

RATE OF CONVERGENCE OF THE NANBU PARTICLE SYSTEM FOR HARD POTENTIALS AND MAXWELL MOLECULES

BY NICOLAS FOURNIER AND STÉPHANE MISCHLER

Université Pierre et Marie Curie and Université Paris Dauphine

We consider the (numerically motivated) Nanbu stochastic particle system associated to the spatially homogeneous Boltzmann equation for true hard potentials and Maxwell molecules. We establish a rate of propagation of chaos of the particle system to the unique solution of the Boltzmann equation. More precisely, we estimate the expectation of the squared Wasserstein distance with quadratic cost between the empirical measure of the particle system and the solution to the Boltzmann equation. The rate we obtain is almost optimal as a function of the number of particles but is not uniform in time.

1. Introduction and main results.

1.1. *The Boltzmann equation.* The Boltzmann equation predicts that in a spatially homogeneous dilute gas, the density $f(t, v)$ of particles with velocity $v \in \mathbb{R}^3$ at time $t \geq 0$ solves

$$(1.1) \quad \partial_t f_t(v) = \frac{1}{2} \int_{\mathbb{R}^3} dv_* \int_{\mathbb{S}^2} d\sigma B(|v - v_*|, \theta) [f_t(v') f_t(v'_*) - f_t(v) f_t(v_*)],$$

where the pre-collisional velocities are given by

$$(1.2) \quad \begin{aligned} v' &= v'(v, v_*, \sigma) = \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \sigma, \\ v'_* &= v'_*(v, v_*, \sigma) = \frac{v + v_*}{2} - \frac{|v - v_*|}{2} \sigma, \end{aligned}$$

and $\theta = \theta(v, v_*, \sigma)$ is the *deviation angle* defined by $\cos \theta = \frac{(v - v_*) \cdot \sigma}{|v - v_*|}$. The *collision kernel* $B(|v - v_*|, \theta) \geq 0$ depends on the nature of the interactions between particles. See Cercignani [11], Desvillettes [13], Villani [43] and Alexandre [2] for physical and mathematical reviews on this equation. Conservation of mass, momentum and kinetic energy hold at least formally for solutions to (1.1), and we classically may assume without loss of generality that $\int_{\mathbb{R}^3} f_0(v) dv = 1$.

Received February 2013; revised May 2014.

MSC2010 subject classifications. 80C40, 60K35.

Key words and phrases. Kinetic theory, stochastic particle systems, propagation of chaos, Wasserstein distance.

We will assume that the collision kernel is of the form

$$(1.3) \quad \begin{aligned} B(|v - v_*|, \theta) \sin \theta &= \Phi(|v - v_*|)\beta(\theta) \\ &\text{with } \beta > 0 \text{ on } (0, \pi/2) \text{ and } \beta = 0 \text{ on } [\pi/2, \pi]. \end{aligned}$$

This last condition $\beta = 0$ on $(\pi/2, \pi]$ is not a restriction: as noted in the introduction of Alexandre et al. [3], one can always reduce to this case for symmetry reasons.

When particles behave like hard spheres, it holds that $\Phi(z) = z$ and $\beta \equiv 1$. When particles interact through a repulsive force in $1/r^s$, with $s \in (2, \infty)$, one has

$$\begin{aligned} \Phi(z) &= z^\gamma \quad \text{with } \gamma = \frac{s-5}{s-1} \in (-3, 1) \quad \text{and} \\ \beta(\theta) &\overset{0}{\sim} \text{cst } \theta^{-1-\nu} \quad \text{with } \nu = \frac{2}{s-1} \in (0, 2). \end{aligned}$$

One classically names *hard potentials* the case when $\gamma \in (0, 1)$ [i.e., $s > 5$ and $\nu \in (0, 1/2)$], *Maxwell molecules* the case when $\gamma = 0$ (i.e., $s = 5$ and $\nu = 1/2$) and *soft potentials* the case when $\gamma \in (-3, 0)$ [i.e., $s \in (2, 5)$ and $\nu \in (1/2, 2)$]. The present paper concerns Maxwell molecules, hard potentials as well as hard spheres, so that we always assume $\gamma \in [0, 1]$.

1.2. *Stochastic particle systems.* As a step to the rigorous derivation of the Boltzmann equation, Kac [28] proposed to show the convergence of a stochastic particle system to the solution to (1.1). Kac’s particle system is a $(\mathbb{R}^3)^N$ -valued Markov process with infinitesimal generator $\tilde{\mathcal{L}}_N$ defined, for $\phi : (\mathbb{R}^3)^N \mapsto \mathbb{R}$ sufficiently regular and $\mathbf{v} = (v_1, \dots, v_N) \in (\mathbb{R}^3)^N$, by

$$\begin{aligned} \tilde{\mathcal{L}}_N \phi(\mathbf{v}) &= \frac{1}{2(N-1)} \\ &\times \sum_{i \neq j} \int_{\mathbb{S}^2} [\phi(\mathbf{v} + (v'(v_i, v_j, \sigma) - v_i)\mathbf{e}_i \\ &\quad + (v'_*(v_i, v_j, \sigma) - v_j)\mathbf{e}_j) - \phi(\mathbf{v})] B(|v_i - v_j|, \theta) d\sigma. \end{aligned}$$

For $h \in \mathbb{R}^3$, we note $h\mathbf{e}_i = (0, \dots, 0, h, 0, \dots, 0) \in (\mathbb{R}^3)^N$ with h at the i th place. Roughly speaking, the system is constituted of N particles entirely characterized by their velocities (v_1, \dots, v_N) , and each couple of particles with velocities (v_i, v_j) are modified, for each $\sigma \in \mathbb{S}^2$, at rate $B(|v_i - v_j|, \theta)/(2(N-1))$ and are then replaced by particles with velocities $v'(v_i, v_j, \sigma)$ and $v'_*(v_i, v_j, \sigma)$.

In the present paper, we will consider a slightly modified and nonsymmetric particle system introduced by Nanbu [36]. The Nanbu stochastic particle system corresponds to the generator \mathcal{L}_N defined, for $\phi : (\mathbb{R}^3)^N \mapsto \mathbb{R}$ sufficiently regular

and $\mathbf{v} = (v_1, \dots, v_N) \in (\mathbb{R}^3)^N$, by

$$\begin{aligned} \mathcal{L}_N \phi(\mathbf{v}) = & \frac{1}{N} \sum_{i \neq j} \int_{\mathbb{S}^2} [\phi(\mathbf{v} + (v'(v_i, v_j, \sigma) - v_i)\mathbf{e}_i) - \phi(\mathbf{v})] \\ (1.4) \quad & \times B(|v_i - v_j|, \theta) d\sigma. \end{aligned}$$

This system still describes N particles characterized by their velocities (v_1, \dots, v_N) , but now each couple of particles with velocities (v_i, v_j) are modified, for each $\sigma \in \mathbb{S}^2$, at rate $B(|v_i - v_j|, \theta)/N$ and are then replaced by particles with velocities $v'(v_i, v_j, \sigma)$ and v_j . Thus only one particle is modified at each ‘‘collision,’’ but the rate of collision is multiplied by 2. All in all, the asymptotic behavior, as $N \rightarrow \infty$, should be the same.

1.3. *Aims.* Our aim is to prove that as N tends to ∞ , the Nanbu stochastic system is asymptotically constituted of independent particles with identical law governed by the Boltzmann equation, and better, to quantify this convergence.

There are two main motivations for such a study. (i) From a physical point of view, we want to know how well the Boltzmann equation approximates true particles. Of course, true particles are subjected to classical (nonrandom) dynamics, so that studying the Kac (or Nanbu) particle system does not provide any rigorous information on how well the Boltzmann equation approximates true particles. However, as already mentioned, Kac proposed this problem as an intermediate step. (ii) From a numerical point of view, we want to know how well the particle system approximates the Boltzmann equation. It is then important to get rates of convergence, to know how to choose the number of particles (and the cutoff parameter) to reach a given accuracy.

The main difficulty lies in the fact that even if the particle system is initially constituted of independent particles, they do not remain independent for later times because of interactions. Hence to answer the convergence issue, we have to prove that particles asymptotically become independent, and at the same time to identify their common law, we have to prove that the system is chaotic in the sense of Kac [28].

We are able to prove and quantify the chaotic property for Nanbu’s particle system. Unfortunately, our study does really not seem to work for Kac’s particle system. From the physical point of view, Nanbu’s system is less pertinent. However, we believe that the behaviors of the two systems are very similar, so that our results should also hold true for Kac’s particle system. From the numerical point of view, both systems are expected to approximate the solution to (1.1) with an error of the same order, so that the system under study is as interesting as Kac’s system.

We will also study a cutoff version of Nanbu’s system, where we remove collisions generating small deviations. For technical reasons, we will not use the standard cutoff procedure where $B(z, \theta)$ is replaced by $B_K(z, \theta) = B(z, \theta)\mathbb{1}_{\{\theta > 1/K\}}$ for some large $K > 0$. We will rather use some cutoff of the form $B_K(z, \theta) =$

$B(z, \theta) \mathbb{1}_{\{\theta > \varphi(K, z)\}}$, where the positive function φ is chosen in such a way that $\int_{\mathbb{S}^2} B_K(z, \theta) d\sigma$ does not depend on z . This will simplify the argument in several places. This cutoff procedure is motivated by two reasons. From a numerical point of view, the particle system with generator \mathcal{L}_N cannot be directly simulated because each particle collides with infinitely many others on each time interval (except for hard spheres). Thus we have to introduce a cutoff. From a technical point of view, we are not able to prove directly our estimates for the particle system without cutoff: we have to study first the particle system with cutoff and then to pass to the limit.

1.4. *Assumptions.* We assume that the collision kernel is of the form (1.3) with

$$(1.5) \quad \exists \gamma \in [0, 1], \forall z \geq 0, \quad \Phi(z) = z^\gamma,$$

and either

$$(1.6) \quad \forall \theta \in (0, \pi/2), \quad \beta(\theta) = 1$$

or

$$(1.7) \quad c_0 \theta^{-1-\nu} \leq \beta(\theta) \leq c_1 \theta^{-1-\nu} \\ \exists \nu \in (0, 1), \exists 0 < c_0 < c_1, \forall \theta \in (0, \pi/2).$$

This work could probably be extended to $\nu \in (0, 2)$, since the *important* computations on which it relies also hold in this case. However, this would introduce several technical difficulties. Since Maxwell molecules and hard potentials, which we study, satisfy (1.7) with $\nu \in (0, 1)$, we decided to avoid these technical complications.

The propagation of exponential moments requires the following additional condition:

$$(1.8) \quad \beta(\theta) = b(\cos \theta) \quad \text{with } b \text{ nondecreasing, convex and } C^1 \text{ on } [0, 1).$$

In practice, all these assumptions are satisfied for Maxwell molecules ($\gamma = 0$ and $\nu = 1/2$), hard potentials [$\gamma \in (0, 1)$ and $\nu \in (1, 1/2)$] and hard spheres ($\gamma = 1$ and $\beta \equiv 1$).

1.5. *Notation.* For $\theta \in (0, \pi/2)$ and $z \in [0, \infty)$ we introduce

$$(1.9) \quad H(\theta) = \int_\theta^{\pi/2} \beta(x) dx \quad \text{and} \quad G(z) = H^{-1}(z).$$

Under (1.7), H is a continuous decreasing bijection from $(0, \pi/2)$ into $(0, \infty)$, and its inverse function $G : (0, \infty) \mapsto (0, \pi/2]$ is defined by $G(H(\theta)) = \theta$, and $H(G(z)) = z$. It is immediately checked that under (1.7), there are some constants $0 < c_2 < c_3$ such that

$$(1.10) \quad \forall z > 0, \quad c_2(1+z)^{-1/\nu} \leq G(z) \leq c_3(1+z)^{-1/\nu}$$

and, as checked in [20], Lemma 1.1, there is a constant $c_4 > 0$ such that for all $x, y \in \mathbb{R}_+$,

$$(1.11) \quad \int_0^\infty (G(z/x) - G(z/y))^2 dz \leq c_4 \frac{(x - y)^2}{x + y}.$$

Under (1.6), we have $G(z) = (\pi/2 - z)_+$ (with the common notation $x_+ = \max\{x, 0\}$), and a direct computation shows that (1.11) also holds true.

1.6. *Well-posedness.* Let $\mathcal{P}_k(\mathbb{R}^3)$ be the set of all probability measures f on \mathbb{R}^3 such that $\int_{\mathbb{R}^3} |v|^k f(dv) < \infty$. We first recall known well-posedness results for the Boltzmann equation, as well as some properties of solutions we will need. A precise definition of weak solutions is stated in the next section.

THEOREM 1.1. *Assume (1.3), (1.5) and (1.6) or (1.7). Let $f_0 \in \mathcal{P}_2(\mathbb{R}^3)$.*

(i) *If $\gamma = 0$, there exists a unique weak solution $(f_t)_{t \geq 0} \in C([0, \infty), \mathcal{P}_2(\mathbb{R}^3))$ to (1.1). If $f_0 \in \mathcal{P}_p(\mathbb{R}^3)$ for some $p \geq 2$, then $\sup_{[0, \infty)} \int_{\mathbb{R}^3} |v|^p f_t(dv) < \infty$. If $\int_{\mathbb{R}^3} f_0(v) \log f_0(v) dv < \infty$ or if $f_0 \in \mathcal{P}_4(\mathbb{R}^3)$ and is not a Dirac mass, then f_t has a density for all $t > 0$.*

(ii) *If $\gamma \in (0, 1]$, assume additionally (1.8) and that*

$$(1.12) \quad \exists p \in (\gamma, 2), \quad \int_{\mathbb{R}^3} e^{|v|^p} f_0(dv) < \infty.$$

There is a unique weak solution $(f_t)_{t \geq 0} \in C([0, \infty), \mathcal{P}_2(\mathbb{R}^3))$ to (1.1) such that

$$(1.13) \quad \forall q \in (0, p), \quad \sup_{[0, \infty)} \int_{\mathbb{R}^3} e^{|v|^q} f_t(dv) < \infty.$$

Under (1.7) and if f_0 is not a Dirac mass, then f_t has a density for all $t > 0$. Under (1.6) and if f_0 has a density, then f_t has a density for all $t > 0$.

Concerning well-posedness, see Toscani and Villani [42] for Maxwell molecules, [15, 24] for hard potentials and [5, 16, 29, 30, 35] for hard spheres. The propagation of moments in the Maxwell case is standard; see, for example, Villani [43], Theorem 1, page 74. The propagation of exponential moments for hard potentials and hard spheres, initiated by Bobylev [7], is checked in [24, 30]. Finally, the existence of a density for f_t has been proved in [18] [under (1.7) and when f_0 is not a Dirac mass and belongs to $\mathcal{P}_4(\mathbb{R}^3)$], in [35] [under (1.6) when f_0 has a density] and is very classical by monotonicity of the entropy when f_0 has a finite entropy; see, for example, Arkeryd [4].

We now introduce our particle system with cutoff.

PROPOSITION 1.2. *Assume (1.3), (1.5) and (1.6) or (1.7). Let $f_0 \in \mathcal{P}_2(\mathbb{R}^3)$ and a number of particles $N \geq 1$ be fixed. Let $(V_0^i)_{i=1, \dots, N}$ be i.i.d. with common law f_0 .*

(i) For each cutoff parameter $K \in [1, \infty)$, there exists a unique (in law) Markov process $(V_t^{i,N,K})_{i=1,\dots,N,t \geq 0}$ with values in $(\mathbb{R}^3)^N$, starting from $(V_0^i)_{i=1,\dots,N}$ and with generator $\mathcal{L}_{N,K}$ defined, for all bounded measurable $\phi : (\mathbb{R}^3)^N \mapsto \mathbb{R}$ and any $\mathbf{v} = (v_1, \dots, v_N) \in \mathbb{R}^3$, by

$$\begin{aligned} \mathcal{L}_{N,K}\phi(\mathbf{v}) &= \frac{1}{N} \sum_{i \neq j} \int_{\mathbb{S}^2} [\phi(\mathbf{v} + (v'(v_i, v_j, \sigma) - v_i)\mathbf{e}_i) - \phi(\mathbf{v})] \\ &\quad \times B(|v_i - v_j|, \theta) \mathbb{1}_{\{\theta \geq G(K/|v_i - v_j|^\nu)\}} d\sigma, \end{aligned}$$

with G defined by (1.9) and, for $h \in \mathbb{R}^3$, $h\mathbf{e}_i = (0, \dots, h, \dots, 0) \in (\mathbb{R}^3)^N$ with h at the i th place.

(ii) There exists a unique (in law) Markov process $(V_t^{i,N,\infty})_{i=1,\dots,N,t \geq 0}$ with values in $(\mathbb{R}^3)^N$, starting from $(V_0^i)_{i=1,\dots,N}$ and with generator \mathcal{L}_N defined, for all Lipschitz bounded functions $\phi : (\mathbb{R}^3)^N \mapsto \mathbb{R}$ and any $\mathbf{v} = (v_1, \dots, v_N) \in \mathbb{R}^3$, by (1.4).

Let us emphasize that the cut-off used for defining the generator $\mathcal{L}_{N,K}$ is not the usual one since it depends not only on the deviation angle $\theta \in (0, 2\pi)$ but also of the relative velocity $|v - v_*|$. It is more convenient in order to perform the computations we want to do. It might also be convenient for practical simulations. Indeed, the total rate of collision of the particle system does not depend on the configuration of the velocities: it always equals $2\pi(N - 1)K$. Hence, the (mean) simulation cost of the particle system on a time interval $[0, T]$ is proportional to $(N - 1)KT$.

1.7. *Wasserstein distance.* For $g, \tilde{g} \in \mathcal{P}_2(\mathbb{R}^3)$, let $\mathcal{H}(g, \tilde{g})$ be the set of probability measures on $\mathbb{R}^3 \times \mathbb{R}^3$ with first marginal g and second marginal \tilde{g} . We then set

$$\mathcal{W}_2(g, \tilde{g}) = \inf \left\{ \left(\int_{\mathbb{R}^3 \times \mathbb{R}^3} |v - \tilde{v}|^2 \eta(dv, d\tilde{v}) \right)^{1/2}; \eta \in \mathcal{H}(g, \tilde{g}) \right\}.$$

This is the Wasserstein distance with quadratic cost. It is well known that the inf is reached. We refer to Villani ([44], Chapter 2), for more details on this distance. A remarkable result, due to Tanaka [40, 41], is that in the case of Maxwell molecules, $t \mapsto \mathcal{W}_2(f_t, \tilde{f}_t)$ is nonincreasing for each pair of reasonable solutions f, \tilde{f} to the Boltzmann equation. The present work is strongly inspired by the ideas of Tanaka.

