

## A CURIE–WEISS MODEL OF SELF-ORGANIZED CRITICALITY

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We try to design a simple model exhibiting self-organized criticality, which is amenable to a rigorous mathematical analysis. To this end, we modify the generalized Ising Curie–Weiss model by implementing an automatic control of the inverse temperature. For a class of symmetric distributions whose density satisfies some integrability conditions, we prove that the sum  $S_n$  of the random variables behaves as in the typical critical generalized Ising Curie–Weiss model. The fluctuations are of order  $n^{3/4}$ , and the limiting law is  $C \exp(-\lambda x^4) dx$  where  $C$  and  $\lambda$  are suitable positive constants.

**1. Introduction.** In their famous article [4], Bak, Tang and Wiesenfeld showed that certain complex systems are naturally attracted by critical points, without any external intervention. The amplification of small internal fluctuations can lead to a critical state and cause a chain reaction leading to a radical change of the system behavior. These systems exhibit the phenomenon of self-organized criticality (SOC). Although there is no universal SOC theory, it can be well understood with the archetype of SOC: the sandpile model, first introduced in [4]. We consider a pile of sand and the constant drop of new sand grains, which randomly slide down the slope of sand. We observe local avalanches with different and unpredictable sizes which are not proportional to the input. Such phenomenon can be observed in nature (e.g., forest fires, earthquakes, species evolution).

In general SOC can be observed empirically or simulated on a computer in various models. However, the mathematical analysis of these models turns out to be extremely difficult, even for the sandpile model whose definition is yet simple. Self-organized criticality has been reviewed in recent works [2, 3, 9, 17, 21]. Other challenging models are the models for forest fires [18], which are built with the help of percolation process. Some simple models of evolutions also lead to critical behaviors [8].

Our goal here is to design a model exhibiting self-organized criticality, which is as simple as possible, and which is amenable to a rigorous mathematical analysis. The simplest models exhibiting SOC are obtained by forcing standard critical transitions into a self-organized state; see Section 15.4.2 of [20]. The idea is to start with a model presenting a phase transition and to create a feedback from the configuration to the control parameters in order to converge toward a critical

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point. The most widely studied model in statistical mechanics, which exhibits a phase transition and presents critical states, is the Ising model. Its mean field version is called the Ising Curie–Weiss model; see Sections IV.4 and V.9 of [11]. It has been extended to real-valued spins by Ellis and Newman [12], in the so called generalized Ising Curie–Weiss model. This model is our starting point, and we will modify it in order to build a system of interacting random variables, which exhibits a phenomenon of SOC.

Let us first recall the definition and some results on the generalized Ising Curie–Weiss model. Let  $\rho$  be a symmetric probability measure on  $\mathbb{R}$  with positive variance  $\sigma^2$  and such that

$$\forall t \geq 0 \quad \int_{\mathbb{R}} \exp(tx^2) d\rho(x) < \infty.$$

The generalized Ising Curie–Weiss model associated to  $\rho$  and the inverse temperature  $\beta > 0$  is defined through an infinite triangular array of real-valued random variables  $(X_n^k)_{1 \leq k \leq n}$  such that, for all  $n \geq 1$ ,  $(X_n^1, \dots, X_n^n)$  has the distribution

$$d\mu_{n,\rho,\beta}(x_1, \dots, x_n) = \frac{1}{Z_n(\beta)} \exp\left(\frac{\beta}{2} \frac{(x_1 + \dots + x_n)^2}{n}\right) \prod_{i=1}^n d\rho(x_i),$$

where  $Z_n(\beta)$  is a normalization. For any  $n \geq 1$ , we set  $S_n = X_n^1 + \dots + X_n^n$ . When  $\rho = (\delta_{-1} + \delta_1)/2$ , we recover the classical Ising Curie–Weiss model.

We denote by  $L$  the log-Laplace of  $\rho$  (see Appendix A). Ellis and Eisele have shown in [10] that, if  $L^{(3)}(t) \leq 0$  for any  $t \geq 0$ , then there exists a map  $m$  which is null on  $]0, 1/\sigma^2]$ , real analytic and positive on  $]1/\sigma^2, +\infty[$  and such that

$$\frac{S_n}{n} \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \begin{cases} \delta_0, & \text{if } \beta \leq 1/\sigma^2, \\ \frac{1}{2}(\delta_{-m(\beta)} + \delta_{m(\beta)}), & \text{if } \beta > 1/\sigma^2. \end{cases}$$

The point  $1/\sigma^2$  is a critical value, and the function  $m$  cannot be extended analytically around  $1/\sigma^2$ . The main result of [12] states that, if  $\beta < 1/\sigma^2$ , then, under  $\mu_{n,\rho,\beta}$ ,

$$\frac{S_n}{\sqrt{n}} \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \mathcal{N}\left(0, \frac{\sigma^2}{1 - \beta\sigma^2}\right).$$

If  $\beta = 1/\sigma^2$ , then there exists  $k \in \mathbb{N} \setminus \{0, 1\}$  and  $\lambda > 0$  such that, under  $\mu_{n,\rho,\beta}$ ,

$$\frac{S_n}{n^{1-1/2k}} \xrightarrow[n \rightarrow \infty]{\mathcal{L}} C_{k,\lambda} \exp\left(-\lambda \frac{s^{2k}}{(2k)!}\right) ds,$$

where  $C_{k,\lambda}$  is a normalization. This is a consequence of Theorem 2.1 of [12] and some properties of  $m$  explained in [10] implying that the function  $s \mapsto L(s\sqrt{\beta}) - s^2/2$  has a unique maximum at 0 whenever  $\beta \leq 1/\sigma^2$ ; see Section V.2 of [14] for the details.

