde{K}$  are convex and  $1/\rho$ -Hölder, the cases  $R \in \{R^-, \tilde{R}^-\}$  were already handled in the previous example, where we established

$$V_{1,\rho}(R; [s, t]^2) = O(|t - s|^{1/\rho}).$$

We claim that  $R^+$  can be handled with part A.  $\tilde{R}^+$  may then be treated analogously. Condition (A.i) is simple: using convexity of  $K$ ,

$$\partial_{s,t} R^+ = K''(t + s) \geq 0 \quad \text{on } [0, T]^2 \setminus D,$$

so that  $\mu := \partial_{s,t}R^+ = \mu_+$  is a nonnegative (but in general not finite) Radon-measure on  $[0, T]^2 \setminus D$ . Unlike in previous examples, condition (A.ii) is not trivial, since  $R^+$  is not a covariance function in general. Nonetheless, we have

$$(2.11) \quad R^+ \left( \begin{matrix} s, t \\ s, t \end{matrix} \right) = K(2t) + K(2s) - (2K(t + s))$$

$$(2.12) \quad = 2 \left( \frac{K(2t) + K(2s)}{2} - K \left( \frac{2t + 2s}{2} \right) \right)$$

$$(2.13) \quad \geq 0 \quad \forall 0 \leq s \leq t \leq \pi,$$

thanks to convexity of  $K$  on  $[0, 2\pi]$ . This settles condition (A.ii). We conclude that  $R^+$  has finite 1-variation,<sup>14</sup>

$$(2.14) \quad \begin{aligned} V_1(R^+; [s, t]^2) &\leq R \left( \begin{matrix} s, t \\ s, t \end{matrix} \right) = K(2t) + K(2s) - 2K(t + s) \\ &= O(|t - s|^{1/\rho}). \end{aligned}$$

Since  $R = R^- + R^+ + \tilde{R}^- - \tilde{R}^+$ , we can now conclude with  $V_{1,\rho} \leq V_1$  and the triangle inequality to see that  $R$  has (Hölder controlled) mixed  $(1, \rho)$ -variation, in the sense that

$$V_{1,\rho}(R; [s, t]^2) = O(|t - s|^{1/\rho}),$$

for all  $[s, t] \subset [0, \pi]$ . (The extension of this estimate to  $[0, 2\pi]$  is not difficult.<sup>15</sup>)

**EXAMPLE 2.15** (Fourier fractional Brownian bridge). Fourier fractional Brownian bridge is the Gaussian process given by the random Fourier series

$$W_t^\alpha = \sum_{k=1}^\infty \frac{Y_k \sin((k/2)t)}{k^\alpha} \quad \text{for } t \in [0, 2\pi], \alpha \in \left( \frac{1}{2}, 1 \right],$$

with  $Y_k$  as above. This process arises by replacing the covariance operator of Brownian bridge (the Dirichlet Laplacian  $-\Delta$ ) by its fractional power  $(-\Delta)^\alpha$ . Clearly, this is a special case of the previous example.

**EXAMPLE 2.16** (Stationary processes: spectral measure). Let  $X_t$  be a stationary, zero-mean process with covariance

$$R(s, t) = K(|t - s|)$$

<sup>14</sup>The situation here is reminiscent of absolutely continuous paths  $x = x(t)$  on  $[0, T]$  with  $\dot{x} \in L^p$  where  $1/\rho + 1/p = 1$ . Indeed, as may be seen from Hölder's inequality, the  $L^1$ -norm of  $\dot{x}|_{[s,t]}$ , which equals the 1-variation of  $x$  over  $[s, t]$ , is finite and of order  $|t - s|^{1/\rho}$ .

<sup>15</sup>Considering the Fourier series with argument shifted by  $\pi$ , gives the same estimate on  $[\pi, 2\pi]^2$ . In fact, one can also handle the mixed  $(1, \rho)$ -variation of  $R^+$  on  $[0, \pi] \times [\pi, 2\pi]$  by playing it back to the mixed variation of  $R^-$  on  $[0, \pi] \times [0, \pi]$ , using the fact that  $K$  is given by cosine series, hence is even around  $\pi$ .

for some continuous function  $K$ . By a well-known theorem of Bochner,

$$K(t) = \int \cos(t\xi)\mu(d\xi),$$

$$\sigma^2(t) := \sigma^2(0, t) = 2(K(0) - K(t)) = 4 \int \sin^2(t\xi/2)\mu(d\xi),$$

where  $\mu$  is a finite positive symmetric measure on  $\mathbb{R}$  (“spectral measure”). The case of discrete  $\mu$  corresponds to Example 2.13. Another example is given by the *fractional Ornstein–Uhlenbeck process*,

$$X_t = \int_{-\infty}^t e^{-\lambda(t-u)} dB_u^H, \quad t \in \mathbb{R},$$

which should be viewed as the stationary solution to  $dX = -\lambda X dt + dB^H$ . In this case, it is known that  $X$  has a *spectral density* of the form<sup>16</sup>

$$\frac{d\mu}{d\xi} = c_H \frac{|\xi|^{1-2H}}{\lambda^2 + \xi^2}.$$

Clearly, the decay of the density is related to the regularity of  $K$ . More precisely, writing

$$\hat{K}(\xi) := \frac{|\xi|^{1-2H}}{\lambda^2 + \xi^2} \sim \langle \xi \rangle^{-1-2H} \quad \text{where } \langle \xi \rangle = (1 + \xi^2)^{1/2},$$

$$\langle \xi \rangle^s \hat{K}(\xi) \sim \langle \xi \rangle^{s-1-2H} \in L^2 \quad \text{iff } 2(s - 1 - 2H) < -1,$$

that is, if and only if  $s < s^* := 1/2 + 2H$ . It follows that  $K \in H^s$  for any  $s < s^*$  and thus by a standard Sobolev embedding,  $K$  is  $\alpha$ -Hölder for  $\alpha < s^* - 1/2 = 2H$ . Alternatively, and a little sharper, Theorem 7.3.1 in [41] tells us that if  $\hat{K}$  is regularly varying at  $\infty$ , then

$$\sigma^2(t) \sim C\hat{K}(1/t)/t \quad \text{as } t \rightarrow 0.$$

Applied to the situation at hand we see that  $\sigma^2(t) = O(t^{2H})$ , since  $\hat{K}(\xi) \sim (1/\xi)^{1+2H}$ . With focus on the rough case  $H \leq 1/2$ , this gives condition (B.iii) with  $\rho = 1/(2H)$ ,  $\omega(s, t) = t - s$ . Moreover, it can be seen that there is a  $T > 0$  such that  $K$  is convex on  $[0, T]$  (cf. Example 5.3 below), which implies (B.i) and (B.ii) as in Example 2.9. Hence it follows that  $V_{1,\rho}(R; [s, t]^2) = O(|t - s|^{2H})$  for all  $[s, t] \subseteq [0, T]$ .

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<sup>16</sup>This generalizes the well-known fact that the spectral density of the classical OU process is of Cauchy type.

2.4. *Proof of Theorem 2.2, part A.* From (A.i), the distributional mixed derivative of  $R$  on  $(0, T)^2 \setminus D$  is given by

$$(2.15) \quad \frac{\partial^2 R}{\partial s \partial t} = \mu_+ - \mu_-,$$

where  $\mu_-$  (trivially extended to  $[0, T]^2$  whenever convenient) has finite mass. By assumption, the distribution function of  $\mu_-$

$$R^-(s, t) := \mu_-([0, s] \times [0, t]),$$

is continuous. We may then define  $R^+ \in C([0, T])$  by imposing the decomposition

$$R = R^+ - R^-.$$

Clearly, the distributional mixed derivatives of  $R^\pm$  on  $(0, T)^2 \setminus D$  are given by

$$(2.16) \quad \frac{\partial^2 R^\pm}{\partial t \partial s} = \mu_\pm.$$

Noting that all rectangular increments of  $R^-$  are nonnegative,  $R^-(A) = \mu_-(A) \geq 0$ , we immediately have

$$V_1(R^-; A) = R^-(A) = \mu_-(A)$$

for all  $A = [s, t] \times [u, v] \subset [0, T]^2$ . On the other hand, any such rectangle  $A$  may be split up in finitely many “small squares,” say  $Q_i = [t_i, t_{i+1}]^2$  with  $t_{i+1} - t_i \leq h$  for all  $i$ , and a (finite) number of “off-diagonal” rectangles  $A_j$ , whose interior does not intersect the diagonal. Since  $R(Q_i) = \sigma^2(t_i, t_{i+1}) \geq 0$ , by (A.ii), and  $R(A_j) \geq -R^-(A_j) = -\mu_-(A_j)$ , we have

$$\begin{aligned} R(A) &= \sum_i R(Q_i) + \sum_j R(A_j) \\ &\geq -\sum_j \mu_-(A_j) \geq -\mu_-(A), \end{aligned}$$

for all rectangles  $A$ . This implies finite 1-variation over every rectangle  $A = [s, t] \times [u, v]$ . Indeed, for any dissections  $(t_i)$  of  $[s, t]$  and  $(t'_j)$  of  $[u, v]$  we have

$$\begin{aligned} \sum_{t_i, t'_j} \left| R \begin{pmatrix} t_i, t_{i+1} \\ t'_j, t'_{j+1} \end{pmatrix} \right| &\leq \sum_{t_i, t'_j} \left\{ \left| R \begin{pmatrix} t_i, t_{i+1} \\ t'_j, t'_{j+1} \end{pmatrix} + \mu_-([t_i, t_{i+1}] \times [t'_j, t'_{j+1}]) \right| \right. \\ &\quad \left. + \mu_-([t_i, t_{i+1}] \times [t'_j, t'_{j+1}]) \right\} \\ &= R \begin{pmatrix} s, t \\ u, v \end{pmatrix} + 2\mu_-([s, t] \times [u, v]), \end{aligned}$$

and so, for all rectangles  $A$ ,

$$V_1(R; A) \leq R(A) + 2\mu_-(A).$$

2.5. *Proof of Theorem 2.2, part B.* Let us start with a few definitions.

DEFINITION 2.17. For  $\gamma, \rho \geq 1$  set

$$V_{\gamma, \rho}^+(R; [s, t] \times [u, v]) := \sup_{(t'_j) \in \mathcal{D}([u, v])} \left( \sum_{t'_j} \sup_{(t_i) \in \mathcal{D}([s, t])} \left( \sum_{t_i} \left| R \left( \begin{matrix} t_i, t_{i+1} \\ t'_j, t'_{j+1} \end{matrix} \right) \right|^\gamma \right)^{\rho/\gamma} \right)^{1/\rho}$$

and

$$V_{\gamma, \rho}^+(R; U_{[s, t]}) := \sup_{(t'_j) \in \mathcal{D}([s, t])} \left( \sum_{t'_j} \sup_{(t_i) \in \mathcal{D}([s, t'_j])} \left( \sum_{t_i} \left| R \left( \begin{matrix} t_i, t_{i+1} \\ t'_j, t'_{j+1} \end{matrix} \right) \right|^\gamma \right)^{\rho/\gamma} \right)^{1/\rho},$$

$$V_{\gamma, \rho}^+(R; L_{[s, t]}) := \sup_{(t'_j) \in \mathcal{D}([s, t])} \left( \sum_{t'_j} \sup_{(t_i) \in \mathcal{D}([t'_{j+1}, t])} \left( \sum_{t_i} \left| R \left( \begin{matrix} t_i, t_{i+1} \\ t'_j, t'_{j+1} \end{matrix} \right) \right|^\gamma \right)^{\rho/\gamma} \right)^{1/\rho},$$

$$V_{\gamma, \rho}^+(R; D_{[s, t]}) := \sup_{(t'_j) \in \mathcal{D}([s, t])} \left( \sum_{t'_j} \sup_{(t_i) \in \mathcal{D}([t'_j, t'_{j+1}])} \left( \sum_{t_i} \left| R \left( \begin{matrix} t_i, t_{i+1} \\ t'_j, t'_{j+1} \end{matrix} \right) \right|^\gamma \right)^{\rho/\gamma} \right)^{1/\rho}.$$

For any rectangle  $A \subseteq I^2$  it is easy to see that

$$V_{\gamma, \rho}(R; A) \leq V_{\gamma, \rho}^+(R; A)$$

and also (e.g., as a consequence of [18], Theorem 1.i)

$$V_1(R; A) = V_{1,1}^+(R; A).$$

The main reason for introducing  $V^+$  as above is the following lemma:

LEMMA 2.18 (Concatenation Lemma 1). *Let  $R$  be as before. Then*

$$V_{\gamma, \rho}^+(R; [s, t]^2) \leq C(V_{\gamma, \rho}^+(R; U_{[s, t]}) + V_{\gamma, \rho}^+(R; D_{[s, t]}) + V_{\gamma, \rho}^+(R; L_{[s, t]})) \quad \forall [s, t] \subseteq I,$$

for some constant  $C = C(\rho, \gamma)$ .

PROOF. Let  $(t'_j)$  be a partition of  $[s, t]$ . Fix  $[t'_j, t'_{j+1}]$ , and let  $(t_i)$  be a partition of  $[s, t]$ . By subdividing rectangles which lie on the diagonal into at maximum three parts, we see that

$$\begin{aligned} & 3^{1-\gamma} \sum_{t_i} \left| R \left( \begin{matrix} t_i, t_{i+1} \\ t'_j, t'_{j+1} \end{matrix} \right) \right|^\gamma \\ & \leq \sup_{(t_i) \in \mathcal{D}([s, t'_j])} \sum_{t_i} \left| R \left( \begin{matrix} t_i, t_{i+1} \\ t'_j, t'_{j+1} \end{matrix} \right) \right|^\gamma + \sup_{(t_i) \in \mathcal{D}([t'_j, t'_{j+1}])} \sum_{t_i} \left| R \left( \begin{matrix} t_i, t_{i+1} \\ t'_j, t'_{j+1} \end{matrix} \right) \right|^\gamma \\ & \quad + \sup_{(t_i) \in \mathcal{D}([t'_{j+1}, t])} \sum_{t_i} \left| R \left( \begin{matrix} t_i, t_{i+1} \\ t'_j, t'_{j+1} \end{matrix} \right) \right|^\gamma. \end{aligned}$$

Now we take the supremum, sum over  $t'_j$  and take the supremum again to see that

$$\begin{aligned} & \sup_{(t'_j) \in \mathcal{D}([s,t])} \left( \sum_{t'_j} \sup_{(t_i) \in \mathcal{D}([s,t])} \left( \sum_{t_i} \left| R \left( \begin{matrix} t_i, t_{i+1} \\ t'_j, t'_{j+1} \end{matrix} \right) \right|^\gamma \right)^{\rho/\gamma} \right)^{1/\rho} \\ & \leq C(V_{\gamma,\rho}^+(R; U_{[s,t]}) + V_{\gamma,\rho}^+(R; D_{[s,t]}) + V_{\gamma,\rho}^+(R; L_{[s,t]})). \quad \square \end{aligned}$$

LEMMA 2.19 (Concatenation Lemma 2). *Assume that there is an  $h > 0$  such that*

$$V_{\gamma,\rho}(R; [s, t] \times [u, v]) \leq \Phi(s, t; u, v)$$

*holds for all squares  $[s, t] \times [u, v] = [s, t]^2 \subseteq D_h$  and all off-diagonal rectangles  $(s, t) \times (u, v) \subseteq I^2 \setminus D$ , where  $\Phi: \Delta_I \times \Delta_I \rightarrow \mathbb{R}_+$  is a nondecreasing function in  $t - s$  and  $v - u$ . Then there is a constant  $C = C(\gamma, \rho, h, T)$  such that*

$$V_{\gamma,\rho}(R; [s, t] \times [u, v]) \leq C\Phi(s, t; u, v)$$

*holds for all rectangles  $[s, t] \times [u, v]$ . The constant  $C$  can be chosen independently of  $h$  and  $T$  when considering only rectangles  $[s, t] \times [u, v] \subset D_h$ . The same is true if one replaces  $V_{\gamma,\rho}$  by  $V_{\gamma,\rho}^+$ .*

PROOF. *Step 1.* Consider any square of the form  $[s, t]^2 \subseteq I^2$ . Then we can subdivide this square into  $N^2$  smaller squares  $(A_{i,j})_{i,j=1}^N$  with equal side length  $\tilde{h}$ , which can be chosen such that  $h/2 \leq \tilde{h} \leq h$  and  $N \leq M$  where  $M$  is a number depending on  $T$  and  $h$ ; see Figure 1. Each of these small squares does either lie on the diagonal, or its inner part does not intersect with the diagonal. Hence

$$V_{\gamma,\rho}(R; [s, t]^2) \leq c_1(N, \gamma, \rho) \sum_{i,j=1}^N V_{\gamma,\rho}(R; A_{i,j}) \leq c_2(N, \gamma, \rho)\Phi(s, t; u, v)$$

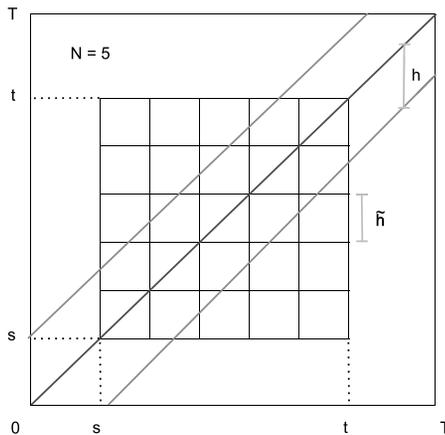


FIG. 1. Subdivision of square as used in step 1 of Lemma 2.19.