1.8. *Empirical law of large numbers.* For $f \in \mathcal{P}_2(\mathbb{R}^3)$ and $N \geq 1$, we define

$$\varepsilon_N(f) := \mathbb{E} \left[\mathcal{W}_2^2 \left(f, N^{-1} \sum_1^N \delta_{X_i} \right) \right] \tag{1.14}$$

with X_1, \dots, X_N independent and f -distributed.

Since a f -chaotic stochastic particle system is asymptotically constituted of i.i.d. f -distributed particles, $\varepsilon_N(f)$ is the best rate (as far as \mathcal{W}_2^2 is concerned) we can hope for such a system. We recall now the estimate proved in [21], Theorem 1 (with $d = 3$ and $p = 2$) and we also refer to Rachev and Rüschendorf [37], Theorem 10.2.1, [33], Lemma 4.2, Boissard–Le Gouic [8] and Dereich, Scheutzow and Schottstedt [12] for earlier (but not optimal) versions.

THEOREM 1.3. *For all $A > 0$, all $k > 2$, all $f \in \mathcal{P}_k(\mathbb{R}^3)$ verifying $\int_{\mathbb{R}^3} |v|^k \times f(dv) \leq A$, all $N \geq 1$,*

$$(1.15) \quad \varepsilon_N(f) \leq \begin{cases} C_{A,k} N^{-(k-2)/k}, & \text{if } k \in (2, 4), \\ C_{A,k} N^{-1/2}, & \text{if } k > 4. \end{cases}$$

This bound is optimal for *general* laws. The convergence might be faster for some regular laws, but this should be quite complicated; see [21], Section 1.2, as well as the discussion in Barthe and Bordenave [6]. We also refer to [27], Theorem 2.13 (and the remarks which follow) for a general discussion about the rate of chaoticity for independent and dependent random arrays.

1.9. Main result. Our study concerns both the particle systems with and without cutoff. It is worth noticing that for true Maxwell molecules and hard potentials, $\nu \in (0, 1/2]$ so that $1 - 2/\nu \leq -3$, and the contribution of the cut-off approximation vanishes rapidly in the limit $K \rightarrow \infty$.

THEOREM 1.4. *Let B be a collision kernel satisfying (1.3), (1.5) and (1.6) or (1.7), and let $f_0 \in \mathcal{P}_2(\mathbb{R}^3)$ not be a Dirac mass. If $\gamma > 0$, assume additionally (1.8) and (1.12). Consider the unique weak solution $(f_t)_{t \geq 0}$ to (1.1) defined in Theorem 1.1 and, for each $N \geq 1$, $K \in [1, \infty]$, the unique Markov process $(V_t^{i,N,K})_{i=1,\dots,N,t \geq 0}$ defined in Proposition 1.2. Let $\mu_t^{N,K} := N^{-1} \sum_1^N \delta_{V_t^{i,N,K}}$.*

(i) *Maxwell molecules. Assume that $\gamma = 0$, (1.7) and either $\int_{\mathbb{R}^3} f_0(v) \times \log f_0(v) dv < \infty$ or $f_0 \in \mathcal{P}_4(\mathbb{R}^3)$. There is a constant C such that for all $T \geq 0$, all $N \geq 1$, all $K \in [1, \infty]$,*

$$(1.16) \quad \sup_{[0,T]} \mathbb{E}[\mathcal{W}_2^2(\mu_t^{N,K}, f_t)] \leq C(1 + T)^2 \sup_{[0,T]} \varepsilon_N(f_t) + CT K^{1-2/\nu}.$$

If $f_0 \in \mathcal{P}_k(\mathbb{R}^3)$ for some $k > 2$, we have $\sup_{[0,\infty)} \int_{\mathbb{R}^3} |v|^k f_t(dv) < \infty$, and we can use Theorem 1.3 to bound $\sup_{[0,T]} \varepsilon_N(f_t)$. In particular if $k > 4$, then for all $T \geq 0$, all $N \geq 1$, all $K \in [1, \infty]$,

$$(1.17) \quad \sup_{[0,T]} \mathbb{E}[\mathcal{W}_2^2(\mu_t^{N,K}, f_t)] \leq C(1 + T)^2 N^{-1/2} + CT K^{1-2/\nu}.$$

(ii) *Hard potentials.* Assume that $\gamma \in (0, 1)$ and (1.7). For all $\varepsilon \in (0, 1)$, all $T \geq 0$, there is a constant $C_{\varepsilon, T}$ such that for all $N \geq 1$, all $K \in [1, \infty]$,

$$(1.18) \quad \sup_{[0, T]} \mathbb{E}[\mathcal{W}_2^2(\mu_t^{N, K}, f_t)] \leq C_{\varepsilon, T} \left(\sup_{[0, T]} \varepsilon_N(f_t) + K^{1-2/\nu} \right)^{1-\varepsilon}.$$

Consequently, for all $\varepsilon \in (0, 1)$, all $T \geq 0$, there is $C_{\varepsilon, T}$ such that for all $N \geq 1$, all $K \in [1, \infty]$,

$$(1.19) \quad \sup_{[0, T]} \mathbb{E}[\mathcal{W}_2^2(\mu_t^{N, K}, f_t)] \leq C_{\varepsilon, T} (N^{-1/2} + K^{1-2/\nu})^{1-\varepsilon}.$$

(iii) *Hard spheres.* Assume finally that $\gamma = 1$, (1.6) and that f_0 has a density. For all $\varepsilon \in (0, 1)$, all $T \geq 0$, all $q \in (1, p)$, there is a constant $C_{\varepsilon, q, T}$ such that for all $N \geq 1$, all $K \in [1, \infty]$,

$$(1.20) \quad \sup_{[0, T]} \mathbb{E}[\mathcal{W}_2^2(\mu_t^{N, K}, f_t)] \leq C_{\varepsilon, q, T} \left(\left(\sup_{[0, T]} \varepsilon_N(f_t) \right)^{1-\varepsilon} + e^{-K^q} \right) e^{C_{\varepsilon, q, T} K}.$$

Thus for all $\varepsilon \in (0, 1)$, all $T \geq 0$, all $q \in (1, p)$, there is $C_{\varepsilon, q, T}$ such that for all $N \geq 1$, all $K \in [1, \infty]$,

$$(1.21) \quad \sup_{[0, T]} \mathbb{E}[\mathcal{W}_2^2(\mu_t^{N, K}, f_t)] \leq C_{\varepsilon, q, T} (N^{-1/2+\varepsilon} + e^{-K^q}) e^{C_{\varepsilon, q, T} K}.$$

Concerning the rate of convergence of the simulation algorithm, we have the following.

REMARK 1.5. Recall that the simulation cost per unit of time is proportional to $(N - 1)K$.

(i) For Maxwell molecules and hard potentials the error is $(N^{-1/6} + K^{1/2-1/\nu})^{1-\varepsilon}$ (for the distance \mathcal{W}_2). For a given simulation cost τ , the best choices are $N \simeq \tau^{(4-2\nu)/(4-\nu)}$ and $K \simeq \tau^{\nu/(4-\nu)}$, which lead to an error in $\tau^{-(2-\nu)/(8-2\nu)+}$. For true hard potentials and Maxwell molecules, this is at worst $\tau^{-3/14+}$ and at best $\tau^{-1/4+}$.

(ii) For hard spheres, make the choice $K \simeq (\log N)^a$ with $a \in (1/q, 1)$. Then $e^{CK} \ll N^\varepsilon$ for any $\varepsilon \in (0, 1)$ and $e^{-K^q} \ll N^{-r}$ for any $r > 1$. With this choice, we thus find an error in $N^{-1/4+\varepsilon}$ for a simulation cost in $N(\log N)^a$. Consequently, for a given simulation cost τ , we find an error in $\tau^{-1/4+}$.

We excluded the case where f_0 is a Dirac mass because we need that f_t has a density and because if $f_0 = \delta_{v_0}$, then the unique solution to (1.1) is given by $f_t = \delta_{v_0}$ and the Markov process of Proposition 1.2 is nothing but $V_t^{1, N, K} = (v_0, \dots, v_0)$ (for any value of $K \in [1, \infty]$), so that $\mu_t^{N, K} = \delta_{v_0}$ and thus $\mathcal{W}_2(f_t, \mu_t^{N, K}) = 0$.

1.10. *Comments.* We thus show that the empirical law of the particle system converges to f_t as fast as i.i.d. f_t -distributed particles (up to an arbitrary small loss if $\gamma \neq 0$). This is thus almost optimal in some sense. However, this is optimal only as far as \mathcal{W}_2 is concerned: we would have preferred to work with another distance and to obtain a rate in $N^{-1/2}$ as is expected for laws of large numbers. Here we obtain a rate in $N^{-1/4}$, since \mathcal{W}_2 is squared. However, \mathcal{W}_2 enjoys several properties that make it quite convenient when studying the Boltzmann equation, mainly because of the role of the kinetic energy. Another default of this work is that we obtain a nonuniform (in time) bound. For Maxwell molecules, the bound is slowly increasing (as T^2) but for hard potentials, it is growing very fast.

Note also that for hard spheres, we are not able to treat the case where $K = \infty$: we need to let K and N go to infinity simultaneously, with some constraints. We believe that this is only a technical problem, but we are not able to solve it. However, we still obtain a very reasonable rate of convergence (as a function of the computational cost).

Our proof is based on a coupling argument: we couple the N -particle system with a family of N i.i.d. Boltzmann processes, in such a way that they remain as close as possible. We prove an accurate control on the increment of the distance between the two systems at each collision. This last computation is similar to those of [20, 24] concerning uniqueness of the solution to (1.1). However, we need to handle much more precise computations: in [24], when studying the distance between two solutions to (1.1), both were supposed to have exponential moments. Such exponential moments are known to propagate for solutions to (1.1) since the seminal work of Bobylev [7], but for the particle system under study, we are not even able to prove the finiteness of a moment of order $2 + \varepsilon$, $\varepsilon > 0$! We thus need a very precise refinement of the computations of [20, 24].

All these problems do not appear when studying Maxwell molecules. Roughly, the collision operator is globally Lipschitz continuous for Maxwell molecules and only locally Lipschitz continuous for hard potentials (which explains why large velocities have to be controlled by using exponential moments). This is why we obtain a better result for Maxwell molecules.

Note that for the (physically more relevant) Kac particle system, moments are known to propagate (uniformly in N) (see Sznitman [38] and also [33]), which would simplify greatly the proof at many places. However, we are not able to exhibit a suitable coupling. This is due to the fact that in Kac's system, each collision modifies the velocity of two particles. In Nanbu's system, the Poisson measures governing two different particles are independent, which is not the case for Kac's system (because each time a particle's velocity is modified, another one has to be also modified), although the larger the number of particles, the lower the correlation. As a consequence, it is more difficult to couple the N -particle symmetric Kac system with N independent copies of the Boltzmann process, and we did not succeed.

1.11. *Known results.* Such a chaos result for the Boltzmann equation with bounded cross section, or for related models, was first established without any rate by Kac [28] (for the so-called Maxwell molecules, Kac's model which is roughly a "toy one-dimensional" Boltzmann equation) and then by McKean [32] and Grünbaum [26]. For unbounded cross section, the chaos property was proved by Sznitman [38] for hard spheres, still without rate.

For Maxwell molecules with Grad's cutoff, a nice rate of convergence (of order $1/N$ in total variation distance on the two-marginal) was obtained by McKean [31] and improved by Graham and Méléard [25]. This was extended by Desvillettes, Graham and Méléard [14] (see also [22]) to true (without Grad's cutoff) Maxwell molecules, but with a rate in $N^{-1}e^{KT} + K^{1-2/\nu}$ (with the notation of the present paper). From a numerical point of view, this leads to a logarithmic convergence as a function of the computational cost.

More recently, a uniform in time rate of chaos convergence of Kac's stochastic particle system to the Boltzmann equation for two unbounded models was established in [10, 33] (see also [34]), by taking up again and improving Grünbaum's approach. For true Maxwell molecules, uniform in time rate of convergence of order $N^{-1/(6+\delta)}$, for any $\delta > 0$, for a weak distance on the two-marginals was proved in [33], Theorem 5.1, when the initial condition f_0 has a compact support. This result was improved and made more precise in [10], step 3 of the proof of Theorem 8, where, still for true Maxwell molecules, uniform in time rate of convergence of order $N^{-1/177}$, for the same \mathcal{W}_2 Wasserstein distance as used in (1.16), was proved for any initial condition f_0 satisfying (1.12). Hard spheres have also been studied in [33], Theorem 6.1: a uniform in time rate of convergence of order $1/(\log N)^\alpha$ with $\alpha > 0$ small, for the \mathcal{W}_1 distance on the two-marginals has been proved. When applying the methods of [10, 33, 34] on finite time intervals, the previous rates cannot really be improved. Finally, let us mention that the present work follows some of the ideas in [19], which concerns the Kac equation.

To summarize:

- We obtain the first rate of convergence for hard potentials, and this rate is reasonable. Recall that hard potentials are twice unbounded (the velocity cross section is unbounded and the angular cross section is nonintegrable), while Maxwell molecules enjoy a bounded velocity cross section and hard spheres an integrable angular cross section.
- For hard spheres and Maxwell molecules, we prove a much faster convergence than [10, 33, 34], but we are restricted to finite time-intervals, and we cannot study Kac's system.

Let us finally mention that we use a coupling method, as is widely used since the famous *cours à l'école d'été de Saint-Flour* by Sznitman [39] for providing rate of chaos convergence for the so-called McKean–Vlasov model and that such methods have been recently adapted to nonglobally Lipschitz coefficients by Bolley, Cañizo and Carrillo in [9], making use of exponential moments.

1.12. *Plan of the paper.* In Section 2, we make precise the notion of weak solutions, rewrite the collision operators in a suitable form and check a accurate version of a lemma due to Tanaka [41]. Section 3 is devoted to the cornerstone estimate on the collision integral. In Section 4 we prove the convergence of the particle system with cutoff. The cutoff is removed in Section 5.

2. Preliminaries.

2.1. *Rewriting equations.* We follow here [22]. For each $X \in \mathbb{R}^3$, we introduce two vectors $I(X), J(X) \in \mathbb{R}^3$ such that $(\frac{X}{|X|}, \frac{I(X)}{|X|}, \frac{J(X)}{|X|})$ is a direct orthonormal basis of \mathbb{R}^3 and, of course, in such a way that I, J are measurable functions. For $X, v, v_* \in \mathbb{R}^3$, for $\theta \in (0, \pi/2)$ and $\varphi \in [0, 2\pi)$, we set

$$(2.1) \quad \begin{cases} \Gamma(X, \varphi) := (\cos \varphi)I(X) + (\sin \varphi)J(X), \\ a(v, v_*, \theta, \varphi) := -\frac{1 - \cos \theta}{2}(v - v_*) + \frac{\sin \theta}{2}\Gamma(v - v_*, \varphi), \\ v'(v, v_*, \theta, \varphi) := v + a(v, v_*, \theta, \varphi), \end{cases}$$

which is a suitable parametrization of (1.2): simply write $\sigma \in \mathbb{S}^2$ as $\sigma = \frac{v-v_*}{|v-v_*|} \cos \theta + \frac{I(v-v_*)}{|v-v_*|} \sin \theta \cos \varphi + \frac{J(v-v_*)}{|v-v_*|} \sin \theta \sin \varphi$. Let us define, classically, weak solutions to (1.1).

DEFINITION 2.1. Assume (1.3), (1.5) and (1.6) or (1.7). A family $(f_t)_{t \geq 0}$ belonging to $C([0, \infty), \mathcal{P}_2(\mathbb{R}^3))$ is called a weak solution to (1.1) if it preserves momentum and energy, that is,

$$(2.2) \quad \int_{\mathbb{R}^3} v f_t(dv) = \int_{\mathbb{R}^3} v f_0(dv) \quad \text{and} \quad \int_{\mathbb{R}^3} |v|^2 f_t(dv) = \int_{\mathbb{R}^3} |v|^2 f_0(dv) \quad \forall t \geq 0,$$

and if for any $\phi : \mathbb{R}^3 \mapsto \mathbb{R}$ bounded and Lipschitz-continuous, any $t \in [0, T]$,

$$(2.3) \quad \begin{aligned} & \int_{\mathbb{R}^3} \phi(v) f_t(dv) \\ &= \int_{\mathbb{R}^3} \phi(v) f_0(dv) + \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \mathcal{A}\phi(v, v_*) f_s(dv_*) f_s(dv) ds, \end{aligned}$$

where

$$(2.4) \quad \begin{aligned} & \mathcal{A}\phi(v, v_*) \\ &= |v - v_*|^\gamma \int_0^{\pi/2} \beta(\theta) d\theta \int_0^{2\pi} d\varphi [\phi(v + a(v, v_*, \theta, \varphi)) - \phi(v)]. \end{aligned}$$

Noting that $|a(v, v_*, \theta, \varphi)| \leq C\theta|v - v_*|$ and that $\int_0^{\pi/2} \theta\beta(\theta) d\theta$, we easily get $|\mathcal{A}\phi(v, v_*)| \leq C_\phi|v - v_*|^{1+\gamma} \leq C_\phi(1 + |v - v_*|^2)$, so that everything makes sense in (2.3).

We next rewrite the collision operator in a way that makes the velocity-dependence $|v - v_*|^\gamma$ disappear in the *rate*. Such a trick was already used in [23] and [20].