We will transform the previous probability distribution in order to obtain a model which presents a phenomenon of self-organized criticality, that is, a model which evolves toward the critical state  $\beta = 1/\sigma^2$  of the previous model. More precisely, the critical generalized Ising Curie–Weiss model is the model where  $(X_n^1, \dots, X_n^n)$  has the distribution

$$\frac{1}{Z_n} \exp\left(\frac{(x_1 + \dots + x_n)^2}{2n\sigma^2}\right) \prod_{i=1}^n d\rho(x_i).$$

We wish to build a model which converges to a critical state for every distribution  $\rho$  and which does not rely on any specific a priori information on  $\rho$ . We search an automatic control of the inverse temperature  $\beta$ , which would be a function of the random variables in the model, so that when  $n$  goes to  $+\infty$ ,  $\beta$  converges toward the critical value of the model. We start with the following observation: if  $(Y_n)_{n \geq 1}$  is a sequence of independent random variables with identical distribution  $\rho$ , then, by the law of large numbers,

$$\frac{Y_1^2 + \dots + Y_n^2}{n} \xrightarrow[n \rightarrow \infty]{} \sigma^2 \quad \text{a.s.}$$

This convergence provides us with an estimator of  $1/\sigma^2$ . If we believe that a similar convergence holds in the generalized Ising Curie–Weiss model, then we are tempted to *replace*  $\beta$  by  $n(x_1^2 + \dots + x_n^2)^{-1}$  in the distribution

$$\frac{1}{Z_n} \exp\left(\frac{\beta}{2} \frac{(x_1 + \dots + x_n)^2}{n}\right) \prod_{i=1}^n d\rho(x_i).$$

Hence the model we consider in this paper is given by the distribution

$$\frac{1}{Z_n} \exp\left(\frac{1}{2} \frac{(x_1 + \dots + x_n)^2}{x_1^2 + \dots + x_n^2}\right) \prod_{i=1}^n d\rho(x_i).$$

The previous considerations suggest that this model should evolve spontaneously toward a critical state. We will prove rigorously that our model indeed exhibits a phenomenon of self-organized criticality. However, our model is a toy model which is certainly much less complex than other famous fundamental models of SOC like the sandpile model.

Our main result (Theorem 2) states that if  $\rho$  has an even density satisfying some integrability condition, then, asymptotically, the sum  $S_n$  of the random variables behaves as in the typical critical generalized Ising Curie–Weiss model: if  $\mu_4$  denotes the fourth moment of  $\rho$ , then

$$\frac{\mu_4^{1/4} S_n}{\sigma^2 n^{3/4}} \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \left(\frac{4}{3}\right)^{1/4} \Gamma\left(\frac{1}{4}\right)^{-1} \exp\left(-\frac{s^4}{12}\right) ds.$$

This fluctuation result shows that our model is a self-organized model exhibiting critical behavior. Indeed it has the same behavior as the critical generalized

Ising Curie–Weiss model, and by construction, it does not depend on any external parameter. In this sense, we can conclude that this is a Curie–Weiss model of self-organized criticality.

Our result presents an unexpected universal feature. For any distribution  $\rho$ , which has an even density satisfying some integrability hypothesis, the fluctuations of  $S_n$  are of order  $n^{3/4}$ . This is in contrast to the situation in the critical generalized Ising Curie–Weiss model: at the critical point, the fluctuations are of order  $n^{1-1/2k}$ , where  $k$  depends on the distribution  $\rho$ . We stress also that our integrability conditions on  $\rho$  are weaker than those of [12]. For instance, our result holds for any centered Gaussian measure on  $\mathbb{R}$ . The Gaussian case of our model can be handled with the help of an explicit computation [15].

The main new technical ingredient of the proof is the following inequality. Let  $Z$  be a random variable with distribution  $\rho$ , and let  $I$  denote the Cramér transform of  $(Z, Z^2)$ , given by

$$\forall (x, y) \in \mathbb{R}^2 \quad I(x, y) = \sup_{(u, v) \in \mathbb{R}^2} \left\{ xu + yv - \ln \int_{\mathbb{R}} e^{uz + vz^2} d\rho(z) \right\}.$$

If  $\rho$  is symmetric and there exists  $v > 0$  such that  $E(\exp(vZ^2)) < +\infty$ , then

$$\forall (x, y) \in \mathbb{R}^2 \quad I(x, y) \geq \frac{x^2}{2y},$$

and the equality holds only at  $(0, \sigma^2)$ . We explain in the heuristics at the end of Section 3 why this inequality is crucial to the proof of our main results.

In Section 2 we properly define our model. We state our main results and the strategy for proving them in Section 3. Next we split the proofs in the remaining Sections 4–7. In the Appendix, we recall some generalities on the Cramér transform and large deviations.