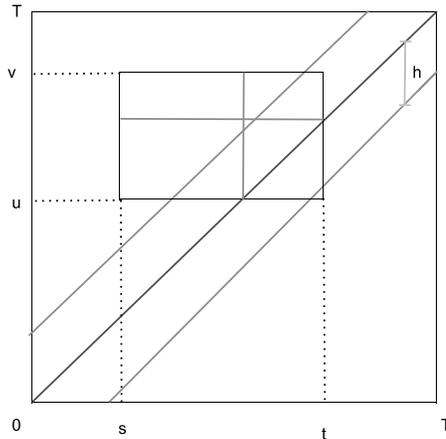


FIG. 2. Subdivision of square as used in step 2 of Lemma 2.19.

by monotonicity.

Step 2. Let  $[s, t] \times [u, v]$  be any rectangle in  $I^2$ . Then we can subdivide it into one square lying on the diagonal and three rectangles for which the inner part does not intersect with the diagonal; see Figure 2. We conclude as in step 1.  $\square$

LEMMA 2.20. Let  $R$  as before and  $\sigma$  as in (2.2). Then the following two assertions are equivalent:

(i)  $\frac{\partial^2 \sigma^2}{\partial_t \partial_s} = -2 \frac{\partial^2 R}{\partial_t \partial_s} \geq 0$  in the sense of distributions, that is, for every nonnegative  $\phi \in C_c^\infty(I^2 \setminus D)$ ,

$$\int_{I^2} \frac{\partial^2 \phi}{\partial_t \partial_s}(s, t) \sigma^2(s, t) ds dt = -2 \int_{I^2} \frac{\partial^2 \phi}{\partial_t \partial_s}(s, t) R(s, t) ds dt \geq 0.$$

(ii) For all off-diagonal rectangles  $(s, t) \times (u, v) \subseteq I^2 \setminus D$ , we have

$$R \begin{pmatrix} s, t \\ u, v \end{pmatrix} \leq 0.$$

In addition, if either of the above conditions is satisfied, then

$$R \begin{pmatrix} s, t \\ u, v \end{pmatrix} \leq \sigma^2(u, v) \quad \forall [u, v] \subseteq [s, t] \subseteq I.$$

All assertions remain true if we substitute  $\leq$  by  $\geq$  in the three inequalities.

PROOF. We will only consider the  $\leq$ -case. Let  $\varphi \in C_c(B_1(0))$  nonnegative with  $\|\varphi\|_{L^1(\mathbb{R}^2)} = 1$ . We then define the standard Dirac sequence  $\varphi^\varepsilon((s, t)) := \varepsilon^{-2} \varphi(\frac{1}{\varepsilon}(s, t))$  and observe  $\text{supp}(\varphi^\varepsilon) \subseteq B_\varepsilon(0)$ . We extend  $R$  by 0 to all of  $\mathbb{R}^2$  and

set  $R^\varepsilon := R * \varphi^\varepsilon$ . Then

$$\frac{\partial^2 R^\varepsilon}{\partial_t \partial_s}(a, b) = \int_{I^2} R(s, t) \frac{\partial^2}{\partial_t \partial_s}(\varphi^\varepsilon(s - a, t - b)) ds dt.$$

For  $(a, b) \in \overset{\circ}{\Delta}_I = \{(s, t) | s < t \in I\}$ , we note

$$\text{supp}(\varphi^\varepsilon(s - a, t - b)) \subseteq B_\varepsilon((a, b)) \subseteq \overset{\circ}{\Delta}_I$$

for all  $\varepsilon$  small enough. Hence,  $\varphi^\varepsilon(\cdot - a, \cdot - b)$  is an admissible test-function for:

(i) and thus  $\frac{\partial^2 R^\varepsilon}{\partial_t \partial_s}(a, b) \leq 0$ . Since

$$R^\varepsilon \left( \begin{matrix} s, t \\ u, v \end{matrix} \right) = \int_{I^2} \mathbb{1}_{[s,t]}(x) \mathbb{1}_{[u,v]}(y) \frac{\partial^2 R^\varepsilon}{\partial_t \partial_s}(x, y) dx dy \quad \forall s \leq t \leq u \leq v \in I,$$

(ii) follows using continuity of  $R$ .

Suppose now that (ii) is satisfied. We may approximate  $R$  by  $R^\varepsilon \in C^\infty(\Delta_I)$  such that

$$\|R - R^\varepsilon\|_{C(\Delta_I)} \leq \frac{\varepsilon}{4}.$$

By (ii) we have

$$\int_{I^2} \mathbb{1}_{[s,t]}(x) \mathbb{1}_{[u,v]}(y) \frac{\partial^2 R^\varepsilon}{\partial_t \partial_s}(x, y) dx dy = R^\varepsilon \left( \begin{matrix} s, t \\ u, v \end{matrix} \right) \leq \varepsilon,$$

for all  $s \leq t \leq u \leq v \in I$ . We note that the set of nonnegative  $f \in L^1(\Delta_I)$  satisfying

$$\int_{\Delta_I} f(x, y) \frac{\partial^2 R^\varepsilon}{\partial_t \partial_s}(x, y) dx dy \leq \varepsilon$$

is a monotone class. By the monotone class theorem, we thus have

$$\int_{\Delta_I} f(x, y) \frac{\partial^2 R^\varepsilon}{\partial_t \partial_s}(x, y) dx dy \leq \varepsilon$$

for all nonnegative  $f \in L^1(\Delta_I)$ . Considering nonnegative  $f \in C_c^\infty(\overset{\circ}{\Delta}_I)$ , a partial integration and letting  $\varepsilon \rightarrow 0$  yields (i).

To prove the remaining inequality we note

$$R \left( \begin{matrix} s, t \\ u, v \end{matrix} \right) = R \left( \begin{matrix} s, u \\ u, v \end{matrix} \right) + R \left( \begin{matrix} u, v \\ u, v \end{matrix} \right) + R \left( \begin{matrix} v, t \\ u, v \end{matrix} \right) \leq R \left( \begin{matrix} u, v \\ u, v \end{matrix} \right). \quad \square$$

We are now able to prove part B of our main theorem.

**PROOF OF THEOREM 2.2, PART B.** We decompose  $R$  as in (2.15), (2.16). We start by proving (2.5) by an application of Lemma 2.18: let  $(t'_j)$  be a partition of

$[s, t]$ . Fix  $[t'_j, t'_{j+1}]$ , and let  $(t_i)$  be a partition of  $[s, t'_j]$ . Apply Lemma 2.20 with  $R$  equal to  $-R^-$  and then  $-R^+$  to get

$$-R^-(A_{i,j}) \leq 0 \leq R^+(A_{i,j}) = \mu_+(A_{i,j})$$

for all  $A_{i,j} = [t'_j, t'_{j+1}] \times [t_i, t_{i+1}]$ . Hence, with condition (B.ii) we have

$$\begin{aligned} \sum_{t_i} \left| R \begin{pmatrix} t_i, t_{i+1} \\ t'_j, t'_{j+1} \end{pmatrix} \right| &\leq \sum_{t_i} \left| R^- \begin{pmatrix} t_i, t_{i+1} \\ t'_j, t'_{j+1} \end{pmatrix} \right| + \left| R^+ \begin{pmatrix} t_i, t_{i+1} \\ t'_j, t'_{j+1} \end{pmatrix} \right| \\ &= R^- \begin{pmatrix} s, t'_j \\ t'_j, t'_{j+1} \end{pmatrix} + R^+ \begin{pmatrix} s, t'_j \\ t'_j, t'_{j+1} \end{pmatrix} \\ &= -R \begin{pmatrix} s, t'_j \\ t'_j, t'_{j+1} \end{pmatrix} + 2R^+ \begin{pmatrix} s, t'_j \\ t'_j, t'_{j+1} \end{pmatrix} \\ &= -R \begin{pmatrix} s, t'_{j+1} \\ t'_j, t'_{j+1} \end{pmatrix} + R \begin{pmatrix} t'_j, t'_{j+1} \\ t'_j, t'_{j+1} \end{pmatrix} + 2R^+ \begin{pmatrix} s, t'_j \\ t'_j, t'_{j+1} \end{pmatrix} \\ &\leq \sigma^2(t'_j, t'_{j+1}) + 2R^+ \begin{pmatrix} s, t'_j \\ t'_j, t'_{j+1} \end{pmatrix} \\ &\leq \omega(t'_j, t'_{j+1})^{1/\rho} + 2\mu_+([s, t'_j] \times [t'_j, t'_{j+1}]). \end{aligned}$$

Taking the supremum over  $(t_i)$ , then the  $\rho$ th power, summing over  $(t'_j)$  and finally taking the supremum over  $(t'_j)$  gives

$$\begin{aligned} V_{1,\rho}^+(R; U_{[s,t]}) &\leq C(\omega(s, t) + \mu_+(\{(u, v) \in [s, t]^2 | u \leq v\})^\rho)^{1/\rho} \\ &\leq C(\omega(s, t)^{1/\rho} + \mu_+([s, t]^2)), \end{aligned}$$

for some constant  $C$  depending on  $\rho$  only. Similarly,

$$V_{1,\rho}^+(R; L_{[s,t]}) \leq C(\omega(s, t)^{1/\rho} + \mu_+([s, t]^2)).$$

Now let  $(t'_j)$  be a partition of  $[s, t]$ , fix  $[t'_j, t'_{j+1}]$  and let  $(t_i)$  be a partition of  $[t'_j, t'_{j+1}]$ . By (B.ii),  $R(A_{i,j}) \geq 0$  for all  $A_{i,j} = [t'_j, t'_{j+1}] \times [t_i, t_{i+1}]$ , thus

$$\sum_{t_i} \left| R \begin{pmatrix} t_i, t_{i+1} \\ t'_j, t'_{j+1} \end{pmatrix} \right| = \left| R \begin{pmatrix} t'_j, t'_{j+1} \\ t'_j, t'_{j+1} \end{pmatrix} \right| = \sigma^2(t'_j, t'_{j+1})$$

and hence

$$V_{1,\rho}^+(R; D_{[s,t]}) \leq \omega(s, t)^{1/\rho}.$$

By Lemma 2.18 we conclude

$$\begin{aligned} V_{1,\rho}^+(R; [s, t]^2) &\leq C(V_{1,\rho}^+(R; U_{[s,t]}) + V_{1,\rho}^+(R; D_{[s,t]}) + V_{1,\rho}^+(R; L_{[s,t]})) \\ &\leq C(\omega(s, t)^{1/\rho} + \mu_+([s, t]^2)) \end{aligned}$$

and (2.5) has been shown.<sup>17</sup>

We now establish (2.6). Let  $(s, t) \times (u, v) \subseteq I^2 \setminus D$ , and let  $(t_i)$  be a partition of  $[s, t]$  and  $(t'_j)$  be a partition of  $[u, v]$ . By nonnegativity of nonoverlapping increments,

$$\begin{aligned} \sum_{t_i, t'_j} \left| R \begin{pmatrix} t_i, t_{i+1} \\ t'_j, t'_{j+1} \end{pmatrix} \right| &\leq \sum_{t_i, t'_j} \left| R^- \begin{pmatrix} t_i, t_{i+1} \\ t'_j, t'_{j+1} \end{pmatrix} \right| + \left| R^+ \begin{pmatrix} t_i, t_{i+1} \\ t'_j, t'_{j+1} \end{pmatrix} \right| \\ &= R^- \begin{pmatrix} s, t \\ u, v \end{pmatrix} + R^+ \begin{pmatrix} s, t \\ u, v \end{pmatrix} \\ &\leq \left| R \begin{pmatrix} s, t \\ u, v \end{pmatrix} \right| + 2R^+ \begin{pmatrix} s, t \\ u, v \end{pmatrix}. \end{aligned}$$

Taking the supremum over all partitions, the Cauchy–Schwarz inequality

$$\left| R \begin{pmatrix} s, t \\ u, v \end{pmatrix} \right| \leq \left| R \begin{pmatrix} s, t \\ s, t \end{pmatrix} \right|^{1/2} \left| R \begin{pmatrix} u, v \\ u, v \end{pmatrix} \right|^{1/2}$$

gives

$$\begin{aligned} V_{1,\rho}(R; [s, t] \times [u, v]) &\leq \left| R \begin{pmatrix} s, t \\ u, v \end{pmatrix} \right| + 2R^+ \begin{pmatrix} s, t \\ u, v \end{pmatrix} \\ &\leq C(\omega(s, t)^{1/(2\rho)} \omega(u, v)^{1/(2\rho)} + \mu_+([s, t] \times [u, v])), \end{aligned}$$

and Lemma 2.19 completes the proof.  $\square$

**3. Random Fourier series.** Let us now consider a (formal) random Fourier series

$$(3.1) \quad \Psi(t) = \frac{\alpha_0 Y_0}{2} + \sum_{k=1}^{\infty} \alpha_k Y^k \sin(kt) + \alpha_{-k} Y^{-k} \cos(kt),$$

where  $Y^k$  are real-valued, centered random variables with  $\mathbb{E}Y^k Y^l = \delta_{k,l}$  for all  $k, l \in \mathbb{Z}$  and  $\alpha_k$  are real-valued coefficients. Since we are interested in properties of the covariance of  $\Psi$ , we will formulate our conditions in terms of the squared coefficients  $a_k := \alpha_k^2, k \in \mathbb{Z}$ .