LEMMA 2.2. *Assume (1.3), (1.5) and (1.6) or (1.7). Recalling (1.9) and (2.1), define, for $z \in (0, \infty)$, $\varphi \in [0, 2\pi)$, $v, v_* \in \mathbb{R}^3$ and $K \in [1, \infty)$,*

$$(2.5) \quad \begin{aligned} c(v, v_*, z, \varphi) &:= a[v, v_*, G(z/|v - v_*|^\gamma), \varphi] \quad \text{and} \\ c_K(v, v_*, z, \varphi) &:= c(v, v_*, z, \varphi)\mathbb{1}_{\{z \leq K\}}. \end{aligned}$$

For any bounded Lipschitz $\phi : \mathbb{R}^3 \mapsto \mathbb{R}$, any $v, v_* \in \mathbb{R}^3$

$$(2.6) \quad \mathcal{A}\phi(v, v_*) = \int_0^\infty dz \int_0^{2\pi} d\varphi [\phi[v + c(v, v_*, z, \varphi)] - \phi[v]].$$

For any $N \geq 1$, $K \in [1, \infty)$, $\mathbf{v} = (v_1, \dots, v_N) \in (\mathbb{R}^3)^N$, any bounded measurable $\phi : (\mathbb{R}^3)^N \mapsto \mathbb{R}$,

$$(2.7) \quad \mathcal{L}_{N,K}\phi(\mathbf{v}) = \frac{1}{N} \sum_{i \neq j} \int_0^\infty dz \int_0^{2\pi} d\varphi [\phi(\mathbf{v} + c_K(v_i, v_j, z, \varphi)\mathbf{e}_i) - \phi(\mathbf{v})].$$

For any $N \geq 1$, any $\mathbf{v} = (v_1, \dots, v_N) \in (\mathbb{R}^3)^N$, any bounded Lipschitz $\phi : (\mathbb{R}^3)^N \mapsto \mathbb{R}$,

$$(2.8) \quad \mathcal{L}_N\phi(\mathbf{v}) = \frac{1}{N} \sum_{i \neq j} \int_0^\infty dz \int_0^{2\pi} d\varphi [\phi(\mathbf{v} + c(v_i, v_j, z, \varphi)\mathbf{e}_i) - \phi(\mathbf{v})].$$

PROOF. To get (2.6), start from (2.4) and use the substitution $\theta = G(z/|v - v_*|^\gamma)$ or equivalently $H(\theta) = z/|v - v_*|^\gamma$, which implies $|v - v_*|^\gamma \beta(\theta) d\theta = dz$. Expressions (2.7) and (2.8) are checked similarly. \square

2.2. *Accurate version of Tanaka’s trick.* As was already noted by Tanaka [41], it is not possible to choose I in such a way that $X \mapsto I(X)$ is continuous. However, he found a way to overcome this difficulty; see also [22], Lemma 2.6. Here we need the following accurate version of Tanaka’s trick.

LEMMA 2.3. *Recall (2.1). There are some measurable functions $\varphi_0, \varphi_1 : \mathbb{R}^3 \times \mathbb{R}^3 \mapsto [0, 2\pi)$, such that for all $X, Y \in \mathbb{R}^3$, all $\varphi \in [0, 2\pi)$,*

$$\begin{aligned} &\Gamma(X, \varphi) \cdot \Gamma(Y, \varphi + \varphi_0(X, Y)) \\ &= X \cdot Y \cos^2(\varphi + \varphi_1(X, Y)) + |X||Y| \sin^2(\varphi + \varphi_1(X, Y)), \\ &|\Gamma(X, \varphi) - \Gamma(Y, \varphi + \varphi_0(X, Y))| \leq |X - Y|. \end{aligned}$$

PROOF. First observe that the second claim follows from the first one: writing $\varphi_i = \varphi_i(X, Y)$

$$\begin{aligned} & |\Gamma(X, \varphi) - \Gamma(Y, \varphi + \varphi_0)|^2 \\ &= |\Gamma(X, \varphi)|^2 + |\Gamma(Y, \varphi + \varphi_0)|^2 - 2\Gamma(X, \varphi) \cdot \Gamma(Y, \varphi + \varphi_0) \\ &= |X|^2 + |Y|^2 - 2(X \cdot Y \cos^2(\varphi + \varphi_1) + |X||Y| \sin^2(\varphi + \varphi_1)) \\ &\leq |X|^2 + |Y|^2 - 2X \cdot Y = |X - Y|^2. \end{aligned}$$

We next check the first claim. Let thus X and Y be fixed. Observe that $\Gamma(X, \varphi)$ goes (at constant speed) all over the circle C_X with radius $|X|$ lying in the plane orthogonal to X . Let $i_X \in C_X$ and $i_Y \in C_Y$ such that X, Y, i_X, i_Y belong to the same plane and $i_X \cdot i_Y = X \cdot Y$ [there are exactly two possible choices for the couple (i_X, i_Y) if X and Y are not collinear, infinitely many otherwise]. Consider φ_X and φ_Y such that $i_X := \Gamma(X, \varphi_X)$ and $i_Y := \Gamma(Y, \varphi_Y)$. Define $j_X := \Gamma(X, \varphi_X + \pi/2)$ and $j_Y := \Gamma(Y, \varphi_Y + \pi/2)$. Then j_X and j_Y are collinear (because both are orthogonal to the plane containing X, Y, i_X, i_Y), satisfy $j_X \cdot j_Y = |j_X||j_Y| = |X||Y|$ and $i_X \cdot j_Y = i_Y \cdot j_X = 0$. Next, observe that $\Gamma(X, \varphi + \varphi_X) = i_X \cos \varphi + j_X \sin \varphi$ while $\Gamma(Y, \varphi + \varphi_Y) = i_Y \cos \varphi + j_Y \sin \varphi$. Consequently, $\Gamma(X, \varphi + \varphi_X) \cdot \Gamma(Y, \varphi + \varphi_Y) = i_X \cdot i_Y \cos^2 \varphi + j_X \cdot j_Y \sin^2 \varphi = X \cdot Y \cos^2 \varphi + |X||Y| \sin^2 \varphi$. The conclusion follows: choose $\varphi_0 := \varphi_Y - \varphi_X$ and $\varphi_1 := -\varphi_X$ (all this modulo 2π). \square

3. Main computations of the paper. The following estimate is our central argument.

LEMMA 3.1. Recall that G was defined in (1.9) and that the deviation functions c and c_K were defined in (2.5). For any $v, v_*, \tilde{v}, \tilde{v}_* \in \mathbb{R}^3$, any $K \in [1, \infty)$,

$$\begin{aligned} & \int_0^\infty \int_0^{2\pi} (|v + c(v, v_*, z, \varphi) - \tilde{v} - c_K(\tilde{v}, \tilde{v}_*, z, \varphi + \varphi_0(v - v_*, \tilde{v} - \tilde{v}_*))|^2 \\ & \qquad \qquad \qquad - |v - \tilde{v}|^2) d\varphi dz \\ & \leq A_1^K(v, v_*, \tilde{v}, \tilde{v}_*) + A_2^K(v, v_*, \tilde{v}, \tilde{v}_*) + A_3^K(v, v_*, \tilde{v}, \tilde{v}_*), \end{aligned}$$

where, setting $\Phi_K(x) = \pi \int_0^K (1 - \cos G(z/x^\gamma)) dz$ and $\Psi_K(x) = \pi \int_K^\infty (1 - \cos G(z/x^\gamma)) dz$,

$$A_1^K(v, v_*, \tilde{v}, \tilde{v}_*) = 2|v - v_*||\tilde{v} - \tilde{v}_*| \int_0^K [G(z/|v - v_*|^\gamma) - G(z/|\tilde{v} - \tilde{v}_*|^\gamma)]^2 dz,$$

$$\begin{aligned} A_2^K(v, v_*, \tilde{v}, \tilde{v}_*) &= -[(v - \tilde{v}) + (v_* - \tilde{v}_*)] \\ & \quad \times [(v - v_*)\Phi_K(|v - v_*|) - (\tilde{v} - \tilde{v}_*)\Phi_K(|\tilde{v} - \tilde{v}_*|)], \end{aligned}$$

$$A_3^K(v, v_*, \tilde{v}, \tilde{v}_*) = (|v - v_*|^2 + 2|v - \tilde{v}||v - v_*|)\Psi_K(|v - v_*|).$$

PROOF. We need to shorten notation. We write $x = |v - v_*|$, $\tilde{x} = |\tilde{v} - \tilde{v}_*|$, $\varphi_0 = \varphi_0(v - v_*, \tilde{v} - \tilde{v}_*)$, $c = c(v, v_*, z, \varphi)$, $\tilde{c} = c(\tilde{v}, \tilde{v}_*, z, \varphi + \varphi_0)$ and $\tilde{c}_K = c_K(\tilde{v}, \tilde{v}_*, z, \varphi + \varphi_0) = \tilde{c}\mathbb{1}_{\{z \leq K\}}$. We start with

$$\begin{aligned} \Delta_K &:= \int_0^\infty \int_0^{2\pi} (|v + c - \tilde{v} - \tilde{c}_K|^2 - |v - \tilde{v}|^2) d\varphi dz \\ &= \int_0^K \int_0^{2\pi} (|c|^2 + |\tilde{c}|^2 - 2c \cdot \tilde{c} + 2(v - \tilde{v}) \cdot (c - \tilde{c})) d\varphi dz \\ &\quad + \int_K^\infty \int_0^{2\pi} (|c|^2 + 2(v - \tilde{v}) \cdot c) d\varphi dz. \end{aligned}$$

First, it holds that $|c|^2 = |-(1 - \cos G(z/x^\gamma))(v - v_*) + (\sin G(z/x^\gamma))\Gamma(v - v_*, \varphi)|^2/4 = (1 - \cos G(z/x^\gamma))|v - v_*|^2/2$. We used that by definition [see (2.1)], $\Gamma(v - v_*, \varphi)$ has the same norm as $v - v_*$ and is orthogonal to $v - v_*$ and that $(1 - \cos \theta)^2 + (\sin \theta)^2 = 2 - 2 \cos \theta$. Consequently, we have

$$\begin{aligned} \int_0^K \int_0^{2\pi} |c|^2 d\varphi dz &= \pi |v - v_*|^2 \int_0^K (1 - \cos G(z/x^\gamma)) dz \\ &= x^2 \Phi_K(x). \end{aligned}$$

Similarly, we also have $\int_0^K \int_0^{2\pi} |\tilde{c}|^2 d\varphi dz = \tilde{x}^2 \Phi_K(\tilde{x})$ and $\int_K^\infty \int_0^{2\pi} |c|^2 d\varphi dz = x^2 \Psi_K(x)$.

Next, using that $c = -(1 - \cos G(z/x^\gamma))(v - v_*)/2 + (\sin G(z/x^\gamma))\Gamma(v - v_*, \varphi)/2$ and that $\int_0^{2\pi} \Gamma(v - v_*, \varphi) d\varphi = 0$,

$$\begin{aligned} \int_0^K \int_0^{2\pi} c d\varphi dz &= -(v - v_*)\pi \int_0^K (1 - \cos G(z/x^\gamma)) dz \\ &= -(v - v_*)\Phi_K(x). \end{aligned}$$

In the same way, $\int_0^K \int_0^{2\pi} \tilde{c} d\varphi dz = -(\tilde{v} - \tilde{v}_*)\Phi_K(\tilde{x})$ and $\int_K^\infty \int_0^{2\pi} c d\varphi dz = -(v - v_*)\Psi_K(x)$.

Finally, $c \cdot \tilde{c} = [(1 - \cos G(z/x^\gamma))(v - v_*) - (\sin G(z/x^\gamma))\Gamma(v - v_*, \varphi)] \cdot [(1 - \cos G(z/\tilde{x}^\gamma))(\tilde{v} - \tilde{v}_*) - (\sin G(z/\tilde{x}^\gamma))\Gamma(\tilde{v} - \tilde{v}_*, \varphi + \varphi_0)]/4$. Since $\int_0^{2\pi} \Gamma(v - v_*, \varphi) d\varphi = \int_0^{2\pi} \Gamma(\tilde{v} - \tilde{v}_*, \varphi + \varphi_0) d\varphi = 0$, we get

$$\begin{aligned} \int_0^{2\pi} c \cdot \tilde{c} d\varphi &= \frac{\pi}{2} (1 - \cos G(z/x^\gamma))(1 - \cos G(z/\tilde{x}^\gamma))(v - v_*) \cdot (\tilde{v} - \tilde{v}_*) \\ &\quad + \frac{1}{4} (\sin G(z/x^\gamma))(\sin G(z/\tilde{x}^\gamma)) \\ &\quad \times \int_0^{2\pi} \Gamma(v - v_*, \varphi) \cdot \Gamma(\tilde{v} - \tilde{v}_*, \varphi + \varphi_0) d\varphi. \end{aligned}$$

Recalling Lemma 2.3 and using that $\int_0^{2\pi} \cos^2(\varphi + \varphi_1) d\varphi = \int_0^{2\pi} \sin^2(\varphi + \varphi_1) d\varphi = \pi$, we obtain

$$\begin{aligned} \int_0^{2\pi} c \cdot \tilde{c} d\varphi &= \frac{\pi}{2}(1 - \cos G(z/x^\gamma))(1 - \cos G(z/\tilde{x}^\gamma))(v - v_*) \cdot (\tilde{v} - \tilde{v}_*) \\ &\quad + \frac{\pi}{4}(\sin G(z/x^\gamma))(\sin G(z/\tilde{x}^\gamma)) \\ &\quad \times [(v - v_*) \cdot (\tilde{v} - \tilde{v}_*) + |v - v_*||\tilde{v} - \tilde{v}_*|]. \end{aligned}$$

But G takes values in $(0, \pi/2)$, so that, since $|v - v_*||\tilde{v} - \tilde{v}_*| \geq (v - v_*) \cdot (\tilde{v} - \tilde{v}_*)$,

$$\begin{aligned} \int_0^{2\pi} c \cdot \tilde{c} d\varphi &\geq \frac{\pi}{2}[(1 - \cos G(z/x^\gamma))(1 - \cos G(z/\tilde{x}^\gamma)) \\ &\quad + (\sin G(z/x^\gamma))(\sin G(z/\tilde{x}^\gamma))](v - v_*) \cdot (\tilde{v} - \tilde{v}_*) \\ &= \frac{\pi}{2}[(1 - \cos G(z/x^\gamma)) + (1 - \cos G(z/\tilde{x}^\gamma))](v - v_*) \cdot (\tilde{v} - \tilde{v}_*) \\ &\quad - \frac{\pi}{2}(1 - \cos(G(z/x^\gamma) - G(z/\tilde{x}^\gamma)))(v - v_*) \cdot (\tilde{v} - \tilde{v}_*). \end{aligned}$$

Using that $\pi(1 - \cos \theta) \leq 2\theta^2$, we thus get

$$\begin{aligned} \int_0^K \int_0^{2\pi} c \cdot \tilde{c} d\varphi dz &\geq (v - v_*) \cdot (\tilde{v} - \tilde{v}_*) \frac{\Phi_K(x) + \Phi_K(\tilde{x})}{2} \\ &\quad - x\tilde{x} \int_0^K (G(z/x^\gamma) - G(z/\tilde{x}^\gamma))^2 dz. \end{aligned}$$

All in all, we find

$$\begin{aligned} \Delta_K &\leq x^2\Phi_K(x) + \tilde{x}^2\Phi_K(\tilde{x}) - (v - v_*) \cdot (\tilde{v} - \tilde{v}_*)[\Phi_K(x) + \Phi_K(\tilde{x})] \\ &\quad + 2(v - \tilde{v}) \cdot [(\tilde{v} - \tilde{v}_*)\Phi_K(\tilde{x}) - (v - v_*)\Phi_K(x)] \\ &\quad + 2x\tilde{x} \int_0^K (G(z/x^\gamma) - G(z/\tilde{x}^\gamma))^2 dz \\ &\quad + x^2\Psi_K(x) - 2(v - \tilde{v}) \cdot (v - v_*)\Psi_K(x). \end{aligned}$$

Recalling that $x = |v - v_*|$, $\tilde{x} = |\tilde{v} - \tilde{v}_*|$, we realize that the third line is nothing but $A_1^K(v, v_*, \tilde{v}, \tilde{v}_*)$ while the fourth one is bounded from above by $A_3^K(v, v_*, \tilde{v}, \tilde{v}_*)$. To conclude, it suffices to note that the sum of the terms on the two first lines equals

$$\begin{aligned} &= (v - v_*) \cdot [(v - v_*) - (\tilde{v} - \tilde{v}_*) - 2(v - \tilde{v})]\Phi_K(x) \\ &\quad + (\tilde{v} - \tilde{v}_*) \cdot [(\tilde{v} - \tilde{v}_*) - (v - v_*) + 2(v - \tilde{v})]\Phi_K(\tilde{x}) \\ &= -(v - v_*) \cdot ((v - \tilde{v}) + (v_* - \tilde{v}_*))\Phi_K(x) \\ &\quad + (\tilde{v} - \tilde{v}_*) \cdot ((v - \tilde{v}) + (v_* - \tilde{v}_*))\Phi_K(\tilde{x}) \end{aligned}$$

which is $A_2^K(v, v_*, \tilde{v}, \tilde{v}_*)$ as desired. \square

Next, we study each term found in the previous inequality. We start with the Maxwell case.

LEMMA 3.2. *Assume (1.3), (1.5) with $\gamma = 0$, (1.7) and adopt the notation of Lemma 3.1. For all $K \in [1, \infty)$, all $v, v_*, \tilde{v}, \tilde{v}_* \in \mathbb{R}^3$:*

- (i) $A_1^K(v, v_*, \tilde{v}, \tilde{v}_*) = 0$,
- (ii) $A_2^K(v, v_*, \tilde{v}, \tilde{v}_*) = \zeta_K[-|v - \tilde{v}|^2 + |v_* - \tilde{v}_*|^2]$ where $\zeta_K = \pi \int_0^K (1 - \cos G(z)) dz$,
- (iii) $A_3^K(v, v_*, \tilde{v}, \tilde{v}_*) \leq C(|v|^2 + |v_*|^2 + |\tilde{v}|^2)K^{1-2/\nu}$.

PROOF. Point (i) is obvious. Point (ii) immediately follows from the fact that $\Psi_k(x) = \zeta_K$ does not depend on x . Point (iii) holds true because $\Psi_K(x) = \pi \int_K^\infty (1 - \cos G(z)) dz \leq \pi \int_K^\infty G^2(z) dz \leq CK^{1-2/\nu}$ by (1.10). \square

The case of hard potentials is much more complicated. The following result gives a possible and useful upper bound on the A_i^K functions.