**2. The model.** Let  $\rho$  be a probability measure on  $\mathbb{R}$ , which is not the Dirac mass at 0. We consider an infinite triangular array of real-valued random variables  $(X_n^k)_{1 \leq k \leq n}$  such that for all  $n \geq 1$ ,  $(X_n^1, \dots, X_n^n)$  has the distribution  $\tilde{\mu}_{n, \rho}$ , where

$$d\tilde{\mu}_{n, \rho}(x_1, \dots, x_n) = \frac{1}{Z_n} \exp\left(\frac{1}{2} \frac{(x_1 + \dots + x_n)^2}{x_1^2 + \dots + x_n^2}\right) \mathbb{1}_{\{x_1^2 + \dots + x_n^2 > 0\}} \prod_{i=1}^n d\rho(x_i),$$

with

$$Z_n = \int_{\mathbb{R}^n} \exp\left(\frac{1}{2} \frac{(x_1 + \dots + x_n)^2}{x_1^2 + \dots + x_n^2}\right) \mathbb{1}_{\{x_1^2 + \dots + x_n^2 > 0\}} \prod_{i=1}^n d\rho(x_i).$$

We define  $S_n = X_n^1 + \dots + X_n^n$  and  $T_n = (X_n^1)^2 + \dots + (X_n^n)^2$ .

The indicator function in the density of the distribution  $\tilde{\mu}_{n, \rho}$  helps to avoid any problem of definition if  $\rho(\{0\})$  is positive, since, if  $\rho(\{0\}) > 0$ , the event  $\{x_1^2 + \dots +$

$x_n^2 = 0$  may occur with positive probability. We notice that, unlike the generalized Ising Curie–Weiss model, our model is defined for any probability measure. Indeed  $x \mapsto x^2$  is a convex function, and therefore

$$\forall (x_1, \dots, x_n) \in \mathbb{R}^n \quad \left( \sum_{i=1}^n x_i \right)^2 = n^2 \left( \sum_{i=1}^n \frac{x_i}{n} \right)^2 \leq n \sum_{i=1}^n x_i^2.$$

Thus for any  $n \geq 1$ ,  $1 \leq Z_n \leq e^{n/2} < +\infty$ .

If we choose  $\rho = (\delta_{-1} + \delta_1)/2$ , we obtain the classical Ising Curie–Weiss model at the critical value.

**3. Convergence theorems.** We state here our main results. By the classical law of large numbers, if  $\rho$  is centered and has variance  $\sigma^2$ , then under  $\rho^{\otimes n}$ ,  $(S_n/n, T_n/n)$  converges in probability toward  $(0, \sigma^2)$ . The next theorem shows that under the law  $\tilde{\mu}_{n,\rho}$ , given certain conditions,  $(S_n/n, T_n/n)$  also converges in probability to  $(0, \sigma^2)$ .

**THEOREM 1.** *Let  $\rho$  be a symmetric probability measure on  $\mathbb{R}$  with positive variance  $\sigma^2$  and such that*

$$\exists v_0 > 0 \quad \int_{\mathbb{R}} e^{v_0 z^2} d\rho(z) < +\infty.$$

*We suppose that one of the following conditions holds:*

- (a)  $\rho$  has a density.
- (b)  $\rho$  is the sum of a finite number of Dirac masses.
- (c) There exists  $c > 0$  such that  $\rho(]0, c]) = 0$ .
- (d)  $\rho(\{0\}) < 1/\sqrt{e}$ .

*Then, under  $\tilde{\mu}_{n,\rho}$ ,  $(S_n/n, T_n/n)$  converges in probability toward  $(0, \sigma^2)$ .*

By the classical central limit theorem, under  $\rho^{\otimes n}$ ,  $S_n/\sqrt{n}$  converges in distribution to a normal distribution with mean zero and variance  $\sigma^2$ . The following theorem shows that given certain conditions, under  $\tilde{\mu}_{n,\rho}$ ,  $S_n/n^{3/4}$  converges toward a specific distribution.

**THEOREM 2.** *Let  $\rho$  be a probability measure on  $\mathbb{R}$  having a density  $f$  which satisfies:*

- (a)  $f$  is even.
- (b) There exists  $v_0 > 0$  such that

$$\int_{\mathbb{R}} e^{v_0 z^2} f(z) dz < +\infty.$$

(c) *There exists  $p \in ]1, 2]$  such that*

$$\int_{\mathbb{R}^2} f^p(x + y) f^p(y) |x|^{1-p} dx dy < +\infty.$$

*Let  $\sigma^2$  be the variance of  $\rho$ , and let  $\mu_4$  be the fourth moment of  $\rho$ . We have*

$$\frac{\mu_4^{1/4} S_n}{\sigma^2 n^{3/4}} \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \left(\frac{4}{3}\right)^{1/4} \Gamma\left(\frac{1}{4}\right)^{-1} \exp\left(-\frac{s^4}{12}\right) ds.$$

The convergence can equivalently be rewritten as

$$\frac{S_n}{n^{3/4}} \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \left(\frac{4\mu_4}{3\sigma^8}\right)^{1/4} \Gamma\left(\frac{1}{4}\right)^{-1} \exp\left(-\frac{\mu_4}{12\sigma^8} s^4\right) ds.$$

We prove this convergence in Section 7. The following corollary is a version of Theorem 2 with a hypothesis which is weaker but easier to check.