**REMARK 3.1.** Assume that  $\alpha_k^2 = O(|k|^{-(1+1/\rho)})$  for some  $\rho > 0$ . Then (3.1) converges uniformly almost surely, and the limit yields a continuous function. Moreover, if the  $Y_k$  are Gaussian,  $\Psi$  has  $\beta$ -Hölder continuous trajectories<sup>18</sup> almost surely for all  $\beta < \frac{1}{2\rho}$ . This follows from [31], Theorems 7.4.3 and 5.8.3.

<sup>17</sup>Note that in fact we may deduce the somewhat stronger conclusion

$$V_{1,\rho}^+(R; [s, t]^2) \leq C(\omega(s, t)^{1/\rho} + V_{1,\rho}^+(R^+; [s, t]^2)) \quad \forall [s, t]^2 \subseteq D_h.$$

<sup>18</sup>If  $\beta = n + \tilde{\beta}$  for some  $\tilde{\beta} \in (0, 1]$ , this means that the trajectories are  $n$ -times differentiable and the  $n$ th derivative is  $\tilde{\beta}$ -Hölder continuous.

Our main theorem on random Fourier series follows:

**THEOREM 3.2.** *Consider the random Fourier series (3.1) with  $(a_k)$  satisfying  $\Delta^2(k^2 a_k) \leq 0$  for all  $k \in \mathbb{Z}$ ,*

$$\lim_{k \rightarrow \pm\infty} k^3 |\Delta^2 a_k| + k^2 |\Delta a_k| = 0,$$

*$a_k = O(|k|^{-(1+1/\rho)})$  for some  $\rho \geq 1$  for  $k \rightarrow \pm\infty$  and  $a_k, a_{-k}$  nonincreasing<sup>19</sup> for  $k \geq 1$ . Then the covariance  $R_\Psi$  of  $\Psi$  has finite Hölder controlled  $(1, \rho)$ -variation, and there is a constant  $C > 0$  such that*

$$(3.2) \quad \begin{aligned} V_{1,\rho}(R_\Psi; [s, t] \times [u, v]) &\leq C |t - s|^{1/(2\rho)} |v - u|^{1/(2\rho)} \\ \forall [s, t] \times [u, v] &\subseteq [0, 2\pi]^2. \end{aligned}$$

*The constant  $C$  depends only on  $\rho$  and  $C_1$ , where  $C_1 \geq \sup_{k \in \mathbb{Z}} a_k |k|^{1+1/\rho}$ .*

Note that the model case  $(a_k) = (|k|^{-2\alpha})$  for  $\alpha \in (\frac{1}{2}, 1]$  is contained as a special case in Theorem 3.2.

**PROOF OF THEOREM 3.2.** Note first that, as already seen in Remark 3.1,  $\Psi$  exists as a uniformly almost sure limit. Since  $(a_k) \in l^1(\mathbb{Z})$  we have  $(\alpha_k) \in l^2(\mathbb{Z})$ . Thus for fixed  $t \in [0, 2\pi]$ ,  $\Psi$  exists also as a convergent sum in  $L^2(\Omega)$ . Set  $Q_1 = [0, \pi]^2$ ,  $Q_2 = [0, \pi] \times [\pi, 2\pi]$ ,  $Q_3 = [\pi, 2\pi]^2$  and  $Q_4 = [\pi, 2\pi] \times [0, \pi]$ . We first show that (3.2) holds provided  $[s, t] \times [u, v] \subseteq Q_i$  for some  $i = 1, \dots, 4$ . Recall from (2.9) that we can decompose the covariance as

$$(3.3) \quad \begin{aligned} R_\Psi(s, t) &= K(|t - s|) + K(|t + s|) + \tilde{K}(|t - s|) + \tilde{K}(|t + s|) \\ &=: R^-(s, t) + R^+(s, t) + \tilde{R}^-(s, t) - \tilde{R}^+(s, t), \end{aligned}$$

where

$$K(t) = \sum_{k=1}^\infty \alpha_k^2 \cos(kt) \quad \text{and} \quad \tilde{K}(t) = \sum_{k=1}^\infty \alpha_{-k}^2 \cos(kt).$$

Using the triangle inequality it is enough to show the estimate (3.2) for  $R^\pm, \tilde{R}^\pm$  separately. From Lemma 3.3 below we know that  $K$  and  $\tilde{K}$  are convex on  $[0, 2\pi]$  and nonincreasing on  $[0, \pi]$ . By Lemma 3.4 below,  $K$  and  $\tilde{K}$  are  $\frac{1}{\rho}$ -Hölder continuous. Convexity implies that

$$\partial_{s,t} R^- = -K'' \leq 0.$$

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<sup>19</sup>The monotonicity of  $a_k, a_{-k}$  is required for the sole purpose of using Lemma 3.4 below. In fact, it can be dropped when we use Sobolev embeddings instead; cf. Remark 3.5 below. However, we may only conclude finite  $(1, \rho')$ -variation for any  $\rho' > \rho$  in this case.

Therefore  $\mu := -\mu_- := \partial_{s,t}^2 R^-$  yields a Radon measure on  $(0, 1)^2 \setminus D$  and condition (B.i) of Theorem 2.2 is satisfied. Condition (B.ii) holds for  $h = \pi$  since  $K$  is nonincreasing.  $(JM)_{\rho,\omega}$  follows from Hölder-continuity with  $\omega(s, t) = C|t - s|$ . Since  $R^-$  is a covariance function, it satisfies the Cauchy–Schwarz inequality. Thus we may apply part B in Theorem 2.2 to conclude that there is a constant  $C$  such that

$$V_{1,\rho}(R^-; [s, t] \times [u, v]) \leq C|t - s|^{1/(2\rho)}|v - u|^{1/(2\rho)}$$

holds for all  $[s, t] \times [u, v] \in Q_1$ . The same reasoning works for  $\tilde{R}^-$ . Using again convexity of  $K$ , we have  $\partial_{s,t}^2 R^+ = K'' \geq 0$  which shows that  $\nu := \nu_+ := \partial_{s,t}^2 R^+$  is a Radon measure on  $(0, T) \setminus D$ . Hence condition (A.i) of Theorem 2.2 holds for  $R^+$ . In (2.13) we have seen that also (A.ii) is satisfied for  $R^+$  on  $Q_1$ , and we may conclude, using part A of Theorem 2.2, that

$$V_{1,\rho}(R^+; [s, t] \times [u, v]) \leq V_1(R^+; [s, t] \times [u, v]) \leq R^+ \left( \begin{matrix} s, t \\ u, v \end{matrix} \right)$$

holds for all  $[s, t] \times [u, v] \in Q_1$ .  $R^+$  will in general not be a covariance function, but we may use the  $2\pi$ -periodicity of  $K$  to deduce the Cauchy–Schwarz inequality for  $R^+$  as well. Indeed,

$$R^+(s, t) = K(t + s) = K(t - (2\pi - s)) = R^-(2\pi - s, t),$$

and using the Cauchy–Schwarz inequality for  $R^-$  implies that

$$\begin{aligned} R^+ \left( \begin{matrix} s, t \\ u, v \end{matrix} \right) &\leq \sqrt{R^+ \left( \begin{matrix} s, t \\ s, t \end{matrix} \right)} \sqrt{R^+ \left( \begin{matrix} u, v \\ u, v \end{matrix} \right)} \\ &\leq \|K\|_{1/\rho\text{-Hö}} |t - s|^{1/(2\rho)} |u - v|^{1/(2\rho)}, \end{aligned}$$

where the second estimate follows from Hölder continuity of  $K$  as seen in (2.14). The same is true for  $\tilde{R}^+$  which shows (3.2) for  $R^+$ ,  $\tilde{R}^+$  and  $[s, t] \times [u, v] \subseteq Q_1$ . The process  $t \mapsto \Psi_{t+\pi}$  has the same covariance as  $\Psi$ . Thus estimate (3.2) also holds for  $[s, t] \times [u, v] \subseteq Q_3$ . By symmetry considerations, if  $E$  is any rectangle in  $Q_2$  or  $Q_4$ , there is a rectangle  $\bar{E}$  in  $Q_1$  (or in  $Q_3$ ) with the same side length such that  $R^+(E) = R^-(\bar{E})$ ,  $\tilde{R}^+(E) = \tilde{R}^-(\bar{E})$  and vice versa for  $R^-$ ,  $\tilde{R}^-$ . Thus (3.2) also holds for  $[s, t] \times [u, v] \subseteq Q_i$  for  $i = 2, 4$ . The general case just follows by subdividing a given rectangle  $[s, t] \times [u, v]$  in at maximum four rectangles lying in  $Q_1, \dots, Q_4$  and using the estimates above (which only leads to a larger constant). This proves the theorem.  $\square$

3.1. *Convexity, monotonicity and Hölder regularity of cosine series.* We start by deriving conditions for convexity and monotonicity of cosine series

$$(3.4) \quad K(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kt).$$

In the following let  $\Delta, \Delta^2$  be the first and second forward-difference operators, that is, for a sequence  $\{a_k\}_{k \in \mathbb{N}}$

$$\Delta a_k := a_{k+1} - a_k$$

and  $\Delta^2 := \Delta \circ \Delta$ . Moreover, let

$$D_n(t) := 1 + 2 \sum_{k=1}^n \cos(kt), \quad t \in \mathbb{R},$$

be the Dirichlet kernel and

$$F_n(t) := \sum_{k=0}^n D_k(t), \quad t \in \mathbb{R},$$

be the unnormalized Fejér kernel.

LEMMA 3.3. *Let  $\{a_k\}_{k \in \mathbb{N}}$  be such that*

$$(3.5) \quad \Delta^2(k^2 a_k) \leq 0, \quad k \in \mathbb{N}$$

and

$$(3.6) \quad \lim_{k \rightarrow \infty} k^3 |\Delta^2 a_k| + k^2 |\Delta a_k| + k |a_k| = 0.$$

Then the cosine series (3.4) exists locally uniformly in  $(0, 2\pi)$ , is convex on  $[0, 2\pi]$  and decreasing on  $[0, \pi]$ .

PROOF. The proof follows ideas from [34]; we include it for the reader’s convenience. We first note that since

$$\Delta(k^2 a_k) = k^2 \Delta a_k + (2k + 1)a_{k+1}$$

and

$$\Delta^2(k^2 a_k) = k^2 \Delta^2 a_k + 2(2k + 1)\Delta a_{k+1} + 2a_{k+2},$$

assumption (3.6) is equivalent to

$$(3.7) \quad \lim_{k \rightarrow \infty} |k \Delta^2(k^2 a_k)| + |\Delta(k^2 a_k)| + k |a_k| = 0.$$

Using the Abel transformation, we observe

$$S_n(t) = \frac{a_0}{2} + \sum_{k=1}^n a_k \cos(kt) = \frac{1}{2} \sum_{k=0}^n \Delta a_{k+1} D_k(t) + \frac{1}{2} a_{n+1} D_n(t).$$

By the assumptions and (3.7) we have  $\sum_{k=1}^\infty |\Delta a_k| < \infty$ . Since  $\sup_{n \in \mathbb{N}} D_n(t)$  is bounded locally uniformly on  $(0, 2\pi)$  and  $a_n \rightarrow 0$ , we observe that

$$K(t) := \frac{a_0}{2} + \sum_{k=1}^\infty a_k \cos(kt) = \frac{1}{2} \sum_{k=0}^\infty \Delta a_k D_k(t)$$

exists locally uniformly and is continuous in  $(0, 2\pi)$ .

The Cesàro means of the sequence  $S_n(t)$  are given by

$$\sigma_n(t) = \frac{a_0}{2} + \sum_{k=1}^n \left(1 - \frac{k}{n+1}\right) a_k \cos(kt).$$

By Fejér’s theorem ([48], Theorem III.3.4) and continuity of  $K$ ,  $\sigma_n \rightarrow K$  locally uniformly in  $(0, 2\pi)$ . Hence,  $\sigma_n'' \rightarrow K''$  in the space of distributions on  $(0, 2\pi)$ . Clearly,

$$\sigma_n''(t) = - \sum_{k=0}^n \left(1 - \frac{k}{n+1}\right) k^2 a_k \cos(kt).$$

Let  $\beta_k := (1 - \frac{k}{n+1})k^2 a_k$ . Using summation by parts twice we obtain

$$\begin{aligned} 2\sigma_n''(t) &= \sum_{k=0}^n \Delta\beta_k D_k(t) \\ &= \Delta\beta_n F_n(t) - \sum_{k=0}^{n-1} \Delta^2\beta_k F_k(t) \\ &= - \sum_{k=0}^{n-1} \Delta^2(k^2 a_k) F_k(t) \\ &\quad - \sum_{k=0}^{n-1} \left(\frac{k\Delta^2(k^2 a_k)}{n+1} - \frac{2\Delta((k+1)^2 a_{k+1})}{n+1}\right) F_k(t) + \frac{n^2}{n+1} a_n F_n(t). \end{aligned}$$

We have  $0 \leq F_n(t) \leq \frac{C}{t^2} + \frac{C}{(2\pi-t)^2}$ , where  $C > 0$  is an absolute constant. Therefore, for every  $\varepsilon$  with  $0 < \varepsilon < 2\pi$ ,

$$(3.8) \quad \sup_{n \geq 0; t \in [\varepsilon, 2\pi - \varepsilon]} F_n(t) = C_\varepsilon < \infty.$$

It follows from (3.5) that for all  $t \in [0, 2\pi]$  and  $n \geq 1$ ,

$$- \sum_{k=0}^{n-1} \Delta^2(k^2 a_k) F_k(t) - \frac{1}{n+1} \sum_{k=0}^{n-1} k\Delta^2(k^2 a_k) F_k(t) \geq 0.$$

Moreover, since  $k|a_k| \rightarrow 0$  as  $k \rightarrow \infty$  [see (3.6)], and (3.8) holds, we have

$$\sup_{t \in [\varepsilon, 2\pi - \varepsilon]} \frac{n^2}{n+1} |a_n| F_n(t) \rightarrow 0$$

as  $n \rightarrow \infty$ . Finally, set

$$(3.9) \quad S_n(t) = \frac{2}{n+1} \sum_{k=0}^{n-1} \Delta((k+1)^2 a_{k+1}) F_k(t).$$

It is easy to see, using (3.8), that for all  $n \geq 1$ ,

$$\sup_{t \in [\varepsilon, 2\pi - \varepsilon]} |S_n(t)| \leq \frac{2C_\varepsilon}{n+1} \sum_{k=0}^{n-1} |\Delta((k+1)^2 a_{k+1})|.$$

Next, taking into account (3.6) and the Cesàro summability theorem for convergent sequences, we obtain

$$\sup_{t \in [\varepsilon, 2\pi - \varepsilon]} |S_n(t)| \rightarrow 0$$

as  $n \rightarrow \infty$ , for all  $t \in [\varepsilon, 2\pi - \varepsilon]$ . Summarizing what was said above, we see that for every  $0 < \varepsilon < 2\pi$ ,

$$\liminf_{n \rightarrow \infty} \inf_{t \in [\varepsilon, 2\pi - \varepsilon]} \sigma_n''(t) \geq 0.$$

For any nonnegative test-function  $\varphi \in C_c^\infty(0, 2\pi)$ , Fatou’s lemma implies

$$K''(\varphi) = \lim_{n \rightarrow \infty} \sigma_n''(\varphi) \geq \int_0^{2\pi} \liminf_{n \rightarrow \infty} \sigma_n''(t) \varphi(t) dt \geq 0;$$

that is,  $K''$  is a nonnegative distribution on  $(0, 2\pi)$ . Thus  $K$  is convex on  $[0, 2\pi]$ .