LEMMA 3.3. *Assume (1.3), (1.5) with $\gamma \in (0, 1)$, (1.7) and adopt the notation of Lemma 3.1.*

- (i) *For all $q > 0$, there is $C_q > 0$ such that for all $M \geq 1$, all $K \in [1, \infty)$, all $v, v_*, \tilde{v}, \tilde{v}_* \in \mathbb{R}^3$,*

$$A_1^K(v, v_*, \tilde{v}, \tilde{v}_*) \leq M(|v - \tilde{v}|^2 + |v_* - \tilde{v}_*|^2) + C_q e^{-Mq/\gamma} e^{C_q(|v|^q + |v_*|^q)}.$$

- (ii) *There is $C > 0$ such that for all $K \in [1, \infty)$, all $v, v_*, \tilde{v}, \tilde{v}_* \in \mathbb{R}^3$ and all $z_* \in \mathbb{R}^3$,*

$$A_2^K(v, v_*, \tilde{v}, \tilde{v}_*) - A_2^K(v, z_*, \tilde{v}, \tilde{v}_*) \leq C[|v - \tilde{v}|^2 + |v_* - \tilde{v}_*|^2 + |v_* - z_*|^2(1 + |v| + |v_*| + |z_*|)^{2\gamma/(1-\gamma)}].$$

- (iii) *There is $C > 0$ such that for all $K \in [1, \infty)$, all $v, v_*, \tilde{v}, \tilde{v}_* \in \mathbb{R}^3$,*

$$A_3^K(v, v_*, \tilde{v}, \tilde{v}_*) \leq C(1 + |v|^{4\gamma/\nu+2} + |v_*|^{4\gamma/\nu+2} + |\tilde{v}|^2 + |\tilde{v}_*|^2)K^{1-2/\nu}.$$

This lemma is very technical. The reason is the following. The solution $(f_t)_{t \geq 0}$ has bounded exponential moments while, on the contrary, the particle system has only a bounded energy (moment of order 2). If $K \in [1, \infty)$, the particle system has all moments finite, which makes all the computations licit, but the moments of order strictly greater than 2 are not uniformly bounded with respect to K (at least, we were not able to show it). We will use the previous estimates with v, v_* (and z_*) taken from the solution f_t and \tilde{v}, \tilde{v}_* taken in the particle system. Thus, it is very important that these estimates do not involve powers greater than 2 of \tilde{v}, \tilde{v}_* . For example, in point (i), only v, v_* appear in the exponential, and this is crucial.

PROOF OF LEMMA 3.3. Using (1.11) and that $|x^\gamma - y^\gamma| \leq 2|x - y|/(x^{1-\gamma} + y^{1-\gamma})$, we get

$$\begin{aligned}
 A_1^K(v, v_*, \tilde{v}, \tilde{v}_*) &\leq 2c_4|v - v_*||\tilde{v} - \tilde{v}_*| \frac{(|v - v_*|^\gamma - |\tilde{v} - \tilde{v}_*|^\gamma)^2}{|v - v_*|^\gamma + |\tilde{v} - \tilde{v}_*|^\gamma} \\
 (3.1) \qquad \qquad \qquad &\leq 8c_4 \frac{|v - v_*| \wedge |\tilde{v} - \tilde{v}_*|}{(|v - v_*| \vee |\tilde{v} - \tilde{v}_*|)^{1-\gamma}} (|v - v_*| - |\tilde{v} - \tilde{v}_*|)^2.
 \end{aligned}$$

Now for any $M \geq 1$, this is bounded from above by

$$\begin{aligned}
 &\frac{M}{2} (|v - v_*| - |\tilde{v} - \tilde{v}_*|)^2 \\
 &+ 8c_4 (|v - v_*| \vee |\tilde{v} - \tilde{v}_*|)^{2+\gamma} \mathbb{1}_{\{8c_4(|v-v_*| \wedge |\tilde{v}-\tilde{v}_*|)/(|v-v_*| \vee |\tilde{v}-\tilde{v}_*|)^{1-\gamma} \geq M/2\}} \\
 &\leq \frac{M}{2} (|v - \tilde{v}| + |v_* - \tilde{v}_*|)^2 + 8c_4 \left[\frac{16c_4}{M} (|v - v_*| \wedge |\tilde{v} - \tilde{v}_*|) \right]^{(2+\gamma)/(1-\gamma)} \\
 &\qquad \qquad \qquad \times \mathbb{1}_{\{(|v-v_*| \wedge |\tilde{v}-\tilde{v}_*|)/(|v-v_*| \vee |\tilde{v}-\tilde{v}_*|)^{1-\gamma} \geq M/(16c_4)\}} \\
 &\leq M (|v - \tilde{v}|^2 + |v_* - \tilde{v}_*|^2) + 8c_4 [16c_4 (|v - v_*| \wedge |\tilde{v} - \tilde{v}_*|)]^{(2+\gamma)/(1-\gamma)} \\
 &\qquad \qquad \qquad \times \mathbb{1}_{\{(|v-v_*| \wedge |\tilde{v}-\tilde{v}_*|)^\gamma \geq M/(16c_4)\}} \\
 &\leq M (|v - \tilde{v}|^2 + |v_* - \tilde{v}_*|^2) + 8c_4 [16c_4 (|v| + |v_*|)]^{(2+\gamma)/(1-\gamma)} \\
 &\qquad \qquad \qquad \times \mathbb{1}_{\{(|v|+|v_*|)^\gamma \geq M/(16c_4)\}}.
 \end{aligned}$$

Fix now $q > 0$, and observe that

$$\begin{aligned}
 x^{(2+\gamma)/(1-\gamma)} \mathbb{1}_{\{x^\gamma \geq M/(16c_4)\}} &\leq x^{(2+\gamma)/(1-\gamma)} e^{-Mq/\gamma} e^{(16c_4)^{q/\gamma} x^q} \\
 &\leq C_q e^{-Mq/\gamma} e^{2(16c_4)^{q/\gamma} x^q}.
 \end{aligned}$$

Point (i) follows.

Point (ii) is quite delicate. First, there is C such that for all $K \in [1, \infty)$, all $x, y > 0$,

$$\Phi_K(x) \leq Cx^\gamma \quad \text{and} \quad |\Phi_K(x) - \Phi_K(y)| \leq C|x^\gamma - y^\gamma|.$$

Indeed, it is enough to prove that for $\Gamma_K(x) = \int_0^K (1 - \cos G(z/x)) dz$, $\Gamma_K(0) = 0$ and $|\Gamma'_K(x)| \leq C$. But $\Gamma_K(x) = x \int_0^{K/x} (1 - \cos G(z)) dz \leq x \int_0^\infty G^2(z) dz$, so that $\Gamma_K(0) = 0$ and $|\Gamma'_K(x)| \leq \int_0^\infty (1 - \cos G(z)) dz + x(K/x^2)(1 - \cos G(K/x)) \leq \int_0^\infty G^2(z) dz + (K/x)G^2(K/x)$, which is uniformly bounded by (1.10). Consequently, for all $X, Y \in \mathbb{R}^3$,

$$\begin{aligned}
 &|X\Phi_K(|X|) - Y\Phi_K(|Y|)| \\
 &\leq C|X - Y|(|X|^\gamma + |Y|^\gamma) + C(|X| + |Y|)||X|^\gamma - |Y|^\gamma|.
 \end{aligned}$$

Using again that $|x^\gamma - y^\gamma| \leq 2|x - y|/(x^{1-\gamma} + y^{1-\gamma})$, we easily conclude that

$$(3.2) \quad |X\Phi_K(|X|) - Y\Phi_K(|Y|)| \leq C|X - Y|(|X|^\gamma + |Y|^\gamma).$$

Now we write

$$(3.3) \quad \begin{aligned} \Delta_2^K &:= A_2^K(v, v_*, \tilde{v}, \tilde{v}_*) - A_2^K(v, z_*, \tilde{v}, \tilde{v}_*) \\ &= -[(v - \tilde{v}) + (v_* - \tilde{v}_*)] \\ &\quad \times [(v - v_*)\Phi_K(|v - v_*|) - (\tilde{v} - \tilde{v}_*)\Phi_K(|\tilde{v} - \tilde{v}_*|)] \\ &\quad + [(v - \tilde{v}) + (z_* - \tilde{v}_*)] \\ &\quad \times [(v - z_*)\Phi_K(|v - z_*|) - (\tilde{v} - \tilde{v}_*)\Phi_K(|\tilde{v} - \tilde{v}_*|)] \\ &= -[(v - \tilde{v}) + (v_* - \tilde{v}_*)] \\ &\quad \times [(v - v_*)\Phi_K(|v - v_*|) - (v - z_*)\Phi_K(|v - z_*|)] \\ &\quad + (z_* - v_*) \cdot [(v - z_*)\Phi_K(|v - z_*|) - (\tilde{v} - \tilde{v}_*)\Phi_K(|\tilde{v} - \tilde{v}_*|)]. \end{aligned}$$

By (3.2) and the Young inequality, we deduce that

$$\begin{aligned} \Delta_2^K &\leq C(|v - \tilde{v}| + |v_* - \tilde{v}_*|)|v_* - z_*|(|v - v_*|^\gamma + |v - z_*|^\gamma) \\ &\quad + C|z_* - v_*|(|v - \tilde{v}| + |z_* - \tilde{v}_*|)(|v - z_*|^\gamma + |\tilde{v} - \tilde{v}_*|^\gamma) \\ &\leq C[(|v - \tilde{v}| + |v_* - \tilde{v}_*|)^2 + |v_* - z_*|^2(|v - v_*|^\gamma + |v - z_*|^\gamma)^2] \\ &\quad + C|z_* - v_*|(|v - \tilde{v}| + |z_* - v_*| + |v_* - \tilde{v}_*|) \\ &\quad \times (|v - z_*|^\gamma + (|v - \tilde{v}| + |v - v_*| + |v_* - \tilde{v}_*|)^\gamma). \end{aligned}$$

The first term is clearly bounded by $C(|v - \tilde{v}|^2 + |v_* - \tilde{v}_*|^2 + |v_* - z_*|^2(1 + |v| + |v_*| + |z_*|)^{2\gamma})$ which fits the statement, since $2\gamma \leq 2\gamma/(1 - \gamma)$. We next bound the second term by

$$\begin{aligned} &C|z_* - v_*|^2(|v - z_*| + |v - v_*|)^\gamma \\ &\quad + C|z_* - v_*|^2(|v - \tilde{v}| + |v_* - \tilde{v}_*|)^\gamma \\ &\quad + C|z_* - v_*|(|v - \tilde{v}| + |v_* - \tilde{v}_*|)(|v - z_*| + |v - v_*|)^\gamma \\ &\quad + C|z_* - v_*|(|v - \tilde{v}| + |v_* - \tilde{v}_*|)^{1+\gamma}. \end{aligned}$$

Using that $x^2y^\gamma \leq x^{4/(2-\gamma)} + y^2$ (for the second line), that $xyz^\gamma \leq (xz^\gamma)^2 + y^2$ (for the third line) and that $xy^{1+\gamma} \leq x^{2/(1-\gamma)} + y^2$, we obtain the upper-bound

$$\begin{aligned} &C|z_* - v_*|^2(1 + |v| + |z_*| + |v_*|)^\gamma \\ &\quad + C(|v - \tilde{v}| + |v_* - \tilde{v}_*|)^2 + |z_* - v_*|^{4/(2-\gamma)} \\ &\quad + C(|v - \tilde{v}| + |v_* - \tilde{v}_*|)^2 + |z_* - v_*|^2(|v - z_*| + |v - v_*|)^{2\gamma} \\ &\quad + C(|v - \tilde{v}| + |v_* - \tilde{v}_*|)^2 + |z_* - v_*|^{2/(1-\gamma)}, \end{aligned}$$

which is bounded by

$$\begin{aligned}
 & C(|v - \tilde{v}|^2 + |v_* - \tilde{v}_*|^2) \\
 & + C|z_* - v_*|^2 \{ (1 + |v| + |z_*| + |v_*|)^\gamma + |z_* - v_*|^{4/(2-\gamma)-2} \\
 & \quad + (|v - z_*| + |v - v_*|)^{2\gamma} + |z_* - v_*|^{2/(1-\gamma)-2} \}.
 \end{aligned}$$

One easily concludes, using that $\max\{\gamma, 4/(2 - \gamma) - 2, 2\gamma, 2/(1 - \gamma) - 2\} = 2\gamma/(1 - \gamma)$.

We finally check point (iii). Using (1.10), we deduce that $1 - \cos(G(z/x^\gamma)) \leq G^2(z/x^\gamma) \leq C(z/x^\gamma)^{-2/\nu}$, whence

$$\Psi_K(x) \leq Cx^{2\gamma/\nu} \int_K^\infty z^{-2/\nu} dz = Cx^{2\gamma/\nu} K^{1-2/\nu}.$$

Thus

$$(3.4) \quad A_3^K(v, v_*, \tilde{v}, \tilde{v}_*) \leq C(|v - v_*|^2 + |v - v_*||\tilde{v} - \tilde{v}_*|)|v - v_*|^{2\gamma/\nu} K^{1-2/\nu},$$

from which we easily conclude, using that $|\tilde{v} - \tilde{v}_*||v - v_*|^{1+2\gamma/\nu} \leq |\tilde{v} - \tilde{v}_*|^2 + |v - v_*|^{2+4\gamma/\nu}$. \square

We conclude with the hard spheres case.

LEMMA 3.4. *Assume (1.3), (1.5) with $\gamma = 1$, (1.6) and adopt the notation of Lemma 3.1.*

(i) *For all $q > 0$, there is $C_q > 0$ such that for all $M \geq 1$, all $K \in [1, \infty)$, all $v, v_*, \tilde{v}, \tilde{v}_* \in \mathbb{R}^3$,*

$$\begin{aligned}
 & A_1^K(v, v_*, \tilde{v}, \tilde{v}_*) \\
 & \leq M(|v - \tilde{v}|^2 + |v_* - \tilde{v}_*|^2) + C_q K (|\tilde{v}| + |\tilde{v}_*|) e^{-M^q} e^{C_q(|v|^q + |v_*|^q)}.
 \end{aligned}$$

(ii) *For all $q > 0$, there is $C_q > 0$ such that for all $M \geq 1$, all $K \in [1, \infty)$, all $v, v_*, \tilde{v}, \tilde{v}_* \in \mathbb{R}^3$,*

$$\begin{aligned}
 & A_2^K(v, v_*, \tilde{v}, \tilde{v}_*) - A_2^K(v, z_*, \tilde{v}, \tilde{v}_*) \\
 & \leq M(|v - \tilde{v}|^2 + |v_* - \tilde{v}_*|^2) \\
 & \quad + C|v_* - z_*|^2(1 + |v| + |v_*| + |z_*|)^2 \\
 & \quad + C_q(1 + |\tilde{v}| + |\tilde{v}_*|)K e^{-M^q} e^{C_q(|v|^q + |v_*|^q + |z_*|^q)}.
 \end{aligned}$$

(iii) *For all $q > 0$, there is $C_q > 0$ such that for all $K \in [1, \infty)$, all $v, v_*, \tilde{v}, \tilde{v}_* \in \mathbb{R}^3$,*

$$A_3^K(v, v_*, \tilde{v}, \tilde{v}_*) \leq C_q(1 + |\tilde{v}|)e^{-K^q} e^{C_q(|v|^q + |v_*|^q + |z_*|^q)}.$$

PROOF. On the one hand, (1.11) implies

$$\begin{aligned} A_1^K(v, v_*, \tilde{v}, \tilde{v}_*) &\leq 2c_4|v - v_*||\tilde{v} - \tilde{v}_*| \frac{(|v - v_*| - |\tilde{v} - \tilde{v}_*|)^2}{|v - v_*| + |\tilde{v} - \tilde{v}_*|} \\ &\leq 4c_4(|v - v_*| \wedge |\tilde{v} - \tilde{v}_*|)(|v - \tilde{v}|^2 + |v_* - \tilde{v}_*|^2). \end{aligned}$$

On the other hand, since G takes values in $(0, \pi/2)$, we obviously have

$$A_1^K(v, v_*, \tilde{v}, \tilde{v}_*) \leq \frac{\pi^2}{2} K |v - v_*| |\tilde{v} - \tilde{v}_*|.$$

Consequently, we may write

$$\begin{aligned} A_1^K(v, v_*, \tilde{v}, \tilde{v}_*) &\leq M(|v - \tilde{v}|^2 + |v_* - \tilde{v}_*|^2) \\ &\quad + \frac{\pi^2}{2} K |v - v_*| |\tilde{v} - \tilde{v}_*| \mathbb{1}_{\{4c_4(|v - v_*| \wedge |\tilde{v} - \tilde{v}_*|) \geq M\}}. \end{aligned}$$

Point (i) easily follows, using that $|v - v_*| \mathbb{1}_{\{4c_4(|v - v_*| \wedge |\tilde{v} - \tilde{v}_*|) \geq M\}} \leq |v - v_*| \times \mathbb{1}_{\{4c_4|v - v_*| \geq M\}} \leq |v - v_*| e^{-M^q} e^{(4c_4|v - v_*|)^q} \leq C_q e^{-M^q} e^{2(4c_4|v - v_*|)^q} \leq C_q e^{-M^q} \times e^{2^{q+1}(4c_4)^q(|v|^q + |v_*|^q)}$.

Using all the computations of the proof of Lemma 3.3(ii), except the one that makes the power $2/(1 - \gamma)$ appear, we see that for $\Delta_2^K := A_2^K(v, v_*, \tilde{v}, \tilde{v}_*) - A_2^K(v, z_*, \tilde{v}, \tilde{v}_*)$

$$\begin{aligned} \Delta_2^K &\leq C[|v - \tilde{v}|^2 + |v_* - \tilde{v}_*|^2 + |v_* - z_*|^2(1 + |v| + |v_*| + |z_*|)^2 \\ &\quad + |z_* - v_*|(|v - \tilde{v}|^2 + |v_* - \tilde{v}_*|^2)] \\ &\leq C(1 + |z_* - v_*|)(|v - \tilde{v}|^2 + |v_* - \tilde{v}_*|^2) \\ &\quad + C|v_* - z_*|^2(1 + |v| + |v_*| + |z_*|)^2. \end{aligned}$$

On the other hand, starting from (3.3) and using that $\phi_K(x) \leq \pi K$, we realize that

$$\Delta_2^K \leq CK(1 + |\tilde{v}| + |\tilde{v}_*|)(1 + |v|^2 + |v_*|^2 + |z_*|^2).$$

Hence we can write, for any $M > 1$,

$$\begin{aligned} \Delta_2^K &\leq M(|v - \tilde{v}|^2 + |v_* - \tilde{v}_*|^2) + C|v_* - z_*|^2(1 + |v| + |v_*| + |z_*|)^2 \\ &\quad + CK(1 + |\tilde{v}| + |\tilde{v}_*|)(1 + |v|^2 + |v_*|^2 + |z_*|^2) \mathbb{1}_{\{C(1 + |z_* - v_*|) \geq M\}}. \end{aligned}$$

But $(1 + |v|^2 + |v_*|^2 + |z_*|^2) \mathbb{1}_{\{C(1 + |z_* - v_*|) \geq M\}} \leq (1 + |v| + |v_*| + |z_*|)^2 \times \mathbb{1}_{\{C(1 + |v| + |v_*| + |z_*|) \geq M\}} \leq (1 + |v| + |v_*| + |z_*|)^2 e^{-M^q} e^{C^q(1 + |v| + |v_*| + |z_*|)^q} \leq C_q e^{-M^q} e^{C_q(|v|^q + |v_*|^q + |z_*|^q)}$. Point (ii) is checked.