**COROLLARY 3.** *Let  $\rho$  be a probability measure on  $\mathbb{R}$  with an even and bounded density  $f$  such that*

$$\exists v_0 > 0 \quad \int_{\mathbb{R}} e^{v_0 z^2} d\rho(z) < +\infty.$$

*Let  $\sigma^2$  be the variance of  $\rho$ , and let  $\mu_4$  be the fourth moment of  $\rho$ . Then*

$$\frac{\mu_4^{1/4} S_n}{\sigma^2 n^{3/4}} \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \left(\frac{4}{3}\right)^{1/4} \Gamma\left(\frac{1}{4}\right)^{-1} \exp\left(-\frac{s^4}{12}\right) ds.$$

**PROOF.** We check that the hypotheses of the corollary imply the condition (c) of Theorem 2. We have

$$\begin{aligned} & \int_{\mathbb{R}^2} f^{3/2}(x + y) f^{3/2}(y) |x|^{-1/2} dx dy \\ &= \int_{[-1, 1] \times \mathbb{R}} \frac{f^{3/2}(x + y) f^{3/2}(y)}{|x|^{1/2}} dx dy \\ & \quad + \int_{[-1, 1]^c \times \mathbb{R}} \frac{f^{3/2}(x + y) f^{3/2}(y)}{|x|^{1/2}} dx dy \\ & \leq \|f\|_{\infty}^{3/2} \int_{[-1, 1] \times \mathbb{R}} \frac{f^{3/2}(y)}{|x|^{1/2}} dx dy + \int_{[-1, 1]^c \times \mathbb{R}} f^{3/2}(x + y) f^{3/2}(y) dx dy \\ & \leq \|f\|_{\infty}^{3/2} \left( \int_{\mathbb{R}} |f(x)|^{3/2} dx \right) \left( \int_{-1}^1 \frac{dx}{|x|^{1/2}} \right) + \left( \int_{\mathbb{R}} |f(x)|^{3/2} dx \right)^2. \end{aligned}$$

The second inequality is obtained by applying Fubini’s theorem. These terms are finite since

$$\int_{\mathbb{R}} |f(x)|^{3/2} dx \leq \|f\|_{\infty}^{1/2} \int_{\mathbb{R}} f(x) dx = \|f\|_{\infty}^{1/2} < +\infty.$$

Thus, with  $p = 3/2 \in ]1, 2]$ , the function  $(x, y) \mapsto f^p(x + y)f^p(y)|x|^{1-p}$  is integrable.  $\square$

For instance, if  $\rho$  has a bounded support and a density which is even and continuous on it, then the hypotheses of the theorem are fulfilled.

We end this section by computing the law of  $(S_n/n, T_n/n)$  under  $\tilde{\mu}_{n,\rho}$  and explaining the strategy for proving these results. We denote by  $\tilde{\nu}_{n,\rho}$  the law of  $(S_n/n, T_n/n)$  under  $\rho^{\otimes n}$ . We have

$$\forall (x_1, \dots, x_n) \in \mathbb{R}^n \quad \frac{(x_1 + \dots + x_n)^2}{x_1^2 + \dots + x_n^2} = n \frac{((x_1 + \dots + x_n)/n)^2}{(x_1^2 + \dots + x_n^2)/n}.$$

Hence, for any bounded measurable function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ ,

$$\mathbb{E}_{\tilde{\mu}_{n,\rho}} \left( f \left( \frac{S_n}{n}, \frac{T_n}{n} \right) \right) = \frac{1}{Z_n} \int_{\mathbb{R}^2} f(x, y) \exp \left( \frac{nx^2}{2y} \right) \mathbb{1}_{\{y>0\}} d\tilde{\nu}_{n,\rho}(x, y).$$

By convexity of  $t \mapsto t^2$ , we have  $S_n^2 \leq nT_n$  for any  $n \geq 1$ . We define

$$\Delta = \{(x, y) \in \mathbb{R}^2 : x^2 \leq y\} \quad \text{and} \quad \Delta^* = \Delta \setminus \{(0, 0)\}.$$

Thus  $\tilde{\nu}_{n,\rho}(\Delta^c) = 0$ . Therefore we have the following proposition:

**PROPOSITION 4.** *Under  $\tilde{\mu}_{n,\rho}$ , the law of  $(S_n/n, T_n/n)$  is*

$$\frac{\exp((nx^2)/(2y)) \mathbb{1}_{\Delta^*}(x, y) d\tilde{\nu}_{n,\rho}(x, y)}{\int_{\Delta^*} \exp((ns^2)/(2t)) d\tilde{\nu}_{n,\rho}(s, t)}.$$

We denote by  $\nu_\rho$  the law of  $(Z, Z^2)$  where  $Z$  is a random variable with distribution  $\rho$ . The log-Laplace  $\Lambda$  of  $\nu_\rho$  is the map defined on  $\mathbb{R}^2$  by

$$\forall (u, v) \in \mathbb{R}^2 \quad \Lambda(u, v) = \ln \int_{\mathbb{R}^2} e^{us+vt} d\nu_\rho(s, t) = \ln \int_{\mathbb{R}} e^{uz+vz^2} d\rho(z),$$

and the Cramér transform  $I$  of  $\nu_\rho$  is defined on  $\mathbb{R}^2$  by

$$\forall (x, y) \in \mathbb{R}^2 \quad I(x, y) = \sup_{(u,v) \in \mathbb{R}^2} (xu + yv - \Lambda(u, v)).$$

For  $n \geq 1$ , under  $\rho^{\otimes n}$ ,  $(S_n/n, T_n/n)$  is the sum of  $n$  independent and identically distributed random variables with distribution  $\nu_\rho$ . We refer to Appendix B for some definitions and results on large deviations, especially Cramér’s theorem (Theorem B.4) which states that if  $\Lambda$  is finite in the neighborhood of  $(0, 0)$ , then  $I$  is a good rate function, and  $(\tilde{\nu}_{n,\rho})_{n \geq 1}$  satisfies the large deviations principle with speed  $n$ , governed by  $I$ .