Assume now that  $K$  is not decreasing on  $[0, \pi]$ ; that is, there are  $s < t \in [0, \pi]$  such that  $K(s) < K(t)$ . Since  $K$  is given as a cosine series, we have  $K(s) = K(s')$  and  $K(t) = K(t')$  for  $s' = 2\pi - s$  and  $t' = 2\pi - t$ . Choose  $\lambda \in (0, 1)$  such that  $\lambda s + (1 - \lambda)s' = t$ . Then

$$K(\lambda s + (1 - \lambda)s') = K(t) > K(s) = \lambda K(s) + (1 - \lambda)K(s')$$

which is a contradiction to the convexity of  $K$ .  $\square$

Concerning Hölder regularity of cosine series we recall the following:

LEMMA 3.4 ([37], Satz 8). *A cosine series (3.4) with nonincreasing coefficients  $a_k \downarrow 0$  for  $k \rightarrow \infty$  is  $\frac{1}{\rho}$ -Hölder continuous if and only if  $a_k = O(k^{-(1+(1/\rho))})$  for  $k \rightarrow \infty$ .*

REMARK 3.5. The above lemma gives a sharper result than what is obtained by usual Sobolev embeddings. Indeed: recall that an  $L^2$  function on the torus with Fourier coefficients  $(a_k)$  is in the Sobolev space  $H^s$  if and only if  $((1 + |k|^s)a_k) \in l^2$ . By a standard Sobolev embedding (here in dimension 1), such functions are  $(s - 1/2)$ -Hölder, provided  $s > 1/2$ . Hence, a cosine series (3.4) with coefficients  $a_k = O(k^{-(1+(1/\rho))})$  for  $k \rightarrow \infty$  is  $\alpha$ -Hölder for all  $\alpha < 1/\rho$ .

3.2. *Stability under approximation.* We now aim to prove stability of the estimates provided in Theorem 3.2 under approximations of  $\Psi$ . These stability properties will be used in Section 3.4 to prove the convergence (in rough path topology) of Galerkin and hyper-viscosity approximations of random Fourier series. Let us consider

$$(3.10) \quad \tilde{\Psi}(t) = \frac{\alpha_0 \beta_0 Y^0}{2} + \sum_{k=1}^{\infty} \alpha_k \beta_k Y^k \sin(kt) + \alpha_{-k} \beta_{-k} Y^{-k} \cos(kt),$$

with  $Y^k$  as above and  $(\alpha_k), (\beta_k)$  real-valued sequences. In the applications, the multiplication of the coefficients by  $\beta_k$  will correspond to a smoothing of  $\Psi$ . We thus aim to prove that the estimates given in Theorem 3.2 remain true uniformly for  $(b_k) = (\beta_k^2)$  in an appropriate class of sequences. This will naturally lead to the following:

DEFINITION 3.6. (1) A sequence  $(b_k)_{k \in \mathbb{Z}}$  is negligible if there are finite, signed, real Borel measures  $\mu_1, \mu_2$  on  $S^1 := \mathbb{R}/2\pi\mathbb{Z}$  such that

$$b_k = \int_0^{2\pi} \cos(kr) \mu_1(dr), \quad b_{-k} = \int_0^{2\pi} \cos(kr) \mu_2(dr) \quad \forall k \in \mathbb{N}.$$

(2) A family of sequences  $(b_k^r)$  is uniformly negligible if each  $(b_k^r)$  is negligible with associated measures  $\mu_1^r, \mu_2^r$  being uniformly bounded in total variation norm.

(3) For two bounded sequences  $(a_k), (c_k)$  we write  $(a_k) \leq (c_k)$  if there is a negligible sequence  $(b_k)$  such that  $a_k = c_k b_k$  for every  $k \in \mathbb{Z}$ .

EXAMPLE 3.7. Some (simple) examples of negligible sequences are:

- (1)  $(b_k) \equiv C$ , with  $\mu_1 = \mu_2 = C \delta_0$ ,
- (2)  $(b_k) \in l^1(\mathbb{Z})$ , with  $\mu_1 = \sum_{k=1}^{\infty} b_k \cos(kt) dt$  and  $\mu_2 = \sum_{k=1}^{\infty} b_{-k} \cos(kt) dt$ .

In the forthcoming Lemmas 3.13 and 3.14, we will give sufficient conditions for (uniform) negligibility.

As will be seen below, our results are uniform relative to “negligible” perturbations as in (3.10).

PROPOSITION 3.8. Consider the random Fourier series (3.10) with  $(a_k)$  satisfying the assumptions of Theorem 3.2. Let  $(b_k)$  be negligible. Then

$$V_{1,\rho}(R_{\tilde{\Psi}}; [s, t]^2) \leq C |t - s|^{1/\rho} \quad \forall [s, t]^2 \subseteq [0, 2\pi]^2.$$

The constant  $C$  depends only on  $\rho$ , the constant  $C_1 = \sup_{k \in \mathbb{Z}} a_k |k|^{1+1/\rho}$  and a constant  $C_2$  which bounds  $\|\mu_1\|_{TV}$  and  $\|\mu_2\|_{TV}$  with  $\mu_1, \mu_2$  corresponding to  $(b_k)$ ; cf. Definition 3.6.

This proposition is a special case of Proposition 3.9 below. Consider another random Fourier series

$$\Phi(t) = \frac{\gamma_0 Z^0}{2} + \sum_{k=1}^{\infty} \gamma_k Z^k \sin(kt) + \gamma_{-k} Z^{-k} \cos(kt),$$

and assume that the  $Z^k$  fulfill the same conditions as the  $Y^k$ . Furthermore, assume that  $\{Y^k, Z^k\}_{k \in \mathbb{Z}}$  are uncorrelated random variables, and set  $c_k := \gamma_k^2, \varrho_k := \mathbb{E}Y^k Z^k$  and

$$R_{\Psi, \Phi}(s, t) := \mathbb{E}\Psi(s)\Phi(t).$$

Then the following holds:

PROPOSITION 3.9. *Assume that there is a sequence  $(d_k)$  satisfying the assumptions of Theorem 3.2 such that*

$$(b_k) := \left( \frac{\alpha_k \gamma_k \varrho_k}{d_k} \right)$$

is negligible with associated measures  $\mu_1, \mu_2$ . Then

$$V_\rho(R_{\Psi, \Phi}; [s, t]^2) \leq V_{1, \rho}(R_{\Psi, \Phi}; [s, t]^2) \leq C|t - s|^{1/\rho} \quad \forall [s, t]^2 \subseteq [0, 2\pi]^2.$$

The constant  $C$  depends only on  $\rho$ , the constant  $C_1 = \sup_{k \in \mathbb{Z}} d_k |k|^{1+1/\rho}$  and a constant  $C_2$  which bounds  $\|\mu_1\|_{TV}$  and  $\|\mu_2\|_{TV}$ .

PROOF. Arguing as for Theorem 3.2 we observe

$$\begin{aligned} &V_{1, \rho}(R_{\Psi, \Phi}; [s, t] \times [u, v]) \\ &\lesssim V_{1, \rho}(R^-; [s, t] \times [u, v]) + V_{1, \rho}(R^+; [s, t] \times [u, v]) \\ &\quad + V_{1, \rho}(\tilde{R}^-; [s, t] \times [u, v]) + V_{1, \rho}(\tilde{R}^+; [s, t] \times [u, v]), \end{aligned}$$

with  $R^-(s, t) = K(t - s), R^+(s, t) = K(t + s), \tilde{R}^-(s, t) = \tilde{K}(t - s)$  and  $\tilde{R}^+(s, t) = \tilde{K}(t + s)$

$$K(t) := \frac{1}{2} \sum_{k=1}^{\infty} d_{-k} b_{-k} \cos(kt),$$

$$\tilde{K}(t) := \frac{1}{2} \sum_{k=1}^{\infty} d_k b_k \cos(kt).$$

We thus need to estimate the mixed  $(1, \rho)$ -variation of cosine series under multiplication with negligible sequences. In the following we consider  $R^\pm, \tilde{R}^\pm$  can be treated analogously. Let

$$R_0^\pm(t, s) := \frac{d_0}{2} + \sum_{k=1}^{\infty} d_{-k} \cos(k(t \pm s)).$$

We then apply Proposition 3.12 below with  $R_\mu^\pm = R^\pm$ ,  $R^\pm = R_0^\pm$ ,  $a_k = d_{-k}$ ,  $b_k = b_{-k}$  to obtain

$$V_{1,\rho}(R^\pm; [s, t]^2) \leq \|\mu\|_{\text{TV}} \sup_{0 \leq z \leq 2\pi} V_{1,\rho}(R_0^\pm; [s - z, t - z] \times [s, t])$$

for every  $[s, t] \subseteq [0, 2\pi]$ . By Theorem 3.2 applied to  $R_0^\pm$ , we have

$$\sup_{0 \leq z \leq 2\pi} V_{1,\rho}(R_0^\pm; [s - z, t - z] \times [s, t]) \leq C|t - s|^{1/\rho},$$

which completes the proof.  $\square$

In the following let  $\mathcal{M}(\mathcal{S}^1)$  be the space of signed, real Borel-measures on the circle  $\mathcal{S}^1$  with finite total variation  $\|\cdot\|_{\text{TV}}$ . Define  $\mathcal{M}^w(\mathcal{S}^1)$  to be  $\mathcal{M}(\mathcal{S}^1)$  endowed with the topology of weak convergence. For  $B \in L^1(\mathcal{S}^1)$  we set  $\mu_B := B dt \in \mathcal{M}(\mathcal{S}^1)$  to be the associated measure with density  $B$ .

LEMMA 3.10. *Let  $\mu \in \mathcal{M}(\mathcal{S}^1)$ ,  $R: \mathcal{S}^1 \times I \rightarrow \mathbb{R}$  and set  $R_\mu(s, t) := (R(\cdot, t) * \mu)(s)$ . Then*

$$V_{\gamma,\rho}(R_\mu; [s, t] \times [u, v]) \leq \|\mu\|_{\text{TV}} \sup_{x \in \mathcal{S}^1} V_{\gamma,\rho}(R; [s - x, t - x] \times [u, v])$$

for all  $[s, t] \times [u, v] \subseteq \mathcal{S}^1 \times I$  and  $1 \leq \gamma \leq \rho$ .

PROOF. Let  $(t_i), (t'_j)$  be partitions of  $[s, t]$ , respectively,  $[u, v]$ . From Jensen's inequality,

$$\begin{aligned} \left| R_\mu \left( \begin{matrix} t_i, t_{i+1} \\ t'_j, t'_{j+1} \end{matrix} \right) \right|^\gamma &\leq \left( \int_{\mathcal{S}^1} \left| R \left( \begin{matrix} t_i - x, t_{i+1} - x \\ t'_j, t'_{j+1} \end{matrix} \right) \right| d|\mu|(x) \right)^\gamma \\ &\leq \|\mu\|_{\text{TV}}^\gamma \int_{\mathcal{S}^1} \left| R \left( \begin{matrix} t_i - x, t_{i+1} - x \\ t'_j, t'_{j+1} \end{matrix} \right) \right|^\gamma d \frac{|\mu|(x)}{\|\mu\|_{\text{TV}}}. \end{aligned}$$

Summing over  $t_i$  and using again Jensen's inequality for  $\frac{\rho}{\gamma}$  yields

$$\begin{aligned} &\sum_{t'_j} \left( \sum_{t_i} \left| R_\mu \left( \begin{matrix} t_i, t_{i+1} \\ t'_j, t'_{j+1} \end{matrix} \right) \right|^\gamma \right)^{\rho/\gamma} \\ &\leq \|\mu\|_{\text{TV}}^\rho \int_{\mathcal{S}^1} \sum_{t'_j} \left( \sum_{t_i} \left| R \left( \begin{matrix} t_i - x, t_{i+1} - x \\ t'_j, t'_{j+1} \end{matrix} \right) \right|^\gamma \right)^{\rho/\gamma} d \frac{|\mu|(x)}{\|\mu\|_{\text{TV}}} \\ &\leq \|\mu\|_{\text{TV}}^\rho \int_{\mathcal{S}^1} V_{\gamma,\rho}^\rho(R; [s - x, t - x] \times [u, v]) d \frac{|\mu|(x)}{\|\mu\|_{\text{TV}}} \\ &\leq \|\mu\|_{\text{TV}}^\rho \sup_{x \in \mathcal{S}^1} V_{\gamma,\rho}^\rho(R; [s - x, t - x] \times [u, v]). \end{aligned}$$

Taking the supremum over all partitions yields the inequality.  $\square$

REMARK 3.11. In many cases,  $x \mapsto V_{\gamma,\rho}(R; [s-x, t-x] \times [s, t])$  attains its maximum at  $x = 0$ . In this case our inequality above reads

$$V_{\gamma,\rho}(R * \mu; [s, t]^2) \leq \|\mu\|_{TV} V_{\gamma,\rho}(R; [s, t]^2)$$

for all squares  $[s, t]^2 \subseteq [0, 2\pi]^2$ . Lemma 3.10 can thus be interpreted as a Young-inequality for the mixed  $(\gamma, \rho)$ -variation of a function with two arguments. If  $\mu = \delta_0$ , we have  $b_k = 1$  for every  $k$  and the estimate is thus sharp.

PROPOSITION 3.12. Let  $R_\mu^+, R_\mu^- : [0, 2\pi]^2 \rightarrow \mathbb{R}$  be continuous functions of the form

$$R_\mu^\pm(s, t) = \frac{a_0 b_0}{2} + \sum_{k=1}^\infty a_k b_k \cos(k(s \pm t))$$

with  $a_k, b_k$  being real-valued coefficients such that  $\sum_{k=1}^\infty |a_k| < \infty$ , and assume that there is a measure  $\mu \in \mathcal{M}(\mathcal{S}^1)$  such that

$$b_k = \int_0^{2\pi} \cos(kr) \mu(dr).$$

Set

$$R^\pm(t, s) = \frac{a_0}{2} + \sum_{k=1}^\infty a_k \cos(k(t \pm s)).$$

Then for every  $1 \leq \gamma \leq \rho$ ,

$$V_{\gamma,\rho}(R_\mu^\pm; [s, t] \times [u, v]) \leq \|\mu\|_{TV} \sup_{0 \leq z \leq 2\pi} V_{\gamma,\rho}(R^\pm; [s-z, t-z] \times [u, v])$$

for every  $[s, t] \times [u, v] \subseteq [0, 2\pi]^2$ .