Finally, we observe that $\Psi_K(x) \leq \pi \int_K^\infty G^2(z/x) dz$. But here, $G(z) = (\pi/2 - z)_+$ whence $\Psi_K(x) \leq (\pi^4/24)x \mathbb{1}_{\{x \geq 2K/\pi\}} \leq 5x \mathbb{1}_{\{x \geq K/2\}}$. Thus for any $q > 0$,

$\Psi_K(x) \leq 5xe^{-K^q} e^{2^q x^q}$, so that

$$\begin{aligned} A_3^K(v, v_*, \tilde{v}, \tilde{v}_*) &\leq C(1 + |\tilde{v}|)(1 + |v|^2 + |v_*|^2)e^{-K^q} |v - v_*|e^{2^q |v - v_*|^q} \\ &\leq C_q(1 + |\tilde{v}|)e^{-K^q} e^{C_q(|v|^q + |v_*|^q)} \end{aligned}$$

as desired. \square

4. Convergence of the particle system with cutoff. To build a suitable coupling between the particle system and the solution to (1.1), we need to introduce the (stochastic) paths associated to (1.1). To do so, we follow the ideas of Tanaka [40, 41] and make use of two probability spaces. The main one is an abstract $(\Omega, \mathcal{F}, \Pr)$, on which the random objects are defined when nothing is made precise. But we will also need an auxiliary one, $[0, 1]$ endowed with its Borel σ -field and its Lebesgue measure. In order to avoid confusion, a random variable defined on this latter probability space will be called an α -random variable, expectation on $[0, 1]$ will be denoted by \mathbb{E}_α , etc.

4.1. *A SDE for the Boltzmann equation.* First, we recall the classical probabilistic interpretation of the Boltzmann equation initiated by Tanaka [40, 41] in the Maxwell molecules case.

PROPOSITION 4.1. *Assume (1.3), (1.5), (1.6) or (1.7), and let $f_0 \in \mathcal{P}_2(\mathbb{R}^3)$. If $\gamma \in (0, 1]$, assume additionally (1.8) and that f_0 satisfies (1.12). Let $(f_t)_{t \geq 0}$ be the corresponding unique weak solution to (1.1). Consider any f_0 -distributed random variable W_0 and any independent Poisson measure $M(ds, d\alpha, dz, d\varphi)$ on $[0, \infty) \times [0, 1] \times [0, \infty) \times [0, 2\pi)$ with intensity measure $ds d\alpha dz d\varphi$. Consider also, for each $t \geq 0$, a f_t -distributed α -random variable W_t^* , in such a way that $(t, \alpha) \mapsto W_t^*(\alpha)$ is measurable. Then there is a unique (càdlàg adapted) strong solution to*

$$(4.1) \quad W_t = W_0 + \int_0^t \int_{\mathbb{R}^3} \int_0^\infty \int_0^{2\pi} c(W_{s-}, W_s^*(\alpha), z, \varphi) M(ds, d\alpha, dz, d\varphi).$$

Furthermore, W_t is f_t -distributed for each $t \geq 0$.

We will note $(W_t)_{t \geq 0}$ such a Boltzmann process. It can be viewed as the time-evolution of the velocity of a typical particle in the gas.

PROOF OF PROPOSITION 4.1. The proof is very similar to that of [18], Proposition 5.1 (see also [20], Section 4) and is omitted. In [18], Proposition 5.1, the same Boltzmann equation is studied, with fewer assumptions on f_0 [so that uniqueness is not known for (1.1)]. But the formulation of the SDE is different (it is equivalent in law). The same proof as in [18], Proposition 5.1, works here, with several difficulties avoided due to the facts that f_0 has exponential moments and that uniqueness is known to hold for (1.1). \square

4.2. *A SDE for the particle system.* Here we write down a Poisson stochastic differential equation corresponding to Nanbu’s particle system, and we prove Proposition 1.2(i).

PROPOSITION 4.2. *Assume (1.3), (1.5), (1.6) or (1.7), and let $f_0 \in \mathcal{P}_2(\mathbb{R}^3)$, $N \geq 1$ and $K \in [1, \infty)$. Consider a family $(V_0^i)_{i=1, \dots, N}$ of i.i.d. f_0 -distributed random variables and an independent family $(O_i^N(ds, dj, dz, d\varphi))_{i=1, \dots, N}$ of Poisson measures on $[0, \infty) \times \{1, \dots, N\} \times [0, \infty) \times [0, 2\pi)$ with intensity measures $ds(N^{-1} \sum_{k=1}^N \delta_k(dj)) dz d\varphi$. There exists a unique (càdlàg and adapted) strong solution to*

$$(4.2) \quad \begin{aligned} V_t^{i,N,K} &= V_0^i + \int_0^t \int_j \int_0^\infty \int_0^{2\pi} c_K(V_{s-}^{i,N,K}, V_{s-}^{j,N,K}, z, \varphi) \\ &\quad \times O_i^N(ds, dj, dz, d\varphi), \quad i = 1, \dots, N. \end{aligned}$$

Furthermore, $(V_t^{i,N,K})_{i=1, \dots, N, t \geq 0}$ is Markov with generator $\mathcal{L}_{N,K}$. We have $\mathbb{E}[|V_t^{1,N,K}|^2] = \int_{\mathbb{R}^3} |v|^2 f_0(dv)$, and if $\int_{\mathbb{R}^3} |v|^p f_0(dv) < \infty$ for some $p \geq 2$, $\sup_{[0,T]} \mathbb{E}[|V_t^{1,N,K}|^p] \leq C_{p,T,f_0,K}$.

PROOF. First of all, observe that we actually deal with finite Poisson measures, since c_K vanishes for $z \geq K$. Thus, strong existence and uniqueness for (4.2) is trivial: it suffices to work recursively on the instants of jumps (which are discrete) of the family $(O_i^N(ds, dj, dz, d\varphi))_{i=1, \dots, N}$. Consequently, $\mathbf{V}_t^{N,K} = (V_t^{1,N,K}, \dots, V_t^{N,N,K})$ is a Markov process, since it solves a well-posed time-homogeneous SDE. Its infinitesimal generator is classically defined by (2.7), with actually a sum over all couples $(i, j) \in \{1, \dots, N\}^2$, but this changes nothing since the terms with $i = j$ vanish because $c_K(v, v, z, \varphi) = 0$ for all $v \in \mathbb{R}^3$. Next, a simple computation shows that

$$\begin{aligned} &\mathbb{E}[|V_t^{1,N,K}|^2] \\ &= \mathbb{E}[|V_0^1|^2] \\ &\quad + \frac{1}{N} \sum_{j=1}^N \int_0^t \int_0^\infty \int_0^{2\pi} \mathbb{E}(|V_s^{1,N,K} + c_K(V_s^{1,N,K}, V_s^{j,N,K}, z, \varphi)|^2 \\ &\quad \quad \quad - |V_s^{1,N,K}|^2) d\varphi dz ds \\ &= \mathbb{E}[|V_0^1|^2] \\ &\quad + \frac{N-1}{N} \int_0^t \int_0^K \int_0^{2\pi} \mathbb{E}(|c(V_s^{1,N,K}, V_s^{2,N,K}, z, \varphi)|^2 \\ &\quad \quad \quad + 2V_s^{1,N,K} \cdot c(V_s^{1,N,K}, V_s^{2,N,K}, z, \varphi)) d\varphi dz ds \end{aligned}$$

by exchangeability. But, as seen in the proof of Lemma 3.1,

$$\int_0^K \int_0^{2\pi} (|c(v, v_*, z, \varphi)|^2 + 2v \cdot c(v, v_*, z, \varphi)) d\varphi dz = [|v - v_*|^2 - 2v \cdot (v - v_*)] \Phi_K(|v - v_*|),$$

whence, using again exchangeability,

$$\begin{aligned} & \mathbb{E}[|V_t^{1,N,K}|^2] \\ &= \mathbb{E}[|V_0^1|^2] + \frac{N-1}{N} \int_0^t \mathbb{E}[(|V_s^{1,N,K} - V_s^{2,N,K}|^2 - 2V_s^{1,N,K} \cdot (V_s^{1,N,K} - V_s^{2,N,K})) \times \Phi_K(|V_s^{1,N,K} - V_s^{2,N,K}|)] ds \\ &= \mathbb{E}[|V_0^1|^2] + \frac{N-1}{N} \int_0^t \mathbb{E}[(|V_s^{1,N,K} - V_s^{2,N,K}|^2 - V_s^{1,N,K} \cdot (V_s^{1,N,K} - V_s^{2,N,K}) - V_s^{2,N,K} \cdot (V_s^{2,N,K} - V_s^{1,N,K})) \times \Phi_K(|V_s^{1,N,K} - V_s^{2,N,K}|)] ds. \end{aligned}$$

In this last expression, the integrand is zero, so that, as claimed, $\mathbb{E}[|V_t^{1,N,K}|^2] = \mathbb{E}[|V_0^1|^2] = \int_{\mathbb{R}^3} |v|^2 f_0(dv)$. Recalling finally (2.1) and (2.5), we see that $|c(v, v_*, z, \varphi)| \leq |v - v_*|$. Thus for $p \geq 2$,

$$\int_0^K \int_0^{2\pi} (|v + c(v, v_*, z, \varphi)|^p - |v|^p) d\varphi dz \leq C_p K (|v| + |v_*|^p).$$

Consequently, we obtain as previously

$$\mathbb{E}[|V_t^{1,N,K}|^p] \leq \mathbb{E}[|V_0^1|^p] + \frac{C_p K}{N} \sum_{j=1}^N \int_0^t \mathbb{E}[|V_s^{1,N,K}|^p + |V_s^{j,N,K}|^p] ds$$

and conclude, using again exchangeability, that $\mathbb{E}[|V_t^{1,N,K}|^p] \leq \mathbb{E}[|V_0^1|^p] e^{2C_p K t}$ as desired. \square

This allows us to deduce the following:

PROOF OF PROPOSITION 1.2(i). The strong existence and uniqueness for the SDE (4.2) classically implies the existence and uniqueness of a Markov process with generator $\mathcal{L}_{N,K}$. \square

4.3. *The coupling.* Here we explain how we couple our particle system with a family of i.i.d. Boltzmann processes. For example, we want to couple $V_t^{1,N,K}$ with a Boltzmann process W_t^1 . The main difficulty is that at each collision, W_t^1 is collided by an independent particle (using W_t^*) while $V_t^{1,N,K}$ is collided by some $V_t^{j,N,K}$. We thus have to choose j in such a way that $V_t^{j,N,K}$ is as close as possible to W_t^* , but j has to remain uniformly chosen.

A technical problem obliges us to introduce the set $(\mathbb{R}^3)_\bullet^N := \{\mathbf{w} \in (\mathbb{R}^3)^N : w_i \neq w_j \forall i \neq j\}$.

LEMMA 4.3. *Let $f_t \in C([0, \infty), \mathcal{P}_2(\mathbb{R}^3))$ be such that f_t has a density for all $t > 0$. Let also $N \geq 1$ be fixed. For $\mathbf{v} = (v_1, \dots, v_N) \in (\mathbb{R}^3)^N$, we denote by $\mu_{\mathbf{v}}^N := N^{-1} \sum_1^N \delta_{v_i}$ the empirical measure associated to \mathbf{v} . There exists a measurable map $(t, \mathbf{w}, \mathbf{v}, \alpha) \mapsto (W_t^*(\alpha), Z_t^*(\mathbf{w}, \alpha), V_t^*(\mathbf{v}, \mathbf{w}, \alpha))$ from $(0, \infty) \times (\mathbb{R}^3)_\bullet^N \times (\mathbb{R}^3)^N \times [0, 1]$ into $\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$ enjoying the following properties:*

- (a) for all $t \geq 0$, the α -law of W_t^* is f_t ;
- (b) for all $t \geq 0$, $\mathbf{w} \in (\mathbb{R}^3)_\bullet^N$, the α -law of $Z_t^*(\mathbf{w}, \cdot)$ is $\mu_{\mathbf{w}}^N$;
- (c) for all $t \geq 0$, $\mathbf{w} \in (\mathbb{R}^3)_\bullet^N$, $\mathbf{v} \in (\mathbb{R}^3)^N$, the α -law of $V_t^*(\mathbf{v}, \mathbf{w}, \cdot)$ is $\mu_{\mathbf{v}}^N$;
- (d) for all $t \geq 0$, $\mathbf{w} \in (\mathbb{R}^3)_\bullet^N$, $\mathbf{v} \in (\mathbb{R}^3)^N$, the α -law of $(Z_t^*(\mathbf{w}, \cdot), V_t^*(\mathbf{v}, \mathbf{w}, \cdot))$ is $N^{-1} \sum_1^N \delta_{(w_i, v_i)}$;
- (e) for all $t \geq 0$, all $\mathbf{w} \in (\mathbb{R}^3)_\bullet^N$, $\int_0^1 |W_t^*(\alpha) - Z_t^*(\mathbf{w}, \alpha)|^2 d\alpha = \mathcal{W}_2^2(f_t, \mu_{\mathbf{w}}^N)$.

PROOF. We first consider, for each $t > 0$, W_t^* such that point (a) holds true and such that $(t, \alpha) \mapsto W_t^*(\alpha)$ is measurable.

Next, we recall that by Brenier’s theorem (see, e.g., Villani [44], Theorem 2.12, page 66) for each $t > 0$ and each $\mathbf{w} \in (\mathbb{R}^3)^N$, since f_t does not charge small sets (because it has a density by [18]), there exists a unique map $F_{t,\mathbf{w}}: \mathbb{R}^3 \mapsto \mathbb{R}^3$ such that, setting $Z_t^*(\mathbf{w}, \alpha) := F_{t,\mathbf{w}}(W_t^*(\alpha))$, points (b) and (e) hold true. In other words, $(W_t^*(\cdot), Z_t^*(\mathbf{w}, \cdot))$ is an optimal coupling for f_t and $\mu_{\mathbf{w}}^N$. Furthermore, Fontbona, Guérin and Méléard [17] have shown that $F_{t,\mathbf{w}}(x)$ is a measurable function of (t, \mathbf{w}, x) . Consequently, $Z_t^*(\mathbf{w}, \alpha)$ is a measurable function of (t, \mathbf{w}, α) .

Finally, we define, for any $\mathbf{w} \in (\mathbb{R}^3)_\bullet^N$ and any $\mathbf{v} \in (\mathbb{R}^3)^N$, the application $G_{\mathbf{w},\mathbf{v}}: \{w_1, \dots, w_N\} \mapsto \{v_1, \dots, v_N\}$ by $G_{\mathbf{w},\mathbf{v}}(w_i) = v_i$ [here we need that $\mathbf{w} \in (\mathbb{R}^3)_\bullet^N$]. We then put $V_t^*(\mathbf{v}, \mathbf{w}, \alpha) = G_{\mathbf{w},\mathbf{v}}(Z_t^*(\mathbf{w}, \alpha))$, which is clearly measurable (in all its variables). Point (d) follows from (b) and the definition of $G_{\mathbf{w},\mathbf{v}}$, and finally (c) follows from (d). \square

Here is the coupling we propose.

LEMMA 4.4. *Assume (1.3), (1.5), (1.6) or (1.7). Let $f_0 \in \mathcal{P}_2(\mathbb{R}^3)$. Assume additionally (1.8) and (1.12) if $\gamma \in (0, 1]$. Let $(f_t)_{t \geq 0}$ be the unique weak solution*

to (1.1), and assume that f_t has a density for all $t > 0$; see Theorem 1.1. Consider $N \geq 1$ and $K \in [1, \infty)$ fixed. Let $(V_0^i)_{i=1, \dots, N}$ be i.i.d. with common law f_0 , and let $(M_i(ds, d\alpha, dz, d\varphi))_{i=1, \dots, N}$ be an i.i.d. family of Poisson measures on $[0, \infty) \times [0, 1] \times [0, \infty) \times [0, 2\pi)$ with intensity measures $ds d\alpha dz d\varphi$, independent of $(V_0^i)_{i=1, \dots, N}$.

(i) The following SDE's, for $i = 1, \dots, N$, define N independent copies of the Boltzmann process:

$$W_t^i = V_0^i + \int_0^t \int_0^1 \int_0^\infty \int_0^{2\pi} c(W_{s-}^i, W_s^*(\alpha), z, \varphi) M_i(ds, d\alpha, dz, d\varphi).$$

In particular, for each $t \geq 0$, $(W_t^i)_{i=1, \dots, N}$ are i.i.d. with common law f_t . Consequently, since f_t has a density for all $t > 0$, $(W_t^i)_{i=1, \dots, N} \in (\mathbb{R}^3)^\bullet_N$ a.s.

(ii) Next, we consider the system of SDE's, for $i = 1, \dots, N$,

$$V_t^{i,N,K} = V_0^i + \int_0^t \int_0^1 \int_0^\infty \int_0^{2\pi} c_K(V_{s-}^{i,N,K}, V_s^*(\mathbf{V}_{s-}^{N,K}, \mathbf{W}_{s-}, \alpha), z, \varphi + \varphi_{i,\alpha,s}) \times M_i(ds, d\alpha, dz, d\varphi).$$

We used the notation $\mathbf{V}_{s-}^{N,K} = (V_{s-}^{1,N,K}, \dots, V_{s-}^{N,N,K}) \in (\mathbb{R}^3)^N$, $\mathbf{W}_{s-} = (W_{s-}^1, \dots, W_{s-}^N) \in (\mathbb{R}^3)^\bullet_N$, and we have set $\varphi_{i,\alpha,s} := \varphi_0(W_{s-}^i - W_s^*(s, \alpha), V_{s-}^{i,N,K} - V_s^*(\mathbf{V}_{s-}^{N,K}, \mathbf{W}_{s-}, \alpha))$ for simplicity. This system of SDEs has a unique solution, and this solution is a Markov process with generator $\mathcal{L}_{N,K}$ and initial condition $(V_0^i)_{i=1, \dots, N}$.

(iii) The family $((W_t^1, V_t^{1,N,K})_{t \geq 0}, \dots, (W_t^N, V_t^{N,N,K})_{t \geq 0})$ is exchangeable.