Here is a classical heuristic on large deviations, suggested by a consequence of Varadhan’s lemma (see Theorem II.7.2 of [11]): as  $n$  goes to  $+\infty$ , the law of

$(S_n/n, T_n/n)$  under  $\tilde{\mu}_{n,\rho}$  concentrates exponentially fast on the minima on  $\Delta^*$  of the function

$$G = I - F - \inf_{\Delta^*}(I - F),$$

where  $F$  is the map defined by

$$\forall (x, y) \in \mathbb{R} \times \mathbb{R} \setminus \{0\} \quad F(x, y) = \frac{x^2}{2y}.$$

If  $G$  has a unique minimum at  $(x_0, y_0) \in \Delta^*$ , then under  $\tilde{\mu}_{n,\rho}$ ,  $(S_n/n, T_n/n)$  converges in probability to  $(x_0, y_0)$ . Moreover, the large deviations principle suggests that for  $n$  large enough,  $\tilde{\nu}_{n,\rho}$  can roughly be approximated by the distribution  $C_n \exp(-nI(x, y)) dx dy$  where  $C_n$  is a normalizing constant. Thus, for each bounded continuous function  $h$  and  $\alpha, \beta > 0$ ,

$$\begin{aligned} \mathbb{E}_{\tilde{\mu}_n} \left( h \left( \frac{S_n - nx_0}{n^{1-\alpha}} \right) \right) &\approx \frac{\int_{\Delta^*} h((x - x_0)n^\alpha) \exp(-nG(x, y)) dx dy}{\int_{\Delta^*} \exp(-nG(x, y)) dx dy} \\ &\approx \frac{\int_{\Delta^*} h(x) \exp(-nG(xn^{-\alpha} + x_0, yn^{-\beta} + y_0)) dx dy}{\int_{\Delta^*} \exp(-nG(xn^{-\alpha} + x_0, yn^{-\beta} + y_0)) dx dy}. \end{aligned}$$

We use then Laplace’s method. The key point is the study of the function  $G$  in the neighborhood of its minimum  $(x_0, y_0)$ . We find four positive values  $A, B, a \in \mathbb{N}$  and  $b \in \mathbb{N}$  such that, uniformly on a neighborhood of  $(x_0, y_0)$ ,

$$-nG(xn^{-1/a} + x_0, yn^{-1/b} + y_0) \xrightarrow[n \rightarrow \infty]{} -Ax^a - By^b.$$

We prove that  $I - F$  has a unique minimum at  $(0, \sigma^2)$  on  $\Delta^*$  in Section 4. Next we give the proof of Theorem 1 in Section 5, with the help of a variant of Varadhan’s lemma. Finally we compute the expansion of  $I - F$  around  $(0, \sigma^2)$  in Section 6, and we prove Theorem 2 with Laplace’s method in Section 7. Throughout these proofs we use some general results on the Cramér transform, stated in Appendix A.

**4. Minimum of  $I - F$  on  $\Delta^*$ .** Let  $\rho$  be a symmetric probability measure on  $\mathbb{R}$ . In this section, we will use Proposition A.4 in the Appendix to show an inequality between  $I$  and  $F$ .

We denote by  $\nu_\rho$  the distribution of  $(Z, Z^2)$  when  $Z$  is a random variable with law  $\rho$ . If the support of  $\rho$  contains at least three points, then  $\nu_\rho$  is a nondegenerate measure on  $\mathbb{R}^2$ ; see the first paragraphs of Appendix A. We denote by  $\mathcal{C}$  the convex hull of the set  $\{(x, x^2) : x \text{ is in the support of } \rho\}$ . The function

$$\Lambda : (u, v) \in \mathbb{R}^2 \mapsto \ln \int_{\mathbb{R}} e^{uz+ vz^2} d\rho(z)$$

is the log-Laplace of  $\nu_\rho$ , and its domain of definition  $D_\Lambda$  contains  $\mathbb{R} \times ]-\infty, 0[$ ; thus its interior is nonempty. Let  $I$  be the Cramér transform of  $\nu_\rho$ . We denote

by  $D_I$  its domain of definition and by  $A_I = \nabla \Lambda(\overset{\circ}{D}_\Lambda)$  its admissible domain; see Definition A.3 in the Appendix.

Using Jensen’s inequality, we get that  $I(0, \sigma^2) = 0$ . Moreover the infimum of  $I - F$  on  $\Delta^*$  belongs to  $[-1/2, 0]$ . The function  $I$  is even in the first variable. Indeed, if  $(x, y) \in \mathbb{R}^2$ , then

$$\begin{aligned} I(-x, y) &= \sup_{(u,v) \in \mathbb{R}^2} \left( -xu + yv - \ln \int_{\mathbb{R}} e^{uz+vy} d\rho(z) \right) \\ &= \sup_{(u,v) \in \mathbb{R}^2} \left( xu + yv - \ln \int_{\mathbb{R}} e^{-uz+vy} d\rho(z) \right) = I(x, y). \end{aligned}$$

Assume that  $I - F$  has a unique minimum  $(x_0, y_0)$  on  $\Delta^*$ . Then  $(-x_0, y_0)$  is also a minimum of  $I - F$ . The uniqueness of the minimum implies that  $x_0 = 0$  so that  $I - F$  is nonnegative on  $\Delta^*$ . Finally, since  $I(0, \sigma^2) = 0$ , we have  $y_0 = \sigma^2$ .