PROOF. Let  $a_{-k} := a_k, b_{-k} := b_k$  for  $k \in \mathbb{N}$ . Since  $\sum_{k=1}^\infty |a_k| < \infty$ , we observe

$$R_\mu^\pm(s, t) = \frac{1}{2} \sum_{k \in \mathbb{Z}} a_k b_k e^{ik(t \pm s)} = (R^\pm(\cdot, t) * \mu)(s)$$

and the estimate is thus a direct consequence from Lemma 3.10.  $\square$

3.3. (Uniform) negligibility. In order to use Proposition 3.12 to control the  $(\gamma, \rho)$ -variation of  $R(s, t)$ , we need to control  $\|\mu\|_{TV}$ . We recall the following:

LEMMA 3.13. Let  $\{b_k\}_{k \in \mathbb{N}}$  be a sequence satisfying  $b_k \rightarrow b \in \mathbb{R}$  for  $k \rightarrow \infty$ , and let  $S_n(t) := \frac{b_0}{2} + \sum_{k=1}^n b_k \cos(kt)$ . Assume one of the following conditions:

- (1)  $\sum_{k=1}^\infty |b_k - b| < \infty$ ;
- (2) there exists a nonincreasing sequence  $A_k$  such that  $\sum_{k=0}^\infty A_k < \infty$  and  $|\Delta b_k| \leq A_k$  for all  $k \geq 0$ ;

(3)  $b_k$  is quasi-convex, that is,

$$\sum_{k=0}^{\infty} (k+1) |\Delta^2 b_k| < \infty.$$

Then,  $B(t) = \frac{b_0}{2} + \sum_{k=1}^{\infty} b_k \cos(kt)$  exists locally uniformly on  $(0, 2\pi)$ , and the right-hand side is the Fourier series of  $B$ . Moreover,

$$\mu_{S_n} \rightharpoonup \mu_B + b\delta_0 =: \mu \quad \text{weakly in } \mathcal{M}(\mathcal{S}^1)$$

and  $b_k = \int_0^{2\pi} \cos(kr) \mu(dr)$ . Moreover, there is a numerical constant  $C > 0$  such that

$$(3.11) \quad \|\mu\|_{\text{TV}} \leq |b| + C \begin{cases} \sum_{k=0}^{\infty} |b_k - b|, & \text{in case (1),} \\ \sum_{k=0}^{\infty} A_k, & \text{in case (2),} \\ \sum_{k=0}^{\infty} (k+1) |\Delta^2 b_k|, & \text{in case (3).} \end{cases}$$

PROOF. The case  $b = 0$  is classical [(1) is trivial; cf. [45] for (2) and [32] for (3)]. The case  $b \neq 0$  may be reduced to  $b = 0$  by noting that  $bD_n(t) \rightarrow 2\pi b\delta_0$  in  $\mathcal{M}^w(\mathcal{S}^1)$ , where  $D_n$  is the Dirichlet kernel.  $\square$

Lemma 3.13 in combination with Proposition 3.12 allows us to derive bounds on the  $\rho$ -variation of covariance functions of the type discussed here, depending on  $\mu$  only via its total variation norm. Since we will use this to prove uniform estimates, we will need the following uniform estimates on the  $L^1$ -norm of cosine series.

LEMMA 3.14. Let  $b \in C^1(0, \infty)$  with  $b(r) \rightarrow 0$  for  $r \rightarrow \infty$  and  $b_k^\tau := b(\tau^m k)$  for some  $\tau, m > 0$ . If:

- (1)  $b$  is convex, nonincreasing, then  $b_k^\tau$  satisfies the assumptions of Lemma 3.13;
- (2)  $B^\tau(t) = \frac{b_0^\tau}{2} + \sum_{k=1}^{\infty} b_k^\tau \cos(kt)$  exists locally uniformly in  $(0, 2\pi)$  and

$$\|B^\tau\|_{L^1([0, 2\pi])} \leq C b_0,$$

for some  $C > 0$ ;

- (3)  $b \in C^2(0, \infty)$  with  $r \mapsto r|b''(r)|$  being integrable, then  $b_k^\tau$  satisfies the assumptions of Lemma 3.13, (3) and

$$\|B^\tau\|_{L^1([0, 2\pi])} \leq C \int_0^\infty r|b''(r)| dr,$$

for some  $C > 0$  with  $B^\tau$  as in (1).

PROOF. (1) Since  $b, |b'|$  are nonincreasing  $\Delta b_k^\tau \leq 0$  and  $-\Delta b_k$  is nonincreasing. We set  $A_k := -\Delta b_k$ . Clearly,  $\sum_{k=0}^\infty A_k = 2b_0$ , and the claim follows from Lemma 3.13.

(2) Let  $b^\tau(r) := b(\tau^m r)$ , and observe

$$\Delta^2 b_k^\tau = \int_{k+1}^{k+2} \int_{s-1}^s (b^\tau)''(r) dr ds.$$

Since  $(b^\tau)'' dr = \tau^m b''(\tau^m r) d(\tau^m r)$ , elementary calculations show

$$\sum_{k=0}^\infty (k+1) |\Delta^2 b_k^\tau| \leq 2 \int_0^\infty r |b''(r)| dr,$$

and Lemma 3.13 completes the proof.  $\square$

EXAMPLE 3.15. As an application of Lemma 3.13, we see that the sequence  $(b_k) = (k^{-\alpha})$ ,  $\alpha > 0$ , is negligible. Furthermore, the sequence  $(b_k) = (e^{-\tau k^\alpha})$ ,  $\alpha, \tau > 0$ , is uniformly negligible in  $\tau$  which follows from Lemma 3.14.

3.4. *Random Fourier series as rough paths.* We now return to the initial problem of showing the existence of a lift to vector-valued versions of (3.1) to a process with values in a rough paths space.

Recall that we write  $(a_k) \leq (b_k)$  for two sequences  $(a_k)$  and  $(b_k)$  if there is a negligible sequence  $(c_k)$  such that  $a_k = c_k b_k$ ; cf. Definition 3.6. We will extend this notation as follows: if  $(A_k) = (a_k^{i,j})$  is a sequence of matrices, and  $(b_k)$  is a sequence of real numbers,  $(A_k) \leq (b_k)$  means that  $(a_k^{i,j}) \leq (b_k)$  for every  $i, j$ . If  $(A_k) = (A_k^1, \dots, A_k^m)$  is a sequence of vectors whose entries are matrices or real numbers, we will write  $(A_k) \leq (b_k)$  if  $(A_k^i) \leq (b_k)$  for all  $i = 1, \dots, m$ .

Let  $\Psi = (\Psi^1, \dots, \Psi^d)$  where the  $\Psi^i$  are given as random Fourier series

$$(3.12) \quad \Psi^i(t) = \frac{\alpha_0^i Y^{0,i}}{2} + \sum_{k=1}^\infty \alpha_k^i Y^{k,i} \sin(kt) + \alpha_{-k}^i Y^{-k,i} \cos(kt),$$

with  $(Y^{k,i})_{k \in \mathbb{Z}, i=1, \dots, d}$  being independent,  $\mathcal{N}(0, 1)$  distributed random variables. As before, set  $a_k^i := (\alpha_k^i)^2$  and  $(a_k) := (a_k^1, \dots, a_k^d)$ . Our main existence result is the following:

THEOREM 3.16. *Assume  $(a_k) \leq (|k|^{-(1+1/\rho)})$  for some  $\rho \in [1, 2)$  with associated measures  $\mu_1^i, \mu_2^i$ ,  $i = 1, \dots, d$ , as in Definition 3.6, and let  $K \geq \max_{i=1, \dots, d} \{\|\mu_1^i\|_{TV}, \|\mu_2^i\|_{TV}\}$ . Then for every  $\beta < \frac{1}{2\rho}$ , there exists a continuous  $G^{[1/\beta]}(\mathbb{R}^d)$ -valued process  $\Psi$  such that:*

(1)  $\Psi$  has geometric  $\beta$ -Hölder rough sample paths, that is,

$$\Psi \in C_0^{0, \beta\text{-Hö}}([0, 2\pi], G^{[1/\beta]}(\mathbb{R}^d))$$

almost surely,

- (2)  $\Psi$  lifts  $\Psi$  in the sense that  $\pi_1(\Psi_t) = \Psi_t - \Psi_0$ ,
- (3) there is a  $C = C(\rho, K)$  such that for all  $s < t$  in  $[0, 2\pi]$  and  $q \in [1, \infty)$ ,

$$|d(\Psi_s, \Psi_t)|_{L^q} \leq C\sqrt{q}|t - s|^{1/(2\rho)},$$

- (4) there exists  $\eta = \eta(\rho, K, \beta) > 0$ , such that

$$\mathbb{E}e^{\eta\|\Psi\|_{\beta\text{-Hö};[0,2\pi]^2}} < \infty.$$

PROOF. By assumption,

$$\begin{aligned} \Psi^i(t) &= \frac{\gamma_0^i Y^{0,i}}{2} + \sum_{k=1}^{\infty} \gamma_k^i |k|^{-(1/2+1/(2\rho))} Y^{k,i} \sin(kt) \\ &\quad + \gamma_{-k}^i |k|^{-(1/2+1/(2\rho))} Y^{-k,i} \cos(kt) \end{aligned}$$

for every  $i = 1, \dots, d$  where  $(c_k^i) = ((\gamma_k^i)^2)$  is a negligible sequence. Hence, we may apply Proposition 3.8 to see that the covariance of  $\Psi^i$  has finite Hölder dominated  $\rho$ -variation for every  $i$ ; thus [17], Theorem 35, applies.  $\square$

We will now compare the lifts of two random Fourier series  $\Psi = (\Psi^1, \dots, \Psi^d)$  and  $\tilde{\Psi} = (\tilde{\Psi}^1, \dots, \tilde{\Psi}^d)$  with

$$\begin{aligned} \Psi^i(t) &= \frac{\alpha_0^i Y^{0,i}}{2} + \sum_{k=1}^{\infty} \alpha_k^i Y^{k,i} \sin(kt) + \alpha_{-k}^i Y^{-k,i} \cos(kt), \\ \tilde{\Psi}^i(t) &= \frac{\tilde{\alpha}_0^i \tilde{Y}^{0,i}}{2} + \sum_{k=1}^{\infty} \tilde{\alpha}_k^i \tilde{Y}^{k,i} \sin(kt) + \tilde{\alpha}_{-k}^i \tilde{Y}^{-k,i} \cos(kt). \end{aligned}$$

We make the following assumption:

$$\{(Y^{k,i}, \tilde{Y}^{k,i}) : k \in \mathbb{Z}, i = 1, \dots, d\}$$

are independent, normally distributed random vectors with  $Y^{k,i}, \tilde{Y}^{k,i} \sim \mathcal{N}(0, 1)$  for all  $k \in \mathbb{Z}$  and  $i = 1, \dots, d$ . It follows that  $\mathbb{E}Y^{k,i} \tilde{Y}^{l,j} = 0$  for  $k \neq l$  or  $i \neq j$ , and we set  $\varrho_k^i := \mathbb{E}Y^{k,i} \tilde{Y}^{k,i}$ . As before, let  $a_k^i := (\alpha_k^i)^2$  and  $\tilde{a}_k^i := (\tilde{\alpha}_k^i)^2$ . Define the matrix

$$A_k^i := \begin{pmatrix} a_k^i & \alpha_k^i \tilde{\alpha}_k^i \varrho_k^i \\ \alpha_k^i \tilde{\alpha}_k^i \varrho_k^i & \tilde{a}_k^i \end{pmatrix},$$

and set  $A_k := (A_k^1, \dots, A_k^d)$ .

**THEOREM 3.17.** Assume that  $(A_k) \leq (|k|^{-(1+1/\rho)})$  for some  $\rho \in [1, 2)$  and that the total variation of all associated measures is bounded by a constant  $K$ . Then we can lift  $\Psi$  and  $\tilde{\Psi}$  to processes with values in a rough paths space as

in Theorem 3.16, and for all  $\gamma < 1 - \frac{\rho}{2}$  and  $\beta < \frac{1}{\rho}(\frac{1}{2} - \gamma)$  there is a constant  $C = C(\rho, K, \beta, \gamma)$  such that

$$(3.13) \quad |\rho_{\beta\text{-H\"{o}l}}(\Psi, \tilde{\Psi})|_{L^q} \leq Cq^{(1/2)\lfloor 1/\beta \rfloor} \left( \sup_{t \in [0, 2\pi]} \mathbb{E}|\Psi(t) - \tilde{\Psi}(t)|^2 \right)^\gamma$$

for all  $q \in [1, \infty)$ .

PROOF. The existence of the lifted processes  $\Psi$  and  $\tilde{\Psi}$  follows from Theorem 3.16. The  $L^q$  norm of the difference of two such processes in rough paths metric can be estimated by the  $\rho$ -variation of the covariance of the difference of the two processes, and an interpolation argument shows that this quantity can actually be bounded by the right-hand side of (3.13) times the  $\rho$ -variation of the covariance of the two processes and their joint covariance function. We aim to apply [15], Theorem 5,<sup>20</sup> where the estimate (3.13) was given for the optimal parameter  $\gamma$ . To obtain a uniform estimate, we need to show that the joint covariance function of the process  $(\Psi, \tilde{\Psi})$  has finite, Hölder dominated  $\rho$ -variation, bounded by a constant depending only on the parameters above. From independence of the components, it suffices to estimate the  $\rho$ -variation of  $R_{\Psi^i, \tilde{\Psi}^i}(s, t) = \mathbb{E}\Psi^i(s)\tilde{\Psi}^i(t)$  for every  $i = 1, \dots, d$ . This can be done using Proposition 3.9.  $\square$

As an application, we consider the *truncated random Fourier series*, that is, we define  $\Psi^N = (\Psi^{1,N}, \dots, \Psi^{d,N})$  by

$$(3.14) \quad \Psi^{i,N}(t) = \frac{\alpha_0^i Y^{0,i}}{2} + \sum_{k=1}^N \alpha_k^i Y^{k,i} \sin(kt) + \alpha_{-k}^i Y^{-k,i} \cos(kt)$$

for  $i = 1, \dots, d$ .

It is then easy to show that convergence also holds for the corresponding rough paths lifts, and we can even give an upper bound for the order of convergence.

COROLLARY 3.18. *Under the assumptions of Theorem 3.16, choose some  $\eta < \frac{1}{\rho} - \frac{1}{2}$  and  $\beta < \frac{1}{2\rho} - \eta$ . Then there is a constant  $C = C(\rho, K, \beta, \eta)$  such that*

$$|\rho_{\beta\text{-H\"{o}l}}(\Psi, \Psi^N)|_{L^q} \leq Cq^{(1/2)\lfloor 1/\beta \rfloor} \left( \frac{1}{N} \right)^\eta$$

for every  $N \in \mathbb{N}$ ,  $q \in [1, \infty)$ . In particular,  $\rho_{\beta\text{-H\"{o}l}}(\Psi, \Psi^N) \rightarrow 0$  for  $N \rightarrow \infty$  almost surely and in  $L^q$  for any  $q \in [1, \infty)$  with rate  $\eta$ .

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<sup>20</sup>Strictly speaking, [15], Theorem 5, assumes that  $\tilde{\Psi}$  is a certain approximation of  $\Psi$ . However, it is shown in [43] that this is not necessary, and ([15], Theorem 5) can be used more generally to give an upper bound for the distance between  $\Psi$  and  $\tilde{\Psi}$  as we need it here.

REMARK 3.19. We emphasize that  $\Psi, \Psi^N$  are lifted to level  $\lfloor 1/\beta \rfloor$  above. In particular, a “good” rate  $\eta$  forces  $\beta$  to be small so that, in general, it is not enough to work with 3 levels, as is the usual setting in Gaussian rough paths theory.