PROOF. Point (i) is a direct consequence of Proposition 4.1, and point (iii) follows from the exchangeability of the family $(V_0^i, M_i)_{i=1, \dots, N}$ and from uniqueness (in law). In point (ii), the existence and uniqueness result is also immediate, since the Poisson measures under consideration are finite (or rather, are finite when z is restricted to $[0, K]$, which is the case since $c_K = c \mathbb{1}_{\{z \leq K\}}$). Finally $(V_t^{1,N,K}, \dots, V_t^{N,N,K})_{t \geq 0}$ is a Markov process with generator $\mathcal{L}_{N,K}$ due to the fact that for all $\mathbf{v} \in (\mathbb{R}^3)^N$, all $\mathbf{w} \in (\mathbb{R}^3)^\bullet_N$, all $s > 0$, all $\varphi_{ij} \in [0, 2\pi)$, for all bounded measurable function $\phi : (\mathbb{R}^3)^N \mapsto \mathbb{R}$,

$$\begin{aligned} & \sum_{i=1}^N \int_0^1 \int_0^\infty \int_0^{2\pi} (\phi(\mathbf{v} + c_K(v_i, V_s^*(\mathbf{v}, \mathbf{w}, \alpha), z, \varphi + \varphi_{ij}) \cdot \mathbf{e}_i) - \phi(\mathbf{v})) d\varphi dz d\alpha \\ &= \sum_{i=1}^N \frac{1}{N} \sum_{j=1}^N \int_0^\infty \int_0^{2\pi} (\phi(\mathbf{v} + c_K(v_i, v_j, z, \varphi + \varphi_{ij}) \cdot \mathbf{e}_i) - \phi(\mathbf{v})) d\varphi dz \\ &= \frac{1}{N} \sum_{i \neq j} \int_0^\infty \int_0^{2\pi} (\phi(\mathbf{v} + c_K(v_i, v_j, z, \varphi) \cdot \mathbf{e}_i) - \phi(\mathbf{v})) d\varphi dz, \end{aligned}$$

which is nothing but $\mathcal{L}_{N,K}\phi(\mathbf{v})$; see (2.7). We used Lemma 4.3(c) for the first equality and the 2π -periodicity of c_K (in φ) and the fact that $c_K(v_i, v_i, z, \varphi) = 0$ for the second one. \square

4.4. *Estimate of the Wasserstein distance.* We can now prove our main result in the case with cutoff. We first study hard potentials.

PROOF OF THEOREM 1.4(ii) WHEN $K \in [1, \infty)$. We thus assume (1.3), (1.5) with $\gamma \in (0, 1)$ and (1.7). We consider $f_0 \in \mathcal{P}_2(\mathbb{R}^3)$ satisfying (1.12) for some $p \in (\gamma, 2)$, and fix $q \in (\gamma, p)$ for the rest of the proof. We also assume that f_0 is not a Dirac mass, so that f_t has a density for all $t > 0$. We fix $N \geq 1$ and $K \in [1, \infty)$ and consider the processes introduced in Lemma 4.4.

Step 1. A direct application of the Itô calculus for jump processes shows that

$$\begin{aligned} \mathbb{E}[|W_t^1 - V_t^{1,N,K}|^2] &= \int_0^t \int_0^1 \int_0^\infty \int_0^{2\pi} \mathbb{E}[|W_s^1 - V_s^{1,N,K} + \Delta^1(s, \alpha, z, \varphi)|^2 \\ &\quad - |W_s^1 - V_s^{1,N,K}|^2] d\varphi dz d\alpha ds, \end{aligned}$$

where

$$\begin{aligned} \Delta^1(s, \alpha, z, \varphi) &= c(W_s^1, W_s^*(s, \alpha), z, \varphi) - c_K(V_s^{1,N,K}, V_s^*(\mathbf{V}_s^{N,K}, \mathbf{W}_s, \alpha), z, \varphi + \varphi_{i,\alpha,s}). \end{aligned}$$

Using Lemma 3.1, we thus obtain

$$\mathbb{E}[|W_t^1 - V_t^{1,N,K}|^2] \leq \int_0^t [B_1^K(s) + B_2^K(s) + B_3^K(s)] ds,$$

where, for $i = 1, 2, 3$,

$$B_i^K(s) := \int_0^1 \mathbb{E}[A_i^K(W_s^1, W_s^*(\alpha), V_s^{1,N,K}, V_s^*(\mathbf{V}_s^{N,K}, \mathbf{W}_s, \alpha))] d\alpha.$$

Step 2. Using Lemma 3.3(i), we see that for all $M \geq 1$ [recall that $q \in (\gamma, p)$ is fixed],

$$\begin{aligned} B_1^K(s) &\leq M \int_0^1 \mathbb{E}[|W_s^1 - V_s^{1,N,K}|^2 + |W_s^*(\alpha) - V_s^*(\mathbf{V}_s^{N,K}, \mathbf{W}_s, \alpha)|^2] d\alpha \\ &\quad + C e^{-M^{q/\gamma}} \int_0^1 \mathbb{E}[\exp(C(|W_s^1|^q + |W_s^*(\alpha)|^q))] d\alpha \\ &\leq M \int_0^1 \mathbb{E}[|W_s^1 - V_s^{1,N,K}|^2 + |W_s^*(\alpha) - V_s^*(\mathbf{V}_s^{N,K}, \mathbf{W}_s, \alpha)|^2] d\alpha \\ &\quad + C e^{-M^{q/\gamma}}. \end{aligned}$$

To get the last inequality, we used that W_s^1 and $W_s^*(\cdot)$ are independent and satisfy $W_s^1 \sim f_s$ and $W_s^*(\cdot) \sim f_s$, whence

$$\int_0^1 \mathbb{E}[\exp(C(|W_s^1|^q + |W_s^*(\alpha)|^q))] d\alpha = \left(\int_{\mathbb{R}^3} e^{C|w|^q} f_s(dw) \right)^2 < \infty$$

by (1.13).

Step 3. Roughly speaking, B_2^K should not be far from zero for reasons of symmetry. We claim that B_2^K would be zero if $W_s^*(\alpha)$ was replaced by $Z_s^*(\mathbf{W}_s, \alpha)$. More precisely, we check here that

$$\tilde{B}_2^K(s) := \int_0^1 \mathbb{E}[A_2^K(W_s^1, Z_s^*(\mathbf{W}_s, \alpha), V_s^{1,N,K}, V_s^*(\mathbf{V}_s^{N,K}, \mathbf{W}_s, \alpha))] d\alpha = 0.$$

By Lemma 4.3(d), we simply have

$$\begin{aligned} \tilde{B}_2^K(s) &= \mathbb{E}\left[\frac{1}{N} \sum_{i=1}^N A_2^K(W_s^1, W_s^i, V_s^{1,N,K}, V_s^{i,N,K}) \right] \\ &= \frac{N-1}{N} \mathbb{E}[A_2^K(W_s^1, W_s^2, V_s^{1,N,K}, V_s^{2,N,K})] \end{aligned}$$

by exchangeability and since $A_2^K(v, v, \tilde{v}, \tilde{v}) = 0$. Finally, we write, using again exchangeability,

$$\begin{aligned} \tilde{B}_2^K(s) &= \frac{N-1}{2N} \mathbb{E}[A_2^K(W_s^1, W_s^2, V_s^{1,N,K}, V_s^{2,N,K}) \\ &\quad + A_2^K(W_s^2, W_s^1, V_s^{2,N,K}, V_s^{1,N,K})]. \end{aligned}$$

This is zero by symmetry of A_2^K : it holds that $A_2^K(v, v_*, \tilde{v}, \tilde{v}_*) + A_2^K(\tilde{v}, \tilde{v}_*, v, v_*) = 0$.

Step 4. By step 3, we thus have

$$\begin{aligned} B_2^K(s) &= \int_0^1 \mathbb{E}[A_2^K(W_s^1, W_s^*(\alpha), V_s^{1,N,K}, V_s^*(\mathbf{V}_s^{N,K}, \mathbf{W}_s, \alpha)) \\ &\quad - A_2^K(W_s^1, Z_s^*(\mathbf{W}_s, \alpha), V_s^{1,N,K}, V_s^*(\mathbf{V}_s^{N,K}, \mathbf{W}_s, \alpha))] d\alpha. \end{aligned}$$

Consequently, Lemma 3.3(ii) implies

$$\begin{aligned} B_2^K(s) &\leq C \int_0^1 \mathbb{E}[|W_s^1 - V_s^{1,N,K}|^2 + |W_s^*(\alpha) - V_s^*(\mathbf{V}_s^{N,K}, \mathbf{W}_s, \alpha)|^2 \\ &\quad + |W_s^*(\alpha) - Z_s^*(\mathbf{W}_s, \alpha)|^2 \\ &\quad \times (1 + |W_s^1| + |W_s^*(\alpha)| + |Z_s^*(\mathbf{W}_s, \alpha)|)^{2\gamma/(1-\gamma)}] d\alpha. \end{aligned}$$

Step 5. Finally, we use Lemma 3.3(iii) to obtain

$$B_3^K(s) \leq CK^{1-2/\nu} \int_0^1 \mathbb{E}[1 + |W_s^1|^{4\gamma/\nu+2} + |W_s^*(\alpha)|^{4\gamma/\nu+2} \\ + |V_s^{1,N,K}|^2 + |V_s^*(\mathbf{V}_s^{N,K}, \mathbf{W}_s, \alpha)|^2] d\alpha.$$

Since $W_s^1 \sim f_s$, we deduce from (1.13) that $\mathbb{E}[|W_s^1|^{4\gamma/\nu+2}] = \int_{\mathbb{R}^3} |v|^{4\gamma/\nu+2} \times f_s(dv) \leq C$. By Lemma 4.3(a), $W_s^*(\cdot) \sim f_s$, whence $\int_0^1 |W_s^*(\alpha)|^{4\gamma/\nu+2} d\alpha = \int_{\mathbb{R}^3} |v|^{4\gamma/\nu+2} f_s(dv) \leq C$. Proposition 4.2 shows that $\mathbb{E}[|V_s^{1,N,K}|^2] = \int_{\mathbb{R}^3} |v|^2 \times f_0(dv)$. We next infer from Lemma 4.3(c) that $\int_0^1 |V_s^*(\mathbf{V}_s^{N,K}, \mathbf{W}_s, \alpha)|^2 d\alpha = N^{-1} \sum_1^N |V_s^{i,N,K}|^2$. Consequently, it holds that $\mathbb{E}[\int_0^1 |V_s^*(\mathbf{V}_s^{N,K}, \mathbf{W}_s, \alpha)|^2 d\alpha] = \mathbb{E}[|V_s^{1,N,K}|^2] = \int_{\mathbb{R}^3} |v|^2 f_0(dv)$. As a conclusion,

$$B_3^K(s) \leq CK^{1-2/\nu}.$$

Step 6. We set $u_t^{N,K} := \mathbb{E}[|W_t^1 - V_t^{1,N,K}|^2]$. Using the previous steps, we see that for all $M \geq 1$,

$$u_t^{N,K} \leq Cte^{-Mq/\gamma} + CtK^{1-2/\nu} \\ + (M + C) \int_0^t \left[u_s^{N,K} + \int_0^1 \mathbb{E}[|W_s^*(\alpha) - V_s^*(\mathbf{V}_s^{N,K}, \mathbf{W}_s, \alpha)|^2] d\alpha \right] ds \\ + C \int_0^t \int_0^1 \mathbb{E}[|W_s^*(\alpha) - Z_s^*(\mathbf{W}_s, \alpha)|^2 \\ \times (1 + |W_s^1| + |W_s^*(\alpha)| + |Z_s^*(\mathbf{W}_s, \alpha))|^{2\gamma/(1-\gamma)}] d\alpha ds.$$

We now write, using Minkowski's inequality and Lemma 4.3(d) and (e),

$$\begin{aligned} & \left[\int_0^1 \mathbb{E}[|W_s^*(\alpha) - V_s^*(\mathbf{V}_s^{N,K}, \mathbf{W}_s, \alpha)|^2] d\alpha \right]^{1/2} \\ & \leq \left[\int_0^1 \mathbb{E}[|W_s^*(\alpha) - Z_s^*(\mathbf{W}_s, \alpha)|^2] d\alpha \right]^{1/2} \\ (4.3) \quad & + \left[\int_0^1 \mathbb{E}[|Z_s^*(\mathbf{W}_s, \alpha) - V_s^*(\mathbf{V}_s^{N,K}, \mathbf{W}_s, \alpha)|^2] d\alpha \right]^{1/2} \\ & = \mathbb{E}[\mathcal{W}_2^2(f_s, \mu_{\mathbf{W}_s}^N)]^{1/2} + \left[\frac{1}{N} \sum_1^N \mathbb{E}[|W_s^i - V_s^{i,N,K}|^2] \right]^{1/2} \\ & = \mathbb{E}[\mathcal{W}_2^2(f_s, \mu_{\mathbf{W}_s}^N)]^{1/2} + (u_s^{N,K})^{1/2} \end{aligned}$$

by exchangeability. We deduce that

$$(4.4) \quad \int_0^1 \mathbb{E}[|W_s^*(\alpha) - V_s^*(\mathbf{V}_s^{N,K}, \mathbf{W}_s, \alpha)|^2] d\alpha \leq 2\mathbb{E}[\mathcal{W}_2^2(f_s, \mu_{\mathbf{W}_s}^N)] + 2u_s^{N,K}.$$

Next, a simple computation shows that for all $\varepsilon \in (0, 1)$,

$$\begin{aligned}
 & \int_0^1 \mathbb{E}[|W_s^*(\alpha) - Z_s^*(\mathbf{W}_s, \alpha)|^2 \\
 & \quad \times (1 + |W_s^1| + |W_s^*(\alpha)| + |Z_s^*(\mathbf{W}_s, \alpha)|)^{(2\gamma)/(1-\gamma)}] d\alpha \\
 & \leq \int_0^1 \mathbb{E}[|W_s^*(\alpha) - Z_s^*(\mathbf{W}_s, \alpha)|^{2-\varepsilon} \\
 (4.5) \quad & \quad \times (1 + |W_s^1| + |W_s^*(\alpha)| + |Z_s^*(\mathbf{W}_s, \alpha)|)^{((2\gamma)/(1-\gamma))+\varepsilon}] d\alpha \\
 & \leq \left(\int_0^1 \mathbb{E}[|W_s^*(\alpha) - Z_s^*(\mathbf{W}_s, \alpha)|^2] d\alpha \right)^{(2-\varepsilon)/2} \\
 & \quad \times \left(\int_0^1 \mathbb{E}[(1 + |W_s^1| + |W_s^*(\alpha)| + |Z_s^*(\mathbf{W}_s, \alpha)|)^{((4\gamma)/(\varepsilon(1-\gamma))+2)}] d\alpha \right)^{\varepsilon/2} \\
 & \leq C_\varepsilon (\mathbb{E}[\mathcal{W}_2^2(f_s, \mu_{\mathbf{W}_s}^N)])^{(2-\varepsilon)/2}.
 \end{aligned}$$

For the last inequality, we used Lemma 4.3(e), the fact that by (1.13),

$$\begin{aligned}
 \mathbb{E}[|W_s^1|^{((4\gamma)/(\varepsilon(1-\gamma))+2)}] &= \int_0^1 |W_s^*(\alpha)|^{((4\gamma)/(\varepsilon(1-\gamma))+2)} d\alpha \\
 &= \int_{\mathbb{R}^3} |v|^{((4\gamma)/(\varepsilon(1-\gamma))+2)} f_s(dv) \leq C_\varepsilon
 \end{aligned}$$

and that, by Lemma 4.3(b),

$$\begin{aligned}
 \int_0^1 \mathbb{E}[|Z_s^*(\mathbf{W}_s, \alpha)|^{((4\gamma)/(\varepsilon(1-\gamma))+2)}] d\alpha &= \mathbb{E}\left[\frac{1}{N} \sum_1^N |W_s^i|^{((4\gamma)/(\varepsilon(1-\gamma))+2)} \right] \\
 &= \mathbb{E}[|W_s^1|^{((4\gamma)/(\varepsilon(1-\gamma))+2)}] \leq C_\varepsilon.
 \end{aligned}$$

We end up with: for all $\varepsilon \in (0, 1)$, all $M \geq 1$,

$$\begin{aligned}
 u_t^{N,K} &\leq Cte^{-Mq/\gamma} + CtK^{1-2/\nu} + 3(M + C) \int_0^t [u_s^{N,K} + \mathbb{E}[\mathcal{W}_2^2(f_s, \mu_{\mathbf{W}_s}^N)]] ds \\
 &\quad + C_\varepsilon \int_0^t (\mathbb{E}[\mathcal{W}_2^2(f_s, \mu_{\mathbf{W}_s}^N)])^{1-\varepsilon/2} ds.
 \end{aligned}$$

Now we observe that $\mathbb{E}[\mathcal{W}_2^2(f_s, \mu_{\mathbf{W}_s}^N)] = \varepsilon_N(f_t)$. Recall (1.14), because W_t^1, \dots, W_t^N are i.i.d. and f_t -distributed. Since $\varepsilon_N(f_t) \leq 2 \int_{\mathbb{R}^3} |v|^2 f_t(dv) = 2 \int_{\mathbb{R}^3} |v|^2 f_0(dv)$, since $M \geq 1$ and $K \in [1, \infty)$, we get

$$u_t^{N,K} \leq C_\varepsilon \left(te^{-Mq/\gamma} + Mt\delta_{N,K,t}^{1-\varepsilon/2} + M \int_0^t u_s^{N,K} ds \right),$$

where we have set

$$\delta_{N,K,t} := K^{1-2/\nu} + \sup_{[0,t]} \varepsilon_N(f_s).$$

Hence by Grönwall’s lemma,

$$\sup_{[0,T]} u_t^{N,K} \leq C_\varepsilon T (e^{-Mq/\gamma} + M \delta_{N,K,T}^{1-\varepsilon/2}) e^{C_\varepsilon MT},$$

this holding for any value of $M \geq 1$. We easily conclude that

$$\sup_{[0,T]} u_t^{N,K} \leq C_{\varepsilon,T} \delta_{N,K,T}^{1-\varepsilon},$$

by choosing $M = 1$ if $\delta_{N,K,T} \geq 1/e$ and $M = |\log \delta_{N,K,T}|^{\gamma/q}$ otherwise, which gives

$$\begin{aligned} \sup_{[0,T]} u_t^{N,K} &\leq C_\varepsilon (T \delta_{N,K,T} + \delta_{N,K,T}^{1-\varepsilon/2} |\log \delta_{N,K,T}|^{\gamma/q}) e^{C_\varepsilon |\log \delta_{N,K,T}|^{\gamma/q} T} \\ &\leq C_{\varepsilon,T} \delta_{N,K,T}^{1-\varepsilon}, \end{aligned}$$

the last inequality following from the fact that $\gamma/q < 1$.