Consider first the case of a Bernoulli distribution for which  $\nu_\rho$  is degenerate. Let  $c > 0$ . Suppose that  $\rho = (\delta_{-c} + \delta_c)/2$ . The law  $\rho$  is centered, and its variance is  $c^2$ . We can compute  $\Lambda$  and  $I$  explicitly in the following way:

$$\forall (u, v) \in \mathbb{R}^2 \quad \Lambda(u, v) = vc^2 + \ln \cosh(uc).$$

For any  $(x, y) \notin [-c, c] \times \{c^2\}$ ,  $I(x, y) = +\infty$  and

$$\forall x \in ]-c, c[ \quad I(x, c^2) = \frac{1}{2c} ((c+x) \ln(c+x) + (c-x) \ln(c-x)) - \ln c.$$

The study of the function  $x \mapsto I(x, c^2) - x^2/(2c^2)$  shows that, in the Bernoulli case,  $I - F$  has a unique minimum at  $(0, \sigma^2)$ . More generally we have the following lemma:

LEMMA 5. *Let  $c > 0$ . We define*

$$\phi_c : x \in \mathbb{R} \mapsto \sup_{u \in \mathbb{R}} (ux - \ln \cosh(uc)).$$

*The function  $x \mapsto \phi_c(x) - x^2/(2c^2)$  is increasing on  $[0, c]$ , decreasing on  $[-c, 0]$  and null at 0.*

Notice that the Bernoulli case is special since if  $X$  is a random variable with distribution  $\rho = (\delta_{-c} + \delta_c)/2$ , then  $X^2 = c^2$  almost surely. Thus

$$\begin{aligned} &\frac{1}{Z_n} \exp\left(\frac{1}{2} \frac{(x_1 + \dots + x_n)^2}{x_1^2 + \dots + x_n^2}\right) \mathbb{1}_{\{x_1^2 + \dots + x_n^2 > 0\}} \prod_{i=1}^n d\rho(x_i) \\ &= \frac{1}{Z_n(1/c^2)} \exp\left(\frac{(x_1 + \dots + x_n)^2}{2nc^2}\right) \prod_{i=1}^n d\rho(x_i). \end{aligned}$$

This is exactly the classical Curie–Weiss model at the critical point.

In the following, we suppose that the support of  $\nu_\rho$  contains at least three distinct points. We first show that if  $D_\Lambda$  is an open subset of  $\mathbb{R}^2$ , then  $I - F$  has a unique minimum at  $(0, \sigma^2)$ . To this end, we use Proposition A.4 in the Appendix which states that  $I$  is differentiable on  $A_I = \overset{\circ}{D}_I = \overset{\circ}{C}$ . Moreover, if  $(x, y) \mapsto (u(x, y), v(x, y))$  is the inverse function of  $\nabla\Lambda$ , then

$$\forall (x, y) \in \overset{\circ}{D}_I \quad \frac{\partial I}{\partial x}(x, y) = u(x, y).$$

If we show that  $u(x, y) > x/y$  for any  $x, y > 0$ , then by integrating this inequality,

$$\forall (x, y) \in \overset{\circ}{D}_I \quad 0 \leq \varepsilon < x \quad \implies \quad I(x, y) - \frac{x^2}{2y} > I(\varepsilon, y) - \frac{\varepsilon^2}{2y}.$$

To obtain that  $I - F$  has a unique minimum at  $(0, \sigma^2)$ , it is enough to extend this inequality to the boundary points of  $D_I$  (if they exist). We conclude by using the fact that  $I$  is even in its first variable.

The following lemma is the key result to establish the uniqueness of the minimum of  $I - F$ , when  $\rho$  is symmetric.

LEMMA 6. *Let  $\rho$  be a symmetric probability measure whose support contains at least three points. For  $(x, y) \in A_I$ , we have  $u(x, y) = 0$  if  $x = 0$  and*

$$\begin{aligned} u(x, y) &> \frac{x}{y} && \text{if } x > 0, \\ u(x, y) &< \frac{x}{y} && \text{if } x < 0. \end{aligned}$$

PROOF. The vector  $(u, v) = (u(x, y), v(x, y))$  verifies

$$(x, y) = \nabla\Lambda(u, v) = \left( \frac{\int_{\mathbb{R}} z e^{uz+ vz^2} d\rho(z)}{\int_{\mathbb{R}} e^{uz+ vz^2} d\rho(z)}, \frac{\int_{\mathbb{R}} z^2 e^{uz+ vz^2} d\rho(z)}{\int_{\mathbb{R}} e^{uz+ vz^2} d\rho(z)} \right).$$

The distribution  $\rho$  is symmetric, thus

$$\int_{\mathbb{R}} z e^{uz+ vz^2} d\rho(z) = \int_0^{+\infty} 2z \sinh(uz) e^{vz^2} d\rho(z).$$

This formula shows that  $u$  and  $x$  have the same sign. Moreover for any  $z \geq 0$ ,  $\tanh(z) \leq z$ . Thus if  $x > 0$ , then  $\sinh(uz) \leq uz \cosh(uz)$ . The equality holds if and only if  $uz = 0$ . Therefore, using the symmetry of  $\rho$ ,

$$x < u \frac{\int_0^{+\infty} 2z^2 \cosh(uz) e^{vz^2} d\rho(z)}{\int_{\mathbb{R}} e^{uz+ vz^2} d\rho(z)} = u \frac{\int_{\mathbb{R}} z^2 e^{uz+ vz^2} d\rho(z)}{\int_{\mathbb{R}} e^{uz+ vz^2} d\rho(z)} = uy.$$