PROOF OF COROLLARY 3.18. We aim to apply Theorem 3.17 with  $\tilde{\alpha}_k^i = \mathbb{1}_{|k| \leq N} \alpha_k^i$  and  $\varrho_k^i \equiv 1$ . We will first show that  $(a_k^i \mathbb{1}_{|k| \leq N}) \leq (|k|^{-(1+1/\rho')})$  for every  $\rho' > \rho$ , uniformly over  $i$  and  $N$ . Indeed, we have

$$a_k^i \mathbb{1}_{|k| \leq N} = (a_k^i |k|^{1+1/\rho}) (|k|^{1/\rho' - 1/\rho} \mathbb{1}_{|k| \leq N}) |k|^{-(1+1/\rho')},$$

and since  $(a_k^i) \leq (|k|^{-(1+1/\rho)})$  for all  $i = 1, \dots, d$ , it suffices to show that  $(|k|^{-\varepsilon} \mathbb{1}_{|k| \leq N})$  is uniformly negligible for every  $\varepsilon > 0$ . Therefore, we need to show that the cosine series

$$B^N(x) = \sum_{k=1}^{\infty} |k|^{-\varepsilon} \mathbb{1}_{|k| \leq N} \cos(kx) = \sum_{k=1}^N |k|^{-\varepsilon} \cos(kx)$$

is uniformly bounded in  $L^1([0, 2\pi])$ . Since  $\Delta k^{-\varepsilon} = O(k^{-\varepsilon-1})$  and  $\lim_{k \rightarrow \infty} \log(k) k^{-\varepsilon} = 0$ , we can apply the Sidon–Telyakovskii theorem (cf. [45], Theorem 4) to obtain  $B^N \rightarrow B$  for  $N \rightarrow \infty$  in  $L^1([0, 2\pi])$  which proves the uniform negligibility, and we may apply Theorem 3.17 for every  $\rho' > \rho$ . Furthermore,

$$\begin{aligned} \mathbb{E}|\Psi(t) - \Psi^N(t)|^2 &= \sum_{k=N+1}^{\infty} a_k \sin^2(kt) + a_{-k} \cos^2(kt) \\ &\leq 2 \sum_{|k| \geq N+1} a_k \lesssim 4 \sum_{k=N+1}^{\infty} k^{-(1+1/\rho)} \lesssim \left(\frac{1}{N}\right)^{1/\rho}. \end{aligned}$$

For given  $\eta$ , we choose  $\rho'$  such that  $\eta < \frac{1}{\rho'} - \frac{1}{2} < \frac{1}{\rho} - \frac{1}{2}$  and apply Theorem 3.17 to complete the  $L^q$  convergence. The almost sure convergence follows by a standard Borel–Cantelli argument; cf. [15], Theorem 6, page 41.  $\square$

**4. Applications to SPDE.** In this section we will apply our results on random Fourier series to construct spatial rough path lifts of stationary Ornstein–Uhlenbeck processes corresponding to the  $\mathbb{R}^d$ -valued (generalized) fractional stochastic heat equation with Dirichlet, Neumann or periodic boundary conditions

$$(4.1) \quad d\Psi_t = -(-\Delta)^\alpha \Psi_t dt + dW_t \quad \text{on } [0, T] \times [0, 2\pi],$$

where the fractional Laplacian  $(-\Delta)^\alpha$  acts on each component of  $\Psi_t$  and  $\alpha \in (0, 1]$ . We will start by first considering the fractional stochastic heat equation with Dirichlet boundary conditions, proving the existence of (continuous) spatial rough paths lifts and stability under approximations. Then we comment on Neumann boundary conditions and on more general equations for periodic boundary conditions.

If a (spatial) rough path lift of (4.1) has been constructed, one can view (4.1) as an evolution in a rough path space, a point of view which has proven extremely fruitful in solving new classes of, until now, ill-posed stochastic PDE [22, 23, 27], arising, for example, in path sampling problems for  $\mathbb{R}^d$ -valued SDE [22, 25, 26].

For a variant of (4.1) with  $\alpha = 1$ , Hairer proved in [22] finite 1-variation of the covariance of the stationary solution to (4.1), that is, of  $(x, y) \mapsto \mathbb{E}\Psi(t, x)\Psi(t, y)$ . This general theory then gives the existence of a “canonical, level 2” rough path  $\Psi$  lifting  $\Psi$ ; cf. Theorem 3.16; see also [21]. It is clear that in the case  $\alpha = 1$  the Brownian-like regularity of  $x \mapsto \Psi(t, x; \omega)$  is due to the competition between the smoothing effects of the Laplacian and the roughness of space–time white noise. Truncation of the higher noise modes (or suitable “coloring”) leads to better spatial regularity; on the other hand, replacing  $\Delta$  by a fractional Laplacian, that is, considering (4.1) for some  $\alpha \in (0, 1)$ , dampens the smoothing effect, and  $x \mapsto \Psi(t, x; \omega)$  will have “rougher” regularity properties than a standard Brownian motion. One thus expects  $\rho$ -variation regularity for the spatial covariance of  $x \mapsto \Psi(t, x; \omega)$  for (4.1) only for some  $\rho > 1$  and subsequently only the existence of a “rougher” rough path, that is, necessarily with higher  $p$  than before.

As we shall see below, (4.1) is handled as a spatial rough path with a number of precise estimates, provided

$$\alpha > \alpha^* = \frac{3}{4}.$$

More precisely, the resulting (geometric rough) path enjoys  $\frac{1}{p}$ -Hölder regularity for any  $p > 2\rho = \frac{2}{2\alpha-1}$ . When  $\alpha > \frac{5}{6}$  we have  $\rho = \frac{1}{2\alpha-1} < \frac{3}{2}$  and can pick  $p < 3$ . The resulting rough path can then be realized as a “level 2” rough path. In the general case (similar to  $H \in (\frac{1}{4}, \frac{1}{3}]$  in the fBm setting) one must go beyond the stochastic area and control the third level iterated integrals. Our approach, which crucially passes through  $\rho$ -variation, combined with existing theory, has many advantages: the notoriously difficult third-level computation need not be repeated in the present context; leave alone the higher level computations needed for rates. A satisfactory approximation theory is also available, based on *uniform*  $\rho$ -variation estimates; cf. Section 4.2 below.

4.1. *Fractional stochastic heat equation with Dirichlet boundary conditions.* We consider

$$(4.2) \quad d\Psi_t = -(-\Delta)^\alpha \Psi_t dt + dW_t \quad \text{on } [0, T] \times [0, 2\pi]$$

on  $[0, 2\pi]$  endowed with Dirichlet boundary conditions. Neumann and periodic boundary conditions may be treated analogously; cf. Section 4.3 below. We have the following orthogonal basis of eigenvectors with corresponding eigenvalues of  $-\Delta$  on  $L^2([0, 2\pi])$ :

$$e_k(x) = \sin\left(\frac{k}{2}x\right), \quad \tau_k = \left(\frac{k}{2}\right)^2, \quad k \in \mathbb{N},$$

and take  $W_t = \sum_{k \in \mathbb{N}} \beta_t^k e_k(x)$ . The fractional Laplacian has eigenvalues  $\lambda_k := \tau_k^\alpha$  for  $k \in \mathbb{N}$  and (informal) Fourier expansion of the stationary solution  $\Psi$  to (4.1) leads to the random Fourier series

$$(4.3) \quad \Psi(t, x) = \sum_{k=1}^{\infty} \alpha_k Y_t^k \sin\left(k \frac{x}{2}\right),$$

with  $\alpha_k = \frac{1}{\sqrt{2\lambda_k}}$  and  $Y_t^k$  being a decoupled, infinite system of  $d$ -dimensional, stationary, normalized Ornstein–Uhlenbeck processes satisfying

$$(4.4) \quad dY_t^k = -\lambda_k Y_t^k dt + \sqrt{2\lambda_k} d\beta_t^k.$$

Clearly (4.3) gives a well-defined and continuous random field and solves (4.2) in the sense of standard SPDE theory; cf., for example, [8, 47]. Note  $\mathbb{E}Y_t^k \otimes Y_s^l = e^{-\lambda_k|t-s|} \delta_{k,l} \text{Id}$ , and set

$$a_k = \alpha_k^2 = \frac{1}{2\lambda_k} = 2^{2\alpha-1} \frac{1}{k^{2\alpha}}, \quad k \in \mathbb{N}.$$

As an immediate consequence of our results on random Fourier series, we get the following:

PROPOSITION 4.1. *Suppose  $\alpha \in (\frac{1}{2}, 1]$ . Then:*

(1) *For every  $t \geq 0$ , the spatial process  $x \mapsto \Psi(t, x)$  is a centered Gaussian process which admits a continuous modification (which we denote by the same symbol) with covariance  $R_\Psi$  of finite mixed  $(1, \rho)$ -variation for all  $\rho \geq \frac{1}{2\alpha-1}$ , and all conclusions of Theorem 3.2 hold.*

(2) *If  $\alpha > \frac{3}{4}$ , the process  $x \mapsto \Psi(t, x)$  lifts to a process with geometric  $\beta$ -Hölder rough paths*

$$\Psi(t) \in C_0^{0, \beta\text{-Hö}}([0, 2\pi], G^{\lfloor 1/\beta \rfloor}(\mathbb{R}^d))$$

*almost surely for every  $\beta < \alpha - \frac{1}{2}$ .*

(3) *Choose  $\gamma$  and  $\beta$  such that*

$$\gamma < 1 - \frac{3}{4\alpha}, \quad \beta < \alpha - \frac{1}{2} - \frac{2\alpha\gamma}{2\alpha - 1}.$$

*Then there is a  $\gamma$ -Hölder continuous modification of the map*

$$(4.5) \quad \begin{aligned} \Psi : [0, T] &\rightarrow C_0^{0, \beta\text{-Hö}}([0, 2\pi], G^{\lfloor 1/\beta \rfloor}(\mathbb{R}^d)), \\ t &\mapsto \Psi(t). \end{aligned}$$

REMARK 4.2. In (3), we observe a “trade-off” between the parameters  $\beta$  and  $\gamma$ : If we want a “good” time regularity (i.e., large  $\gamma$ ), we have to take  $\beta$  small which is tantamount to working in a rough paths space with many “levels”

of formal iterated integrals. For instance, when  $\alpha = 1$ , we can get arbitrarily close to  $\frac{1}{4}$  in time, at the price of working with many arbitrary levels. On the other hand, if we insist to work with the first 3 levels only (or 2 levels in case  $\alpha > 5/6$ ), which is the standard setting in Gaussian rough path theory, we only get poor time regularity of the evolution in rough path space.

PROOF OF PROPOSITION 4.1. Since  $\Psi$  is a rescaling of

$$\tilde{\Psi}(t, x) = \sum_{k=1}^{\infty} \alpha_k Y_t^k \sin(kx) = \Psi(t, 2x),$$

it is enough to consider  $\tilde{\Psi}$ :

(1) Clearly  $x \mapsto \tilde{\Psi}(t, x)$  is centered and Gaussian. Due to (3.3) and Lemma 3.4, we have

$$\sigma_t^2(x, y) = \mathbb{E}|\tilde{\Psi}(t, x) - \tilde{\Psi}(t, y)|^2 \lesssim |x - y|^{2\alpha-1},$$

which implies that there is a continuous modification of  $\tilde{\Psi}$ . Theorem 3.2 implies the claim.

(2) Follows from Theorem 3.16.

(3) We will derive the existence of a continuous modification by application of Kolmogorov’s continuity theorem. Therefore, we need an estimate on a  $q$ th moment of the distance in the  $\rho_{\beta\text{-H}\ddot{o}l}$  metric of the rough paths  $\tilde{\Psi}(t), \tilde{\Psi}(s)$  at different times  $0 \leq s < t \leq T$ . Such an estimate can be obtained by applying Theorem 3.17. Let  $0 \leq s \leq t \leq T$ ,  $\tau := |t - s|$ , and set  $A_k = (A_k^1, \dots, A_k^d)$  where

$$A_k^i := \begin{pmatrix} a_k & a_k e^{-\lambda_k \tau} \\ a_k e^{-\lambda_k \tau} & a_k \end{pmatrix}$$

for  $i = 1, \dots, d$ . We claim that  $(A_k) \leq (|k|^{-2\alpha})$  uniformly in  $\tau$ . Defining  $b(r) := e^{-(r/2)^{2\alpha}}$  we note  $b_k^\tau = e^{-\lambda_k \tau} = b(k\tau^{1/(2\alpha)})$ , and  $b$  is convex, nonincreasing. Lemma 3.14 then implies that  $(e^{-\lambda_k \tau})$  is uniformly negligible which shows the claim. Hence, we can apply Theorem 3.17 and obtain

$$|\rho_{\beta\text{-H}\ddot{o}l}(\tilde{\Psi}(t), \tilde{\Psi}(s))|_{L^q} \leq Cq^{(1/2)\lfloor 1/\beta \rfloor} \left( \sup_{x \in [0, 2\pi]} \mathbb{E}|\tilde{\Psi}(t, x) - \tilde{\Psi}(s, x)|^2 \right)^\theta$$

for all  $\theta < \frac{4\alpha-3}{4\alpha-2}$ ,  $\beta < \alpha - \frac{1}{2} - \theta$  and all  $q \in [1, \infty)$ . In order to estimate the right-hand side, we note

$$\begin{aligned} & \mathbb{E}|\tilde{\Psi}^1(t, x) - \tilde{\Psi}^1(s, x)|^2 \\ &= \mathbb{E}|\tilde{\Psi}^1(t, x)|^2 + \mathbb{E}|\tilde{\Psi}^1(s, x)|^2 - 2\mathbb{E}\tilde{\Psi}^1(t, x)\tilde{\Psi}^1(s, x) \\ &\leq 2 \sum_{k=1}^{\infty} a_k (1 - e^{-\lambda_k \tau}) \sin^2(kx) \leq 2 \sum_{k=1}^{\infty} a_k |1 - e^{-\lambda_k \tau}| \\ &\leq C \sum_{k \leq N} |t - s| + CN^{1-2\alpha'} \sum_{k > N} a_k k^{2\alpha'-1} \leq C(N|t - s| + N^{1-2\alpha'}) \end{aligned}$$

for all  $\alpha' < \alpha$ . We then choose  $N \sim |t - s|^{-1/(2\alpha')}$  to obtain

$$\mathbb{E}|\tilde{\Psi}(t, x) - \tilde{\Psi}(s, x)|^2 \leq C|t - s|^{1-1/(2\alpha')}.$$

Thus we can choose  $\gamma < 1 - \frac{3}{4\alpha}$  and  $\beta < \alpha - \frac{1}{2} - \frac{2\alpha\gamma}{2\alpha-1}$  to obtain

$$|\rho_{\beta\text{-H\"{o}l}}(\tilde{\Psi}(t), \tilde{\Psi}(s))|_{L^q} \leq Cq^{(1/2)\lfloor 1/\beta \rfloor} |t - s|^\gamma,$$

for all  $q \in [1, \infty)$ . Kolmogorov’s continuity theorem gives the result.  $\square$

4.2. *Stability and approximations.* Due to the “contraction principle” in the form of Proposition 3.12, the estimates on the  $\rho$ -variation of the covariance of random Fourier series derived in Section 3 are robust with respect to approximations. In order to emphasize this point, in this section we consider Galerkin and hyperviscosity approximations to  $\Psi$  with  $\Psi$  as in Section 4.1 and prove strong convergence of the corresponding rough paths lifts. Recall that by the general theory of rough paths, this immediately implies the strong convergence of the corresponding stochastic integrals as well as of solutions to rough differential equations; cf., for example, [2, 22].