Final step. We now recall that $\mu_t^{N,K} = \mu_{\mathbf{V}_t^{N,K}}^N$ and write

$$\mathbb{E}[\mathcal{W}_2^2(\mu_t^{N,K}, f_t)] \leq 2\mathbb{E}[\mathcal{W}_2^2(\mu_{\mathbf{V}_t^{N,K}}^N, \mu_{\mathbf{W}_t}^N)] + 2\mathbb{E}[\mathcal{W}_2^2(\mu_{\mathbf{W}_t}^N, f_t)].$$

But $\mathbb{E}[\mathcal{W}_2^2(\mu_{\mathbf{V}_t^{N,K}}^N, \mu_{\mathbf{W}_t}^N)] \leq \mathbb{E}[N^{-1} \sum_1^N |V_t^{i,N,K} - W_t^i|^2] = \mathbb{E}[|V_t^{1,N,K} - W_t^1|^2] = u_t^{N,K}$ by exchangeability, and we have already seen that $\mathbb{E}[\mathcal{W}_2^2(\mu_{\mathbf{W}_t}^N, f_t)] = \varepsilon_N(f_t)$. Consequently, for all $\varepsilon \in (0, 1)$, all $t \in [0, T]$,

$$\mathbb{E}[\mathcal{W}_2^2(\mu_t^N, f_t)] \leq C_{\varepsilon,T} \delta_{N,K,T}^{1-\varepsilon} + 2\varepsilon_N(f_t) \leq C_{\varepsilon,T} \left(K^{1-2/\nu} + \sup_{[0,T]} \varepsilon_N(f_t) \right)^{1-\varepsilon}$$

and this proves (1.18). Using finally (1.13) and applying Theorem 1.3 (with any choice of $k > 4$), (1.19) easily follows. \square

We next study the case of Maxwell molecules.

PROOF OF THEOREM 1.4(i) WHEN $K \in [1, \infty)$. We thus assume (1.3), (1.5) with $\gamma = 0$ and (1.7). We consider $f_0 \in \mathcal{P}_2(\mathbb{R}^3)$ not being a Dirac mass. We also assume that $f_0 \in \mathcal{P}_4(\mathbb{R}^3)$ or that $\int_{\mathbb{R}^3} f_0(v) \log f_0(v) dv < \infty$, so that f_t has a density for all $t > 0$. We fix $N \geq 1$ and $K \in [1, \infty)$ and consider the processes introduced in Lemma 4.4.

Step 1. Exactly as in the case of hard potentials, we find that

$$\mathbb{E}[|W_t^1 - V_t^{1,N,K}|^2] \leq \int_0^t [B_1^K(s) + B_2^K(s) + B_3^K(s)] ds,$$

where $B_i^K(s) := \int_0^1 \mathbb{E}[A_i^K(W_s^1, W_s^*(\alpha), V_s^{1,N,K}, V_s^*(\mathbf{V}_s^{N,K}, \mathbf{W}_s, \alpha))] d\alpha$ for $i = 1, 2, 3$.

Step 2. By Lemma 3.2(i), we have $B_1^K(s) = 0$.

Steps 3 and 4. By Lemma 3.2(ii), it holds that for $\zeta_K = \pi \int_0^K (1 - \cos G(z)) dz$,

$$B_2^K(s) = \zeta_K \int_0^1 \mathbb{E}[-|W_s^1 - V_s^{1,N,K}|^2 + |W_s^*(\alpha) - V_s^*(\mathbf{V}_s^{N,K}, \mathbf{W}_s, \alpha)|^2] d\alpha.$$

Step 5. By Lemma 3.2(iii),

$$B_3^K(s) \leq CK^{1-2/\nu} \int_0^1 \mathbb{E}[|W_s^1|^2 + |W_s^*(\alpha)|^2 + |V_s^{1,N,K}|^2] d\alpha \leq CK^{1-2/\nu},$$

since, as usual, $\mathbb{E}[|W_s^1|^2] = \int_0^1 |W_s^*(\alpha)|^2 d\alpha = \mathbb{E}[|V_s^{1,N,K}|^2] = \int_{\mathbb{R}^3} |v|^2 f_0(dv)$.

Step 6. Setting $u_t^{N,K} := \mathbb{E}[|W_t^1 - V_t^{1,N,K}|^2]$, we thus have

$$\begin{aligned} u_t^{N,K} &\leq CK^{1-2/\nu} t \\ &\quad + \zeta_K \int_0^t \left(-u_s^{N,K} + \int_0^1 \mathbb{E}[|W_s^*(\alpha) - V_s^*(\mathbf{V}_s^{N,K}, \mathbf{W}_s, \alpha)|^2] d\alpha \right) ds \\ &\leq CK^{1-2/\nu} t + \zeta_K \int_0^t \left(2\sqrt{u_s^{N,K}} \sqrt{\mathbb{E}[\mathcal{W}_2^2(f_s, \mu_{\mathbf{W}_s^N}^N)]} + \mathbb{E}[\mathcal{W}_2^2(f_s, \mu_{\mathbf{W}_s^N}^N)] \right) ds \end{aligned}$$

by (4.3). Next we recall that $\varepsilon_N(f_t) = \mathbb{E}[\mathcal{W}_2^2(f_s, \mu_{\mathbf{W}_s^N}^N)]$, we set $\varepsilon_{N,T} = \sup_{[0,T]} \varepsilon_N(f_t)$ and we recall that $\zeta_K \leq \int_0^\infty (1 - \cos G(z)) dz < \infty$. We thus may write, for all $t \in [0, T]$,

$$u_t^{N,K} \leq C(K^{1-2/\nu} + \varepsilon_{N,T}T)T + C\varepsilon_{N,T}^{1/2} \int_0^t (u_s^{N,K})^{1/2} ds =: v_t^{N,K}.$$

Then we have $(v_t^{N,K})' \leq C\varepsilon_{N,T}^{1/2}(v_t^{N,K})^{1/2}$, so that $(v_t^{N,K})^{1/2} \leq (C(K^{1-2/\nu} + \varepsilon_{N,T}T)T)^{1/2} + C\varepsilon_{N,T}^{1/2}t$. We conclude that

$$\begin{aligned} \sup_{[0,T]} u_t^{N,K} &\leq C(K^{1-2/\nu} + \varepsilon_{N,T}T)T + CT^2\varepsilon_{N,T} \\ &\leq CK^{1-2/\nu}T + C(T + T^2)\varepsilon_{N,T}. \end{aligned}$$

Final step. Exactly as in the case of hard potentials, for $t \in [0, T]$,

$$\begin{aligned} \mathbb{E}[\mathcal{W}_2^2(\mu_t^{N,K}, f_t)] &\leq 2\varepsilon_N(f_t) + 2u_t^{N,K} \\ &\leq CK^{1-2/\nu}T + C(1 + T)^2 \sup_{[0,T]} \varepsilon_N(f_t) \end{aligned}$$

whence (1.16). If finally $f_0 \in \mathcal{P}_k(\mathbb{R}^3)$ for some $k > 4$, then $\sup_{[0,\infty)} \int_{\mathbb{R}^3} |v|^k \times f_t(dv) < \infty$, so that (1.17) follows by application of Theorem 1.3. \square

We conclude with hard spheres.

PROOF OF THEOREM 1.4(iii). We thus assume (1.3), (1.5) with $\gamma = 1$ and (1.6). We consider $f_0 \in \mathcal{P}_2(\mathbb{R}^3)$ satisfying (1.12) for some $p \in (\gamma, 2)$ and fix $q \in (\gamma, p)$ for the rest of the proof. We also assume that f_0 has a density, so that f_t has a density for all $t > 0$. We fix $N \geq 1$ and $K \in [1, \infty)$ and consider the processes introduced in Lemma 4.4.

Step 1. Exactly as in the case of hard potentials, we find that

$$u_t^{N,K} := \mathbb{E}[|W_t^1 - V_t^{1,N,K}|^2] \leq \int_0^t [B_1^K(s) + B_2^K(s) + B_3^K(s)] ds,$$

where $B_i^K(s) := \int_0^1 \mathbb{E}[A_i^K(W_s^1, W_s^*(\alpha), V_s^{1,N,K}, V_s^*(\mathbf{V}_s^{N,K}, \mathbf{W}_s, \alpha))] d\alpha$ for $i = 1, 2, 3$.

Steps 2, 3, 4, 5, 6. Following the case of hard potentials, using Lemma 3.4 instead of Lemma 3.3, we deduce that for all $M > 1$,

$$\begin{aligned} & \sum_1^3 B_i^K(s) \\ & \leq 2M \int_0^1 \mathbb{E}[|W_s^1 - V_s^{1,N,K}|^2 + |W_s^*(\alpha) - V_s^*(\mathbf{V}_s^{N,K}, \mathbf{W}_s, \alpha)|^2] d\alpha \\ & \quad + C(K e^{-Mq} + e^{-Kq}) \int_0^1 \mathbb{E}[(1 + |V_s^{1,N,K}| + |V_s^*(\mathbf{V}_s^{N,K}, \mathbf{W}_s, \alpha)|) \\ & \quad \quad \quad \times e^{C(|W_s^1|^q + |W_s^*(\alpha)|^q + |Z_s^*(\mathbf{W}_s, \alpha)|^q)}] d\alpha \\ & \quad + C \int_0^1 \mathbb{E}[|W_s^*(\alpha) - Z_s^*(\mathbf{V}_s^{N,K}, \mathbf{W}_s, \alpha)|^2 \\ & \quad \quad \quad \times (1 + |W_s^1| + |W_s^*(\alpha)| + |Z_s^*(\mathbf{V}_s^{N,K}, \mathbf{W}_s, \alpha)|)^2] d\alpha. \end{aligned}$$

Proceeding as in (4.5), we deduce that the last line is bounded, for all $\varepsilon \in (0, 1)$, by

$$C_\varepsilon (\mathbb{E}[\mathcal{W}_2^2(f_s, \mu_{\mathbf{W}_s}^N)])^{(2-\varepsilon)/\varepsilon},$$

and using (4.4), the first term is bounded by

$$4M \mathbb{E}[\mathcal{W}_2^2(f_s, \mu_{\mathbf{W}_s}^N)] + 6M u_s^{N,K}.$$

Using finally the Cauchy–Schwarz inequality, that by Lemma 4.3(c) and exchangeability, $\mathbb{E}[\int_0^1 |V_s^*(\mathbf{V}_s^{N,K}, \mathbf{W}_s, \alpha)|^2 d\alpha] = \mathbb{E}[N^{-1} \sum_1^N |V_s^{i,N,K}|^2] = \mathbb{E}[|V_s^{1,N,K}|^2] =$

$\int_{\mathbb{R}^3} |v|^2 f_0(dv) < \infty$ and (1.13), we easily bound the second line by $C(Ke^{-M^q} + e^{-K^q})$ [recall that $W_s^i \sim f_s$, that $W_s^*(\cdot) \sim f_s$ and that, by Lemma 4.4(b), $\int_0^1 e^{C|Z_s^*(W_s, \alpha)|^q} d\alpha = N^{-1} \sum_1^N e^{C|W_s^i|^q}$].

Recalling that $\mathbb{E}[\mathcal{W}_2^2(f_s, \mu_{W_s}^N)] = \varepsilon_N(f_s)$ and setting $\varepsilon_{N,t} = \sup_{[0,t]} \varepsilon_N(f_s)$, we thus have, for any $M > 1$, any $\varepsilon \in (0, 1)$,

$$u_t^{N,K} \leq 6M \int_0^t u_s^{N,K} ds + Ct(Ke^{-M^q} + e^{-K^q}) + C_\varepsilon t \varepsilon_{N,t}^{1-\varepsilon/2}.$$

Thus by Grönwall’s lemma,

$$u_t^{N,K} \leq C_\varepsilon t (Ke^{-M^q} + e^{-K^q} + \varepsilon_{N,t}^{1-\varepsilon/2}) e^{6Mt}.$$

Choosing $M = 2K$ and using that $Ke^{-(2K)^q} \leq Ce^{-K^q}$, we deduce that

$$\begin{aligned} \sup_{[0,T]} u_t^{N,K} &\leq C_\varepsilon T (e^{-K^q} + \varepsilon_{N,T}^{1-\varepsilon/2}) e^{12KT} \\ &= C_\varepsilon T \left(e^{-K^q} + \left(\sup_{[0,T]} \varepsilon_N(f_s) \right)^{1-\varepsilon/2} \right) e^{12KT}. \end{aligned}$$

Final step. We conclude as usual, using that $\mathbb{E}[\mathcal{W}_2^2(\mu_t^{N,K}, f_t)] \leq 2\varepsilon_N(f_t) + 2u_t^{N,K}$ to obtain (1.20) and then (1.13) and Theorem 1.3 to deduce (1.21). \square

5. Extension to the particle system without cutoff. It remains to check that the particle system without cutoff is well posed and that we can pass to the limit as $K \rightarrow \infty$ in the convergence estimates (1.16)–(1.17)–(1.18)–(1.19). We will need the following rough computations.

LEMMA 5.1. *Assume (1.3), (1.5) and (1.6) or (1.7). Adopt the notation of Lemma 3.1. There are $C > 0$, $\kappa > 0$ and $\delta > 0$ (depending on γ, v) such that for all $K \in [1, \infty)$, all $v, v_*, \tilde{v}, \tilde{v}_* \in \mathbb{R}^3$,*

$$\begin{aligned} &\sum_{i=1}^3 A_i^K(v, v_*, \tilde{v}, \tilde{v}_*) \\ &\leq C(1 + |v| + |v_*| + |\tilde{v}| + |\tilde{v}_*|)^K (|v - \tilde{v}|^2 + |v_* - \tilde{v}_*|^2 + K^{-\delta}). \end{aligned}$$

PROOF. Concerning A_1^K , we start from (3.1) (this is valid for all $\gamma \in [0, 1]$), and we deduce that

$$\begin{aligned} A_1^K(v, v_*, \tilde{v}, \tilde{v}_*) &\leq 8c_4 (|v - \tilde{v}| \wedge |v_* - \tilde{v}_*|)^\gamma (|v - \tilde{v}| + |v_* - \tilde{v}_*|)^2 \\ &\leq C(1 + |v| + |v_*| + |\tilde{v}| + |\tilde{v}_*|)^\gamma (|v - \tilde{v}|^2 + |v_* - \tilde{v}_*|^2). \end{aligned}$$

We then make use of (3.2) (also valid for all $\gamma \in [0, 1]$) to write

$$\begin{aligned} A_2^K(v, v_*, \tilde{v}, \tilde{v}_*) &\leq C(|v - \tilde{v}| + |v_* - \tilde{v}_*|)^2 (|v - v_*|^\gamma + |\tilde{v} - \tilde{v}_*|^\gamma) \\ &\leq C(1 + |v| + |v_*| + |\tilde{v}| + |\tilde{v}_*|)^\gamma (|v - \tilde{v}|^2 + |v_* - \tilde{v}_*|^2). \end{aligned}$$

For A_3^K , we separate two cases. Under hypothesis (1.7), we immediately deduce from (3.4) that

$$A_3^K(v, v_*, \tilde{v}, \tilde{v}_*) \leq C(1 + |v| + |v_*| + |\tilde{v}| + |\tilde{v}_*|)^{2+2\gamma/\nu} K^{1-2/\nu}.$$

Under hypothesis (1.6), we have seen (when $\gamma = 1$, at the end of the proof of Lemma 3.4) that $\Psi_K(x) \leq 5x^\gamma \mathbb{1}_{\{x \geq K/2\}}$, whence $\Psi_K(x) \leq 10x^{2\gamma}/K$ and thus

$$\begin{aligned} A_3^K(v, v_*, \tilde{v}, \tilde{v}_*) &\leq C(|v - v_*| \vee |\tilde{v} - \tilde{v}_*|)^{2+2\gamma} K^{-1} \\ &\leq C(1 + |v| + |v_*| + |\tilde{v}| + |\tilde{v}_*|)^{2+2\gamma} K^{-1}. \end{aligned}$$

The conclusion follows, choosing $\kappa = 2 + 2\gamma/\nu$ and $\delta = 2/\nu - 1$ under (1.7) and $\kappa = 2 + 2\gamma$ and $\delta = 1$ under (1.6). \square

Now we can give the following:

PROOF OF PROPOSITION 1.2(ii). We only sketch the proof, since it is quite standard. In the whole proof, $N \geq 2$ is fixed, as well as $f_0 \in \mathcal{P}_2(\mathbb{R}^3)$ and a family of i.i.d. f_0 -distributed random variables $(V_0^{i,N})_{i=1,\dots,N}$.

Step 1. Recall (2.8). Classically, $(V_t^{i,N,\infty})_{i=1,\dots,N,t \geq 0}$ is a Markov process with generator \mathcal{L}_N starting from $(V_0^{i,N})_{i=1,\dots,N}$ if it solves

$$\begin{aligned} (5.1) \quad V_t^{i,N,\infty} &= V_0^i + \int_0^t \int_j \int_0^\infty \int_0^{2\pi} c(V_{s-}^{i,N,\infty}, V_{s-}^{j,N,\infty}, z, \varphi) \\ &\quad \times O_i^N(ds, dj, dz, d\varphi), \quad i = 1, \dots, N \end{aligned}$$

for some i.i.d. Poisson measures $O_i^N(ds, dj, dz, d\varphi)_{i=1,\dots,N}$ on $[0, \infty) \times \{1, \dots, N\} \times [0, \infty) \times [0, 2\pi)$ with intensity measures $ds(N^{-1} \sum_{k=1}^N \delta_k(dj)) dz d\varphi$.

Step 2. The existence of a solution (in law) to (5.1) is easily checked, using martingale problems methods (tightness and consistency), by passing to the limit in (4.2). The main estimates to be used are that, uniformly in $K \in [1, \infty)$ (and in $N \geq 1$, but this is not the point here),

$$\mathbb{E}[|V_t^{1,N,K}|^2] = \int_{\mathbb{R}^3} |v|^2 f_0(dv) \quad \text{and} \quad \mathbb{E}\left[\sup_{[0,T]} |V_t^{1,N,K}|\right] \leq C_T$$

for all $T > 0$. This second estimate is immediately deduced from the first one and the fact that $\int_0^\infty \int_0^{2\pi} |c(v, v_*, z, \varphi)| \leq C|v - v_*|^{1+\gamma} \leq C(1 + |v| + |v_*|)^2$. The tightness is easily checked by using Aldous’s criterion [1].