Since  $x > 0, u > 0$  and  $y > 0$ , we conclude that  $u > x/y$ . Similarly, we show that if  $x < 0$ , then  $u < x/y$ .  $\square$

We can now prove the following inequality:

PROPOSITION 7. *If  $\rho$  is a symmetric probability measure on  $\mathbb{R}$  with positive variance  $\sigma^2$  and such that  $D_\Delta$  is an open subset of  $\mathbb{R}^2$ , then*

$$\forall(x, \varepsilon, y) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \setminus \{0\} \quad 0 \leq \varepsilon < x \implies I(x, y) - \frac{x^2}{2y} \geq I(\varepsilon, y) - \frac{\varepsilon^2}{2y}.$$

*This inequality is strict if  $(\varepsilon, y) \in \mathring{D}_I$ .*

The inequality is also true for  $x < \varepsilon \leq 0$  since  $I$  is even in its first variable. In Corollary 12, we shall extend the inequality to any symmetric distribution on  $\mathbb{R}$ .

PROOF OF PROPOSITION 7. We have already treated the Bernoulli case. We assume next that the support of  $\rho$  contains at least three points. The Cramér transform  $I$  is  $C^\infty$  on  $\mathring{D}_I$  and

$$\forall(x, y) \in \mathring{D}_I \quad \frac{\partial I}{\partial x}(x, y) = u(x, y).$$

Let us examine the structure of the set  $D_I$ . We put

$$\forall y > 0 \quad D_{I,y} = \{x \in \mathbb{R} : (x, y) \in D_I\}$$

(see Figure 1). Let  $y > 0$  be such that  $(x, y) \in \mathring{D}_I$  for some  $x \in \mathbb{R}$ . The set  $D_{I,y}$  is a convex subset of  $\mathbb{R}$ . Moreover  $x \mapsto I(x, y)$  is even, therefore  $\mathring{D}_{I,y}$  (the interior of  $D_{I,y}$  as a subset of  $\mathbb{R}$ ) is an open interval  $] -a(y), a(y)[$  with  $a(y) \in [0, \sqrt{y}]$ . Lemma 6 implies that  $u(t, y) > t/y$  for any  $t \in ]0, a(y)[$ . Thus, for any  $x \in \mathring{D}_{I,y} \cap ]0, +\infty[$ ,

$$\forall \varepsilon \in [0, x[ \quad I(x, y) - I(\varepsilon, y) = \int_\varepsilon^x u(t, y) dt > \int_\varepsilon^x \frac{t}{y} dt = \frac{x^2}{2y} - \frac{\varepsilon^2}{2y}.$$

There is no problem of definition at  $y = 0$  since  $\mathring{D}_I \subset \Delta^*$  does not contain  $\mathbb{R} \times \{0\}$  and  $\mathring{D}_{I,0} = \emptyset$ . Moreover

$$x \mapsto \frac{I(x, y) - I(\varepsilon, y)}{x - \varepsilon}$$

is nondecreasing on  $D_{I,y} \setminus \{\varepsilon\}$  since  $I$  is convex. Therefore, if  $-a(y)$  and  $a(y)$  belong to  $D_{I,y}$ , then the previous inequality extends to  $x = -a(y)$  and  $x = a(y)$ .

We have shown that

$$\forall(x, y) \in D_I \quad y > 0, 0 \leq \varepsilon < x \implies I(x, y) - I(\varepsilon, y) > \frac{x^2}{2y} - \frac{\varepsilon^2}{2y},$$

except for the points  $(x, y)$  of the superior and inferior borders of  $D_I$ , if they exist. More precisely, we set

$$K^2 = \inf\{x^2 : x \text{ is in the support of } \rho\} \geq 0$$

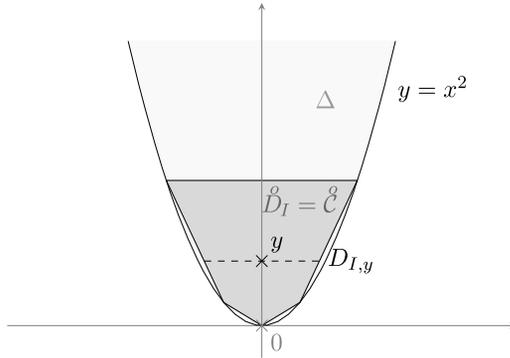


FIG. 1. Case where  $\rho$  is symmetric discrete and charges 5 points.

and

$$L^2 = \sup\{x^2 : x \text{ is in the support of } \rho\} \leq +\infty.$$

If  $K = 0$  and  $L = +\infty$ , then the inequality is already proven on the set  $D_I \setminus \{(0, 0)\}$ . Suppose that  $K^2 > 0$ . Let  $y = K^2$  and  $x \in \mathbb{R}$ . We define

$$f : (u, v) \in \mathbb{R}^2 \mapsto ux + vK^2 - \Lambda(u, v).$$

Denoting  $c_K = \rho(\{K\})$ , we have for all  $(u, v) \in \mathbb{R}^2$ ,

$$f(u, v) = ux - \ln(2c_K \cosh(uK)) - \ln \int_{\mathbb{R} \setminus [-K, K]} e^{uz+v(z^2-K^2)} d\rho(z).$$