4.2.1. *Galerkin approximations.* The Galerkin approximation  $\Psi_t^N$  of  $\Psi_t$  is defined to be the projection of  $\Psi$  onto the  $N$ -dimensional subspace spanned by  $\{e_k\}_{k=1, \dots, N}$ . This process solves the SPDE

$$(4.6) \quad d\Psi_t^N = -(P_N(-\Delta)^\alpha)\Psi_t^N dt + dP_N W_t,$$

where  $P_N(-\Delta)^\alpha$  has the eigenvalues  $(\frac{k}{2})^{2\alpha}\mathbb{1}_{k \leq N}$ , and  $P_N W_t$  has the covariance operator  $Q^N$  given by  $Q^N e_k = \mathbb{1}_{k \leq N} e_k$ . The process  $\Psi^N$  can be written as the truncated Fourier series

$$\Psi^N(t, x) = \sum_{k=1}^N \alpha_k Y_t^k \sin\left(k \frac{x}{2}\right),$$

with  $\alpha_k = 2^{\alpha-1/2} k^{-\alpha}$  and  $Y^k$  Ornstein–Uhlenbeck processes as in Section 4.1.

One easily checks that we can lift the spatial sample paths of  $\Psi_t^N$  to Gaussian rough paths and find continuous modifications of  $t \mapsto \Psi_t^N$ . Moreover, we can prove the following strong convergence result:

PROPOSITION 4.3. *Assume  $\alpha > \frac{3}{4}$ , and choose  $\eta < 2\alpha - \frac{3}{2}$  and  $\beta < \alpha - \frac{1}{2} - \eta$ . Then there is a constant  $C = C(\alpha, \beta, \eta)$  such that*

$$|\rho_{\beta\text{-H\"{o}l}}(\Psi(t), \Psi^N(t))|_{L^q} \leq Cq^{(1/2)\lfloor 1/\beta \rfloor} \left(\frac{1}{N}\right)^\eta$$

for all  $t \in [0, T]$ ,  $N \in \mathbb{N}$ ,  $q \in [1, \infty)$ . In particular, for every  $t \in [0, T]$ ,  $\rho_{\beta\text{-H\"{o}l}}(\Psi(t), \Psi^N(t)) \rightarrow 0$  for  $N \rightarrow \infty$  almost surely and in  $L^q$  for any  $q \in [1, \infty)$  with rate  $\eta$ .

PROOF. The proof follows from Corollary 3.18.  $\square$

4.2.2. *Hyper-viscosity approximations.* The hyper-viscosity approximation  $\Psi^\varepsilon = (\Psi^{\varepsilon,1}, \dots, \Psi^{\varepsilon,d})$  is the solution to

$$(4.7) \quad d\Psi_t^\varepsilon = -((-\Delta)^\alpha + \varepsilon(-\Delta)^\theta)\Psi_t^\varepsilon dt + dW_t,$$

for some (large)  $\theta \geq 1$  and  $\varepsilon > 0$ . Again, it is easy to see that we can lift the spatial sample paths of  $\Psi_t^\varepsilon$  to Gaussian rough paths and find continuous modifications of  $t \mapsto \Psi_t^\varepsilon$ .

PROPOSITION 4.4. *Assume  $\alpha > \frac{3}{4}$  and  $\theta > \alpha$ . Choose  $\beta < \alpha - \frac{1}{2}$ . Then there is a function  $r_{\alpha,\beta,\theta}: \mathbb{R} \rightarrow \mathbb{R}_+$  such that  $r(\varepsilon) \rightarrow 0$  for  $\varepsilon \rightarrow 0$  and a constant  $C = C(\alpha, \beta, \theta)$  such that*

$$|\rho_{\beta\text{-H\"{o}l}}(\Psi(t), \Psi^\varepsilon(t))|_{L^q} \leq Cq^{(1/2)\lfloor 1/\beta \rfloor} r(\varepsilon)$$

for every  $t \in [0, T]$ ,  $\varepsilon > 0$  and  $q \in [1, \infty)$ .

PROOF. As before,  $\Psi_t^\varepsilon$  has the form of a random Fourier series where the  $k$ th Fourier coefficients are given by  $\alpha_k^\varepsilon Y_t^{\varepsilon,k}$  with  $\alpha_k^\varepsilon = \frac{1}{\sqrt{2\lambda_k^\varepsilon}}$ ,

$$\lambda_k^\varepsilon = \left(\frac{k}{2}\right)^{2\alpha} + \varepsilon \left(\frac{k}{2}\right)^{2\theta},$$

and  $t \mapsto Y_t^{\varepsilon,k}$  are  $d$ -dimensional, stationary Ornstein–Uhlenbeck processes with independent components, each component being centered with variance 1 and correlation

$$\mathbb{E}Y_t^{\varepsilon,k} \otimes Y_t^{\varepsilon,l} = 2 \frac{\sqrt{\lambda_k \lambda_l^\varepsilon}}{\lambda_k + \lambda_l^\varepsilon} \delta_{k,l} \text{Id}.$$

From Theorem 3.17, we know that it is sufficient to show that  $(A_k^\varepsilon) \leq (|k|^{-2\alpha})$  uniformly over  $\varepsilon > 0$  where

$$A_k^\varepsilon := \begin{pmatrix} \alpha_k^2 & \alpha_k \alpha_k^\varepsilon \varrho_k^\varepsilon \\ \alpha_k \alpha_k^\varepsilon \varrho_k^\varepsilon & (\alpha_k^\varepsilon)^2 \end{pmatrix}, \quad \varrho_k^\varepsilon := 2 \frac{\sqrt{\lambda_k \lambda_k^\varepsilon}}{\lambda_k + \lambda_k^\varepsilon}$$

and that

$$(4.8) \quad \sup_{t \in [0, T]} \sup_{x \in [0, 2\pi]} \mathbb{E}|\Psi(t, x) - \Psi^\varepsilon(t, x)|^2 \rightarrow 0 \quad \text{for } \varepsilon \rightarrow 0.$$

For the first claim, we have to show that the series

$$\sum_{k=1}^\infty (\alpha_k^\varepsilon)^2 k^{2\alpha} \cos(kx), \quad \sum_{k=1}^\infty \alpha_k \alpha_k^\varepsilon \varrho_k^\varepsilon k^{2\alpha} \cos(kx)$$

are uniformly bounded in  $L^1$  which can be done using Lemma 3.14(2). Showing (4.8) follows by writing down the left-hand side as a Fourier series and bounding it uniformly in  $x$  and  $t$  by an infinite series. Then we send  $\varepsilon \rightarrow 0$ , using the dominated convergence theorem.  $\square$

4.3. *Various generalizations.*

4.3.1. *Generalized fractional stochastic heat equation on periodic domains.* Based on the stability results for the mixed  $(1, \rho)$ -variation of the covariance of random Fourier series developed in Section 3, one may consider more general fractional stochastic heat type equations and different types of boundary conditions. As an example let us consider generalized fractional stochastic heat equations on  $[0, 2\pi]$  with periodic boundary conditions. An orthogonal basis of eigenvectors and corresponding eigenvalues of  $-\Delta$  on  $L^2([0, 2\pi])$  is given by

$$\tau_k = k^2, \quad e_k(x) := \begin{cases} \sin(kx), & k > 0, \\ \frac{1}{2}, & k = 0, \\ \cos(kx), & k < 0. \end{cases}$$

Via the spectral theorem we may define  $A = f(-\Delta)$  for each Borel measurable function  $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ , still having  $e_k$  as a basis of eigenvectors and eigenvalues  $f(\tau_k)$ .

In order to be able to consider stationary Ornstein–Uhlenbeck processes, we need to shift the spectrum of  $A$  to be strictly negative. Hence, we consider  $\mathbb{R}^d$ -valued SPDE of the form

$$(4.9) \quad d\Psi_t = (-A - \lambda)\Psi_t dt + dW_t \quad \text{with } \lambda > 0,$$

where  $W_t$  is a (possibly) colored Wiener process with covariance operator having  $e_k$  as basis of eigenvectors and  $\sigma_k$  as eigenvalues. An (informal) Fourier expansion of the stationary solution  $\Psi$  to (4.9) leads to the random Fourier series

$$(4.10) \quad \Psi(t, x) = \frac{\alpha_0 Y_0}{2} + \sum_{k=1}^{\infty} \alpha_k Y_t^k \sin(kx) + \alpha_{-k} Y_t^{-k} \cos(kx),$$

with  $\alpha_k = \alpha_{-k} = \sqrt{\frac{\sigma_k}{2(\lambda + f(\tau_k))}}$  and  $Y_t^k$  as in (4.4). Suppose  $(a_k)$  to be eventually nonincreasing and  $(a_k) \leq (k^{-2\alpha})$  for some  $\alpha \in (\frac{1}{2}, 1]$ . Then analogous results to Proposition 4.1 may be established under various assumptions on  $\sigma_k$  and  $f(\tau_k)$ , by means of the stability results given in Section 3.

EXAMPLE 4.5. We consider the stochastic fractional heat equation with (possibly) colored noise on the 1-dimensional torus, that is,

$$(4.11) \quad d\Psi_t^i = -((-\Delta)^\alpha \Psi_t^i + \lambda) dt + d(-\Delta)^{-\gamma/2} W_t^i, \quad i = 1, \dots, d,$$

where  $\alpha \in (0, 1]$ ,  $\gamma \geq 0$ ,  $\lambda > 0$  and  $W_t$  is a cylindrical Wiener process. Hence,  $f(\tau_k) = |k|^{2\alpha}$  and  $\sigma_k = |k|^{-2\gamma}$ . By elementary calculations we see  $(\frac{\sigma_k}{\lambda + f(\tau_k)}) \leq (k^{-(2\gamma + 2\alpha)})$  and thus the conclusions of Proposition 4.1 hold if  $2\gamma + 2\alpha > \frac{3}{2}$ .

4.3.2. *Neumann boundary conditions.* In the case of homogeneous Neumann boundary conditions, an orthogonal basis of eigenvectors of  $-\Delta$  on  $L^2([0, 2\pi])$  is given by

$$e_k(x) = \cos\left(\frac{k}{2}x\right), \quad \tau_k = \left(\frac{k}{2}\right)^2, \quad k \in \mathbb{N} \cup \{0\}.$$

In order to be able to consider stationary Ornstein–Uhlenbeck processes, we need to shift the spectrum; that is, we consider

$$d\Psi_t = -((-\Delta)^\alpha + 1)\Psi_t dt + dW_t.$$

We may then proceed as for Dirichlet boundary conditions, resolving additional difficulties due to the shift of the spectrum as in the proof of Proposition 4.1.

**5. The continuous case.** In some cases, the covariance function of a Gaussian process  $X$  is given as the cosine transform of some function  $f$ . For example, this is the case if the spectral measure of a stationary process has a density  $f$  with respect to the Lebesgue measure; cf. Example 2.16 and [41], Chapter 5.6. In this case, we may obtain similar results as for random Fourier series. The key is a continuous version of Lemma 3.3 which we are now going to present. For a (symmetric) function  $f \in L^1(\mathbb{R})$ , let  $\hat{f}$  denote its (real) Fourier transform. Then the following holds:

LEMMA 5.1. *Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be symmetric and in  $L^1(\mathbb{R}) \cap C^2(\mathbb{R} \setminus \{0\})$ . Assume  $\hat{f} \in L^1(\mathbb{R})$  and*

$$\lim_{\xi \rightarrow \infty} |\xi^3 f''(\xi)| + |\xi^2 f'(\xi)| + |\xi f(\xi)| = 0$$

and that there is an  $x_0 \in (0, \infty]$  such that

$$\limsup_{R \rightarrow \infty} \int_0^R \frac{\partial^2}{\partial \xi^2} (f(\xi)\xi^2) F_\xi(x) d\xi \leq 0,$$

for all  $x \in (0, x_0)$  where  $F_\xi(x) = \frac{1 - \cos(\xi x)}{x^2}$  denotes the Féjer kernel. Then  $\hat{f}$  is a convex function on  $[0, x_0)$ .

PROOF. Since the proof is very similar to Lemma 3.3 we just sketch it briefly. By Féjer’s theorem for Fourier transforms (cf. [33], Theorem 49.3),

$$\lim_{R \rightarrow \infty} \frac{1}{2\pi} \int_{-R}^R \left(1 - \frac{|\xi|}{R}\right) \hat{g}(\xi) e^{ix\xi} d\xi = g(x),$$

for all  $x$  provided  $g \in C(\mathbb{R}) \cap L^1(\mathbb{R})$ . Setting  $g = \hat{f}$ , we obtain from Fourier inversion

$$\sigma_R(x) := \int_{-R}^R \left(1 - \frac{|\xi|}{R}\right) f(\xi) e^{ix\xi} d\xi \rightarrow \hat{f}(x) \quad \text{for } R \rightarrow \infty.$$

Applying integration by parts twice, our assumptions imply that

$$\liminf_{R \rightarrow \infty} \sigma_R''(x) \geq 0$$

for all  $x \in (0, x_0)$ . This implies the claim.  $\square$

REMARK 5.2. Note that for a given  $f \in L^1(\mathbb{R})$ , it does not follow that also  $\hat{f} \in L^1(\mathbb{R})$ . However, Bernstein’s theorem states that the Fourier transform of functions  $f$  in the Sobolev space  $H^s(\mathbb{R})$  is contained in  $L^1(\mathbb{R})$  for all  $s > \frac{1}{2}$ ; cf. [28], Corollary 7.9.4.

EXAMPLE 5.3. Consider the covariance  $R$  of a fractional Ornstein–Uhlenbeck process with Hurst parameter  $H \in (0, 1/2]$ ; cf. Example 2.16. Then  $R(s, t) = K(t - s)$  with

$$K(x) = \int f(\xi) \cos(\xi x) d\xi, \quad f(\xi) = c_H \frac{|\xi|^{1-2H}}{\lambda^2 + \xi^2}, \quad \lambda > 0.$$

We prove that there is an  $x_0 > 0$  such that  $K$  is convex on  $[0, x_0)$ . Since  $f(\xi) = O(\xi^{-1-2H})$ ,  $f \in H^s$  for any  $s < 2H + 1/2$  and Bernstein’s theorem implies that  $\hat{f} \in L^1$  for any  $H > 0$ . An easy calculation shows that  $g := \partial_{\xi, \xi}^2 (f(\xi)\xi^2)$  is non-positive on  $[\xi_0, \infty)$  for some  $\xi_0 > 0$  and that  $g(\xi) = O(-\xi^{-1-2H})$ . It follows that

$$\limsup_{R \rightarrow \infty} \int_0^R \frac{\partial^2}{\partial \xi^2} (f(\xi)\xi^2) F_\xi(x) d\xi = \int_0^\infty \frac{\partial^2}{\partial \xi^2} (f(\xi)\xi^2) F_\xi(x) d\xi.$$

Note first that  $\int_0^{\xi_0} g(\xi) F_\xi(x) d\xi$  is uniformly bounded for  $x \searrow 0$ . Furthermore  $\lim_{x \rightarrow 0} F_\xi(x) = \xi^2/2$ , and Fatou’s lemma gives

$$\liminf_{x \rightarrow 0} \int_{\xi_0}^\infty -g(\xi) F_\xi(x) d\xi \geq -\frac{1}{2} \int_{\xi_0}^\infty \xi^2 g(\xi) d\xi = +\infty.$$

Hence

$$\lim_{x \rightarrow 0} \int_0^\infty g(\xi) F_\xi(x) d\xi = -\infty.$$

Thus there is an  $x_0$  such that  $\int_0^\infty g(\xi) F_\xi(x) d\xi \leq 0$  for all  $x \in (0, x_0)$ , and we can apply Lemma 5.1 to conclude that  $K$  is convex on  $[0, x_0)$ .