Step 3. Uniqueness (in law) for (5.1) is more difficult. Consider a (càdlàg and adapted) solution $(V_t^{i,N,\infty})_{i=1,\dots,N,t \geq 0}$ to (5.1). For $K \in [1, \infty)$, consider the solution to

$$\begin{aligned} V_t^{i,N,K} &= V_0^i + \int_0^t \int_j \int_0^\infty \int_0^{2\pi} c_K(V_{s-}^{i,N,K}, V_{s-}^{j,N,K}, z, \varphi + \varphi_{s,i,j}) \\ &\quad \times O_i^N(ds, dj, dz, d\varphi), \end{aligned}$$

$i = 1, \dots, N$, where $\varphi_{s,i,j} := \varphi_0(V_{s-}^{i,N,\infty} - V_{s-}^{j,N,\infty}, V_{s-}^{i,N,K} - V_{s-}^{j,N,K})$. Such a solution obviously exists and is unique, because the involved Poisson measures are finite [recall that $c_K(v, v_*, z, \varphi) = 0$ for $z \geq K$]. Furthermore, this solution $(V_t^{i,N,K})_{i=1,\dots,N,t \geq 0}$ is a Markov process with generator $\mathcal{L}_{N,K}$ starting from $(V_0^{i,N,K})_{i=1,\dots,N}$ [because the only difference with (4.2) is the presence of $\varphi_{s,i,j}$ which does not change the law of the particle system; see Lemma 4.4(ii) for a similar claim]. Hence Proposition 1.2(i) implies that the law of $(V_t^{i,N,K})_{i=1,\dots,N,t \geq 0}$ is uniquely determined.

We next introduce $\tau_{N,K,A} = \inf\{t \geq 0 : \exists i \in \{1, \dots, N\}, |V_t^{i,N,\infty}| + |V_t^{i,N,K}| \geq A\}$. Using, on the one hand, the fact that $(V_t^{i,N,\infty})_{i=1,\dots,N,t \geq 0}$ is a.s. càdlàg (and thus locally bounded), and on the other hand, the (uniform in K) estimate established in step 2, one easily gets convinced that

$$(5.2) \quad \forall T > 0, \quad \lim_{A \rightarrow \infty} \sup_{K \geq 1} \Pr[\tau_{N,K,A} \leq T] = 0.$$

Next, a simple computation shows that

$$\begin{aligned} & \mathbb{E}[|V_{t \wedge \tau_{N,K,A}}^{1,N,\infty} - V_{t \wedge \tau_{N,K,A}}^{1,N,K}|^2] \\ & \leq \frac{1}{N} \sum_{j=1}^N \mathbb{E} \left[\int_0^{t \wedge \tau_{N,K,A}} \int_0^\infty \int_0^{2\pi} (|V_{s-}^{1,N,\infty} - V_{s-}^{1,N,K} \right. \\ & \qquad \qquad \qquad \left. + \Delta_{s-}^{1,j,N,K}(z, \varphi)|^2 \right. \\ & \qquad \qquad \qquad \left. - |V_{s-}^{1,N,\infty} - V_{s-}^{1,N,K}|^2) d\varphi dz \right], \end{aligned}$$

where

$$\begin{aligned} & \Delta_{s-}^{1,j,N,K}(z, \varphi) \\ & := c(V_{s-}^{1,N,\infty}, V_{s-}^{1,N,\infty}, z, \varphi) - c_K(V_{s-}^{1,N,K}, V_{s-}^{1,N,K}, z, \varphi + \varphi_{s,i,j}). \end{aligned}$$

Using Lemmas 3.1 and 5.1 and the fact that all the velocities are bounded by A until $\tau_{N,K,A}$, we easily deduce that

$$\begin{aligned} & \mathbb{E}[|V_{t \wedge \tau_{N,K,A}}^{1,N,\infty} - V_{t \wedge \tau_{N,K,A}}^{1,N,K}|^2] \\ & \leq \frac{C(1+A)^\kappa}{N} \sum_{j=1}^N \mathbb{E} \left[\int_0^{t \wedge \tau_{N,K,A}} (|V_s^{1,N,\infty} - V_s^{1,N,K}|^2 \right. \\ & \qquad \qquad \qquad \left. + |V_s^{j,N,\infty} - V_s^{j,N,K}|^2 + K^{-\delta}) ds \right] \\ & \leq C_T(1+A)^\kappa K^{-\delta} + C(1+A)^\kappa \int_0^t \mathbb{E}[|V_{s \wedge \tau_{N,K,A}}^{1,N,\infty} - V_{s \wedge \tau_{N,K,A}}^{1,N,K}|^2] ds \end{aligned}$$

by exchangeability. We now use the Grönwall lemma and then deduce that for any $A > 0$,

$$(5.3) \quad \lim_{K \rightarrow \infty} \sup_{[0, T]} \mathbb{E}[|V_{t \wedge \tau_{N, K, A}}^{1, N, \infty} - V_{t \wedge \tau_{N, K, A}}^{1, N, K}|^2] = 0.$$

Gathering (5.2) and (5.3), we easily conclude that for all $t \geq 0$, $V_t^{1, N, K}$ tends in probability to $V_t^{1, N, \infty}$ as $K \rightarrow \infty$. Thus for any finite family $0 \leq t_1 \leq \dots \leq t_l$, $(V_{t_j}^{i, N, K})_{i=1, \dots, N, j=1, \dots, l}$ goes in probability to $(V_{t_j}^{i, N, \infty})_{i=1, \dots, N, j=1, \dots, l}$, of which the law is thus uniquely determined. This is classically sufficient to characterize the whole law of the process $(V_t^{i, N, \infty})_{i=1, \dots, N, t \geq 0}$.

Conclusion. We thus have the existence of a unique Markov process $(V_t^{i, N, \infty})_{i=1, \dots, N, t \geq 0}$ with generator \mathcal{L}_N starting from $(V_0^{i, N})_{i=1, \dots, N}$, and it holds that for each $t \geq 0$, each $N \geq 2$, $(V_t^{i, N, \infty})_{i=1, \dots, N}$ is the limit in law, as $K \rightarrow \infty$, of $(V_t^{i, N, K})_{i=1, \dots, N}$. \square

To conclude, we will need the following lemma.

LEMMA 5.2. *Let $N \geq 2$ be fixed. Let $(X^{i, N, K})_{i=1, \dots, N}$ be a sequence of $(\mathbb{R}^3)^N$ -valued random variable going in law, as $K \rightarrow \infty$, to some $(\mathbb{R}^3)^N$ -valued random variable $(X^{i, N})_{i=1, \dots, N}$. Consider the associated empirical measures $\nu^{N, K} := N^{-1} \sum_{i=1}^N \delta_{X^{i, N, K}}$ and $\nu^N := N^{-1} \sum_{i=1}^N \delta_{X^{i, N}}$. Then for any $g \in \mathcal{P}_2(\mathbb{R}^3)$,*

$$\mathbb{E}[\mathcal{W}_2^2(\nu^N, g)] \leq \liminf_{K \rightarrow \infty} \mathbb{E}[\mathcal{W}_2^2(\nu^{N, K}, g)].$$

PROOF. First observe that the map $(x_1, \dots, x_N) \mapsto \mathcal{W}_2(N^{-1} \sum_1^N \delta_{x_i}, g)$ is continuous on $(\mathbb{R}^3)^N$. Indeed, it suffices to use the triangular inequality for \mathcal{W}_2 and the easy estimate

$$\mathcal{W}_2^2\left(\frac{1}{N} \sum_1^N \delta_{x_i}, \frac{1}{N} \sum_1^N \delta_{y_i}\right) \leq \frac{1}{N} \sum_1^N |x_i - y_i|^2.$$

Consequently, $\mathcal{W}_2^2(\nu^{N, K}, g)$ goes in law to $\mathcal{W}_2^2(\nu^N, g)$. Thus for any $A > 1$, we have

$$\mathbb{E}[\mathcal{W}_2^2(\nu^N, g) \wedge A] = \lim_{K \rightarrow \infty} \mathbb{E}[\mathcal{W}_2^2(\nu^{N, K}, g) \wedge A] \leq \liminf_{K \rightarrow \infty} \mathbb{E}[\mathcal{W}_2^2(\nu^{N, K}, g)].$$

It then suffices to let A increase to infinity and to use the monotonic convergence theorem. \square

This allows us to complete the proof of our main results.

PROOF OF THEOREM 1.4(i)–(ii) WHEN $K = \infty$. Recall that (1.16)–(1.19) have already been established when $K \in [1, \infty)$. Since $(V_t^{i, N, \infty})_{i=1, \dots, N}$ is the

limit (in law) of $(V_t^{i,N,K})_{i=1,\dots,N}$ as $K \rightarrow \infty$ for each $t \geq 0$ and each $N \geq 2$ [see the conclusion of the proof of Proposition 1.2(ii)], we can let $K \rightarrow \infty$ in (1.16)–(1.19) using Lemma 5.2. \square

REFERENCES

- [1] ALDOUS, D. (1978). Stopping times and tightness. *Ann. Probab.* **6** 335–340. [MR0474446](#)
- [2] ALEXANDRE, R. (2009). A review of Boltzmann equation with singular kernels. *Kinet. Relat. Models* **2** 551–646. [MR2556715](#)
- [3] ALEXANDRE, R., DESVILLETES, L., VILLANI, C. and WENNERBERG, B. (2000). Entropy dissipation and long-range interactions. *Arch. Ration. Mech. Anal.* **152** 327–355. [MR1765272](#)
- [4] ARKERYD, L. (1972). On the Boltzmann equation. I. Existence. *Arch. Ration. Mech. Anal.* **45** 1–16. [MR0339665](#)
- [5] ARKERYD, L. (1972). On the Boltzmann equation. II. The full initial value problem. *Arch. Ration. Mech. Anal.* **45** 17–34. [MR0339666](#)
- [6] BARTHE, F. and BORDENAVE, C. (2013). Combinatorial optimization over two random point sets. In *Séminaire de Probabilités XLV. Lecture Notes in Math.* **2078** 483–535. Springer, Cham. [MR3185927](#)
- [7] BOBYLEV, A. V. (1997). Moment inequalities for the Boltzmann equation and applications to spatially homogeneous problems. *J. Stat. Phys.* **88** 1183–1214. [MR1478067](#)
- [8] BOISSARD, E. and LE GOUIC, T. (2014). On the mean speed of convergence of empirical and occupation measures in Wasserstein distance. *Ann. Inst. Henri Poincaré Probab. Stat.* **50** 539–563. [MR3189084](#)
- [9] BOLLEY, F., CAÑIZO, J. A. and CARRILLO, J. A. (2011). Stochastic mean-field limit: Non-Lipschitz forces and swarming. *Math. Models Methods Appl. Sci.* **21** 2179–2210. [MR2860672](#)
- [10] CARRAPATOSO, K. (2015). Quantitative and qualitative Kac’s chaos on Boltzmann’s sphere. *Ann. Inst. Henri Poincaré Probab. Stat.* **51** 993–1039. [MR3365971](#)
- [11] CERCIGNANI, C. (1988). *The Boltzmann Equation and Its Applications. Applied Mathematical Sciences* **67**. Springer, New York. [MR1313028](#)
- [12] DEREICH, S., SCHEUTZOW, M. and SCHOTTSTEDT, R. (2013). Constructive quantization: Approximation by empirical measures. *Ann. Inst. Henri Poincaré Probab. Stat.* **49** 1183–1203. [MR3127919](#)
- [13] DESVILLETES, L. (2001). Boltzmann’s kernel and the spatially homogeneous Boltzmann equation. fluid dynamic processes with inelastic interactions at the molecular scale (Torino, 2000). *Riv. Mat. Univ. Parma* (6) **4*** 1–22. [MR1881103](#)
- [14] DESVILLETES, L., GRAHAM, C. and MÉLÉARD, S. (1999). Probabilistic interpretation and numerical approximation of a Kac equation without cutoff. *Stochastic Process. Appl.* **84** 115–135. [MR1720101](#)
- [15] DESVILLETES, L. and MOUHOT, C. (2009). Stability and uniqueness for the spatially homogeneous Boltzmann equation with long-range interactions. *Arch. Ration. Mech. Anal.* **193** 227–253. [MR2525118](#)
- [16] ESCOBEDO, M. and MISCHLER, S. (2010). Scalings for a ballistic aggregation equation. *J. Stat. Phys.* **141** 422–458. [MR2728840](#)
- [17] FONTBONA, J., GUÉRIN, H. and MÉLÉARD, S. (2009). Measurability of optimal transportation and convergence rate for Landau type interacting particle systems. *Probab. Theory Related Fields* **143** 329–351. [MR2475665](#)
- [18] FOURNIER, N. (2015). Finiteness of entropy for the homogeneous Boltzmann equation with measure initial condition. *Ann. Appl. Probab.* **25** 860–897. [MR3313757](#)

- [19] FOURNIER, N. and GODINHO, D. (2012). Asymptotic of grazing collisions and particle approximation for the Kac equation without cutoff. *Comm. Math. Phys.* **316** 307–344. [MR2993918](#)
- [20] FOURNIER, N. and GUÉRIN, H. (2008). On the uniqueness for the spatially homogeneous Boltzmann equation with a strong angular singularity. *J. Stat. Phys.* **131** 749–781. [MR2398952](#)
- [21] FOURNIER, N. and GUILLIN, A. (2015). On the rate of convergence in Wasserstein distance of the empirical measure. *Probab. Theory Related Fields* **162** 707–738. [MR3383341](#)
- [22] FOURNIER, N. and MÉLÉARD, S. (2002). A stochastic particle numerical method for 3D Boltzmann equations without cutoff. *Math. Comp.* **71** 583–604 (electronic). [MR1885616](#)
- [23] FOURNIER, N. and MÉLÉARD, S. (2002). A weak criterion of absolute continuity for jump processes: Application to the Boltzmann equation. *Bernoulli* **8** 537–558. [MR1914702](#)
- [24] FOURNIER, N. and MOUHOT, C. (2009). On the well-posedness of the spatially homogeneous Boltzmann equation with a moderate angular singularity. *Comm. Math. Phys.* **289** 803–824. [MR2511651](#)
- [25] GRAHAM, C. and MÉLÉARD, S. (1997). Stochastic particle approximations for generalized Boltzmann models and convergence estimates. *Ann. Probab.* **25** 115–132. [MR1428502](#)
- [26] GRÜNBAUM, F. A. (1971). Propagation of chaos for the Boltzmann equation. *Arch. Ration. Mech. Anal.* **42** 323–345. [MR0334788](#)
- [27] HAURAY, M. and MISCHLER, S. (2014). On Kac’s chaos and related problems. *J. Funct. Anal.* **266** 6055–6157. [MR3188710](#)
- [28] KAC, M. (1956). Foundations of kinetic theory. In *Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, Vol. III* 171–197. Univ. California Press, Berkeley. [MR0084985](#)
- [29] LU, X. (1999). Conservation of energy, entropy identity, and local stability for the spatially homogeneous Boltzmann equation. *J. Stat. Phys.* **96** 765–796. [MR1716814](#)
- [30] LU, X. and MOUHOT, C. (2012). On measure solutions of the Boltzmann equation, part I: Moment production and stability estimates. *J. Differential Equations* **252** 3305–3363. [MR2871802](#)
- [31] MCKEAN, H. P. (1975). Fluctuations in the kinetic theory of gases. *Comm. Pure Appl. Math.* **28** 435–455. [MR0395662](#)
- [32] MCKEAN, H. P. JR. (1967). An exponential formula for solving Boltzmann’s equation for a Maxwellian gas. *J. Combinatorial Theory* **2** 358–382. [MR0224348](#)
- [33] MISCHLER, S. and MOUHOT, C. (2013). Kac’s program in kinetic theory. *Invent. Math.* **193** 1–147. [MR3069113](#)
- [34] MISCHLER, S., MOUHOT, C. and WENNBURG, B. (2015). A new approach to quantitative propagation of chaos for drift, diffusion and jump processes. *Probab. Theory Related Fields* **161** 1–59. [MR3304746](#)
- [35] MISCHLER, S. and WENNBURG, B. (1999). On the spatially homogeneous Boltzmann equation. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **16** 467–501. [MR1697562](#)
- [36] NANBU, K. (1983). Interrelations between various direct simulation methods for solving the Boltzmann equation. *J. Phys. Soc. Japan* **52** 3382–3388.
- [37] RACHEV, S. T. and RÜSCHENDORF, L. (1998). *Mass Transportation Problems. Vol. I: Theory*. Springer, New York. [MR1619170](#)
- [38] SZNITMAN, A.-S. (1984). Équations de type de Boltzmann, spatialement homogènes. *Z. Wahrsch. Verw. Gebiete* **66** 559–592. [MR0753814](#)
- [39] SZNITMAN, A.-S. (1991). Topics in propagation of chaos. In *École d’Été de Probabilités de Saint-Flour XIX—1989. Lecture Notes in Math.* **1464** 165–251. Springer, Berlin. [MR1108185](#)

- [40] TANAKA, H. (1978). On the uniqueness of Markov process associated with the Boltzmann equation of Maxwellian molecules. In *Proceedings of the International Symposium on Stochastic Differential Equations (Res. Inst. Math. Sci., Kyoto Univ., Kyoto, 1976)* 409–425. Wiley, New York. [MR0536022](#)
- [41] TANAKA, H. (1978/79). Probabilistic treatment of the Boltzmann equation of Maxwellian molecules. *Z. Wahrsch. Verw. Gebiete* **46** 67–105. [MR0512334](#)
- [42] TOSCANI, G. and VILLANI, C. (1999). Probability metrics and uniqueness of the solution to the Boltzmann equation for a Maxwell gas. *J. Stat. Phys.* **94** 619–637. [MR1675367](#)
- [43] VILLANI, C. (2002). A review of mathematical topics in collisional kinetic theory. In *Handbook of Mathematical Fluid Dynamics, Vol. I* 71–305. North-Holland, Amsterdam. [MR1942465](#)
- [44] VILLANI, C. (2003). *Topics in Optimal Transportation. Graduate Studies in Mathematics* **58**. Amer. Math. Soc., Providence, RI. [MR1964483](#)

LABORATOIRE DE PROBABILITÉS
ET MODÈLES ALÉATOIRES
UNIVERSITÉ PIERRE ET MARIE CURIE
UMR 7599, UPMC, CASE 188
4 PL. JUSSIEU
F-75252 PARIS CEDEX 5
FRANCE
E-MAIL: nicolas.fournier@upmc.fr

CEREMADE UMR 7534
UNIVERSITÉ PARIS-DAUPHINE
4 PLACE DU MARÉCHAL DE LATTRE
DE TASSIGNY F-75775, PARIS CEDEX 16
FRANCE
E-MAIL: mischler@ceremade.dauphine.fr