EXAMPLE 5.4. Consider the SPDE

$$d\Psi_t = -((-\Delta)^\alpha + \lambda)\Psi_t dt + dW_t \quad \text{on } \mathbb{R},$$

for some  $\alpha \in (0, 1]$ ,  $\lambda > 0$ . The stationary solution can be written down explicitly (cf. [47]), namely

$$\Psi_t(x) = \int_{-\infty}^t \int_{\mathbb{R}} K_{t-s}(x, y) W(ds, dy),$$

where  $K$  is the fractional heat kernel operator associated to  $-((-\Delta)^\alpha + \lambda)$  with Fourier transform given by

$$\widehat{K}_t(\xi) = e^{-t|\xi|^{2\alpha} - \lambda t}.$$

After some calculations, one sees that the covariance  $R$  of the spatial process  $x \mapsto \Psi_t(x)$  for every time point  $t$  is given by  $R(x, y) = K(x - y)$  where

$$K(x) = \int f(\xi) \cos(\xi x) d\xi, \quad f(\xi) = \frac{1}{2|\xi|^{2\alpha} + 2\lambda}.$$

With a similar calculation as in Example 5.3, one can see that  $K$  is convex in a neighborhood around 0. It is easy to see that  $\sigma^2(x) = 2(K(x) - K(0)) = O(|x|^{2\alpha-1})$  (using, e.g., [41], Theorem 7.3.1). Hence we are in the situation of Example 2.9, and we may conclude that

$$V_{1,\rho}(R_X; [x, y]) = O(|y - x|^{2\alpha-1})$$

for  $|y - x|$  small enough. Applying [20], Theorem 35, we see that  $\Psi_t$  can be lifted, for every fixed time point  $t$ , to a process  $\Psi_t$  with sample paths in  $C_0^{0,\beta\text{-H\"{o}l}}([x, y], G^{[1/\beta]}(\mathbb{R}^d))$ , every  $\beta < \alpha - 1/2$ , provided  $\alpha > 3/4$  and  $|y - x|$  is small enough. By concatenation one has the existence of spatial rough paths lifts on all compact intervals in  $\mathbb{R}$ .

**6. Application to non-Markovian Hörmander theory.** Consider a (rough) differential equation

$$(6.1) \quad dY = V(Y) d\mathbf{X}$$

driven by a (Gaussian) rough path  $\mathbf{X}$  along a vector field  $V = (V_1, \dots, V_d)$ , started at  $Y_0 = y_0 \in \mathbb{R}^e$ . Assume  $V$  to be bounded with bounded derivatives of all orders such that Hörmander’s condition  $\text{Lie}(V_1, \dots, V_d)|_{y_0} = \mathbb{R}^e$  holds.<sup>21</sup> If  $X$  is sufficiently nondegenerate (e.g., fBm) one can hope for a density of  $Y_t$  at positive times  $t > 0$ . This has been achieved in a series of papers starting with Baudoin and Hairer [1] (with  $\mathbf{X}$  fBm for  $H > 1/2$ ), followed by [3, 5, 24] which dealt, respectively, with general Gaussian signals  $\mathbf{X}$  ( $\rho < 2$  subject to CYR<sup>22</sup>), fBm for  $H > 1/4$  and then again general Gaussian signals ( $\rho < 2$  subject to CYR), now with a smoothness result. The general case [3, 5] requires a number of assumptions on  $\mathbf{X}$  that are not always easy to check. To wit, even if  $X$  is fBm-like, in the sense that

$$\sigma^2(s, t) = F(t - s) \geq 0$$

<sup>21</sup>We may also include a drift vector  $V_0$ , in which case we mean the weak Hörmander condition.

<sup>22</sup>Complementary Young regularity for Cameron–Martin paths  $h$ : that is,  $h$  has finite  $q$ -variation and  $X(\omega)$  has finite  $p$ -variation a.s. with  $1/p + 1/q > 1$ .

with  $F$  being concave and  $F(t) = O(t^{2H})$ , already the CYR is unclear in the aforementioned references [3, 5]. Indeed, CYR for fBm (in case  $H > 1/4$ ) relies on the variation embedding theorem [16] which is not applicable in this more general situation. Our results provide a convenient way to check the assumptions of [5]. Let us illustrate how to proceed by the concrete example of an RDE driven by a (Gaussian) process (with i.i.d. components) with stationary increments.

**THEOREM 6.1.** *Assume  $F(t) = O(t^{1/\rho})$  with  $\rho \in [1, 2)$  as  $t \downarrow 0$ ,  $F$  concave and nonzero. Then*

$$F'_-(T) > 0 \quad \text{for some } T > 0,$$

and  $Y_t$  in (6.1) has a smooth density for all  $t \in (0, T]$ .

*This applies in particular when  $X$  is given as a random Fourier series as in Example 2.13 (with  $\rho < 2$ ) or as a fractional Ornstein–Uhlenbeck process with Hurst parameter  $H \in (1/4, 1/2]$  as in Example 2.16.*

**PROOF.** By assumption,  $F$  is not identically equal to zero. In order to see that  $F'_-(T) > 0$  for some  $T$  small enough, assume the opposite, that is,  $F'_-(t) \leq 0$  for all  $t > 0$ . Then

$$\frac{F(t+h) - F(t)}{h} \leq F'_-(t) \leq 0 \quad \forall h, t > 0,$$

and thus  $F$  is nonincreasing. Since  $F(0) = 0$  and  $F \geq 0$ , this implies that  $F$  is trivial and gives the desired contradiction. We now proceed by checking the conditions from [5]. Condition 1 (CYR; [3, 5]) follows from Theorem 2.2, applied as in Example 2.9 which yields mixed  $(1, \rho)$ -variation and thus (cf. part 1) complementary Young regularity. For Condition 2 from [5]<sup>23</sup> we first note that, leaving the imminent estimate (6.2) to the end of the proof,

$$\begin{aligned} & 2 \operatorname{Var}(X_{s,t} | \mathcal{F}_{0,s} \vee \mathcal{F}_{t,T}) \\ & \geq 2R \binom{s, t}{0, T} \\ (6.2) \quad & = \sigma^2(0, t) - \sigma^2(0, s) + \sigma^2(s, T) - \sigma^2(t, T) \\ & = F(t) - F(s) + F(T - s) + F(T - t) \\ & \geq 2F'_-(T)(t - s), \end{aligned}$$

where we used (thanks to concavity) that the left-hand side derivative of  $F$  at  $T$  exists and

$$\inf_{0 \leq s < t \leq T} \frac{F(t) - F(s)}{t - s} = F'_-(T).$$

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<sup>23</sup>For the reader’s convenience we recall (the essence of) Condition 2 in [5]: there exists  $c, \alpha > 0$  such that  $\operatorname{Var}(X_{s,t} | \mathcal{F}_{0,s} \vee \mathcal{F}_{t,T}) \geq c(t - s)^\alpha$  for all  $0 \leq s < t \leq T$ .

By assumption,  $F'_-(T) > 0$ , and so Condition 2 holds with  $\alpha = 1$ . Also note that  $F'_-(T)$  is nonincreasing in  $T$ ; thus Condition 2 remains valid upon decreasing  $T$ .

Next, we prove that ([5], Condition 4, page 10) is satisfied. Due to concavity of  $F$  and Lemma 2.20,  $X$  has nonpositively correlated increments, and it is enough to show that (cf. [5], page 11)

$$(6.3) \quad \mathbb{E}X_{0,s}X_{s,t} = R \begin{pmatrix} 0, S \\ s, t \end{pmatrix} \geq 0 \quad \forall [s, t] \subseteq [0, S] \subseteq [0, T],$$

which is clear from our condition (B.ii) of Theorem 2.2, which was seen to be verified in the present situation in Example 2.9. We also note that ([5], Condition 3, page 10) is implied by Condition 4 (cf. [5], Corollary 6.8). In conclusion, [5], Theorem 3.5, implies the claim.

It remains to prove estimate (6.2). To this end, let  $\mathcal{G} := \mathcal{F}_{0,s} \vee \mathcal{F}_{t,T}$ . Since  $X$  is Gaussian,  $\text{Var}(X_{s,t}|\mathcal{G})$  is deterministic, and by a simple argument (detailed in [5], Lemma 4.1) one has

$$(6.4) \quad \text{Var}(X_{s,t}|\mathcal{G}) = \inf_{Y \in L^2(\Omega, \mathcal{G}, \mathbb{P})} \|X_{s,t} - Y\|_2^2,$$

where the inf is achieved at  $Y = \mathbb{E}[X_{s,t}|\mathcal{G}]$ , an element in the first Wiener–Itô chaos, that is, the  $L^2$ -closure of  $\{X_t : 0 \leq t \leq T\}$  and of course  $\mathcal{G}$ -measurable. As a consequence,  $\mathbb{E}[X_{s,t}|\mathcal{G}] = \lim_n Y_n$  in  $L^2$  for suitable “simple” approximations of the form, with  $[t_i^n, t_{i+1}^n] \subset [0, s] \cup [t, T]$ ,

$$Y_n = \sum_{i=1}^{k_n} a_i^n X_{t_i^n, t_{i+1}^n},$$

and we can replace the inf in (6.4) by the inf taken over such simple elements. In what follows let us write  $(\tilde{t}_i^n)$  for the dissection obtained from  $(t_i^n : 1 \leq i < k_n) \cup \{s, t\}$ . This way, we may condense the expression  $X_{s,t} - \sum_i a_i^n X_{t_i^n, t_{i+1}^n}$  to  $\sum_i \tilde{a}_i^n X_{\tilde{t}_i^n, \tilde{t}_{i+1}^n}$ . Using elementary estimates such as  $\alpha_i \alpha_j \leq (\alpha_i^2 + \alpha_j^2)/2$  and symmetry of  $R$ , we find

$$\begin{aligned} \|X_{s,t} - Y_n\|_2^2 &= \mathbb{E} \left| X_{s,t} - \sum_i a_i^n X_{t_i^n, t_{i+1}^n} \right|^2 = \mathbb{E} \left| \sum_i \tilde{a}_i^n X_{\tilde{t}_i^n, \tilde{t}_{i+1}^n} \right|^2 \\ &= \sum_{i,j} \tilde{a}_i^n \tilde{a}_j^n \mathbb{E} X_{\tilde{t}_i^n, \tilde{t}_{i+1}^n} X_{\tilde{t}_j^n, \tilde{t}_{j+1}^n} = \sum_{i,j} \tilde{a}_i^n \tilde{a}_j^n R \left( \begin{matrix} \tilde{t}_i^n, \tilde{t}_{i+1}^n \\ \tilde{t}_j^n, \tilde{t}_{j+1}^n \end{matrix} \right) \\ &\geq - \sum_i \sum_{j \neq i} |\tilde{a}_i^n| |\tilde{a}_j^n| \left| R \left( \begin{matrix} \tilde{t}_i^n, \tilde{t}_{i+1}^n \\ \tilde{t}_j^n, \tilde{t}_{j+1}^n \end{matrix} \right) \right| + \sum_i (\tilde{a}_i^n)^2 R \left( \begin{matrix} \tilde{t}_i^n, \tilde{t}_{i+1}^n \\ \tilde{t}_i^n, \tilde{t}_{i+1}^n \end{matrix} \right) \\ &\geq - \sum_i \sum_{j \neq i} (\tilde{a}_i^n)^2 \left| R \left( \begin{matrix} \tilde{t}_i^n, \tilde{t}_{i+1}^n \\ \tilde{t}_j^n, \tilde{t}_{j+1}^n \end{matrix} \right) \right| + \sum_i (\tilde{a}_i^n)^2 R \left( \begin{matrix} \tilde{t}_i^n, \tilde{t}_{i+1}^n \\ \tilde{t}_i^n, \tilde{t}_{i+1}^n \end{matrix} \right). \end{aligned}$$

Due to nonpositively correlated increments we may drop the minus and absolute values in the line above and combine both sums. Thanks to (6.3), we can then finish the desired estimate,

$$\begin{aligned} \|X_{s,t} - Y_n\|_2^2 &\geq \sum_i (\tilde{a}_i^n)^2 \sum_j R \begin{pmatrix} \tilde{t}_i^n, \tilde{t}_{i+1}^n \\ \tilde{t}_j^n, \tilde{t}_{j+1}^n \end{pmatrix} = \sum_i (\tilde{a}_i^n)^2 R \begin{pmatrix} \tilde{t}_i^n, \tilde{t}_{i+1}^n \\ 0, T \end{pmatrix} \\ &= R \begin{pmatrix} s, t \\ 0, T \end{pmatrix} + \sum_i (a_i^n)^2 R \begin{pmatrix} t_i^n, t_{i+1}^n \\ 0, T \end{pmatrix} \geq R \begin{pmatrix} s, t \\ 0, T \end{pmatrix}. \quad \square \end{aligned}$$

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P. K. FRIZ  
 INSTITUT FÜR MATHEMATIK  
 TECHNISCHE UNIVERSITÄT BERLIN  
 STRASSE DES 17. JUNI 136  
 10623 BERLIN  
 GERMANY  
 AND  
 WEIERSTRASS-INSTITUT FÜR ANGEWANDTE ANALYSIS  
 UND STOCHASTIK  
 MOHRENSTRASSE 39  
 10117 BERLIN  
 GERMANY  
 E-MAIL: [friz@math.tu-berlin.de](mailto:friz@math.tu-berlin.de)  
[friz@wias-berlin.de](mailto:friz@wias-berlin.de)

A. GULISASHVILI  
 DEPARTMENT OF MATHEMATICS  
 OHIO UNIVERSITY  
 MORTON HALL 321  
 1 OHIO UNIVERSITY  
 ATHENS, OHIO 45701  
 USA  
 E-MAIL: [gulisash@ohio.edu](mailto:gulisash@ohio.edu)

B. GESS  
 DEPARTMENT OF MATHEMATICS  
 UNIVERSITY OF CHICAGO  
 5734 S. UNIVERSITY AVENUE  
 CHICAGO, ILLINOIS 60637  
 USA  
 E-MAIL: [gess@math.tu-berlin.de](mailto:gess@math.tu-berlin.de)

S. RIEDEL  
 INSTITUT FÜR MATHEMATIK  
 TECHNISCHE UNIVERSITÄT BERLIN  
 STRASSE DES 17. JUNI 136  
 10623 BERLIN  
 GERMANY  
 E-MAIL: [riedel@math.tu-berlin.de](mailto:riedel@math.tu-berlin.de)