For any  $z \in \mathbb{R} \setminus ]-K, K[$ , the function  $v \mapsto \exp(v(z^2 - K^2))$  is nondecreasing. Therefore

$$\begin{aligned} & \sup_{v \in \mathbb{R}} f(u, v) - (ux - \ln(2c_K \cosh(uK))) \\ &= -\ln \left( \lim_{v \rightarrow -\infty} \int_{\mathbb{R} \setminus [-K, K]} e^{uz+v(z^2-K^2)} d\rho(z) \right) = 0, \end{aligned}$$

by the dominated convergence theorem. Indeed

$$\forall z \in \mathbb{R} \setminus [-K, K], \forall v < -1 \quad |e^{uz+v(z^2-K^2)}| \leq e^{uz-(z^2-K^2)},$$

and the map  $z \in \mathbb{R} \setminus [-K, K] \mapsto e^{uz-(z^2-K^2)}$  is integrable with respect to  $\rho$  since it is bounded (it is continuous and goes to 0 when  $|z|$  goes to  $+\infty$ ). Hence

$$I(x, K^2) = \sup_{u, v \in \mathbb{R}} f(u, v) = \sup_{u \in \mathbb{R}} \{ux - \ln(2c_K \cosh(uK))\}.$$

In fact, we come back to the Bernoulli case. The reason is that, if we condition on  $T_n = K^2$  in our model, then for any  $i$ ,  $X_n^i = -K$  or  $K$ .

If  $c_K = 0$ , then  $I(x, K^2) = +\infty$  for any  $x \neq 0$ , so that the (large) inequality is verified for  $y = K^2$ . If  $c_K > 0$ , then Lemma 5 implies that, for any  $\varepsilon, x$  in  $\mathbb{R}$  such that  $0 \leq \varepsilon < x \leq K$ ,

$$I(x, K^2) - I(\varepsilon, K^2) = \phi_K(x) - \phi_K(\varepsilon) > \frac{x^2}{2K^2} - \frac{\varepsilon^2}{2K^2}.$$

If  $L < +\infty$ , then we show similarly the inequality for  $y = L^2$ . Therefore

$$\forall (x, y) \in D_I \setminus \{(0, 0)\} \quad 0 \leq \varepsilon < x \implies I(x, y) - \frac{x^2}{2y} \geq I(\varepsilon, y) - \frac{\varepsilon^2}{2y},$$

and this inequality is strict if  $(\varepsilon, y) \in \overset{\circ}{D}_I$ . Finally we notice that for any  $y \in \mathbb{R}$ , by the convexity and the symmetry of  $x \mapsto I(x, y)$ , if  $I(\varepsilon, y) = +\infty$ , then for all  $x > \varepsilon$ ,  $I(x, y) = +\infty$ . Therefore the inequality extends to each subset of  $\mathbb{R}^2$  which does not contain  $\mathbb{R} \times \{0\}$ .  $\square$

From the arguments in the previous proof, we notice that if we take  $x = 0$  and  $y = 0$ , then for any  $u \in \mathbb{R}$ , the function  $v \mapsto \Lambda(u, v)$  is nondecreasing on  $\mathbb{R}$ . Therefore

$$\inf_{v \in \mathbb{R}} \Lambda(u, v) = \lim_{v \rightarrow -\infty} \Lambda(u, v) = \lim_{v \rightarrow -\infty} \left( \ln \rho(\{0\}) + \ln \int_{\mathbb{R} \setminus \{0\}} e^{uz+ vz^2} d\rho(z) \right).$$

By the dominated convergence theorem, the last integral is equal to  $\ln \rho(\{0\})$ . Hence

$$\inf_{u, v \in \mathbb{R}^2} \Lambda(u, v) = \ln \rho(\{0\}).$$

This is valid for any probability measure  $\rho$  on  $\mathbb{R}$ . This yields the following lemma:

LEMMA 8. *If  $\rho$  is a probability measure on  $\mathbb{R}$ , then  $I(0, 0) = -\ln \rho(\{0\})$ .*

A consequence of Proposition 7 and the fact that  $I$  is even in its first variable is that if  $D_\Lambda$  is an open subset of  $\mathbb{R}^2$ , then the function  $I - F$  has a unique minimum on  $\Delta^*$  at  $(0, \sigma^2)$ . Now we will extend this result to any symmetric probability measure such that  $(0, 0) \in \overset{\circ}{D}_\Lambda$ . For this we need Mosco’s theorem, which we restate next.

DEFINITION 9. Let  $f$  and  $f_n, n \in \mathbb{N}$ , be convex functions from  $\mathbb{R}^d$  to  $[-\infty, +\infty]$ . The sequence  $(f_n)_{n \in \mathbb{N}}$  is said to Mosco converge to  $f$  if for any  $x \in \mathbb{R}^d$ , we have:

- ★ for each sequence  $(x_n)_{n \in \mathbb{N}}$  in  $\mathbb{R}^d$  converging to  $x$ ,

$$\liminf_{n \rightarrow +\infty} f_n(x_n) \geq f(x);$$

★ there exists a sequence  $(x_n)_{n \in \mathbb{N}}$  in  $\mathbb{R}^d$  converging to  $x$  and such that

$$\limsup_{n \rightarrow +\infty} f_n(x_n) \leq f(x).$$

If  $f$  is a convex function from  $\mathbb{R}^d$  to  $[-\infty, +\infty]$ , we denote by  $f^*$  its Fenchel–Legendre transform  $f^*$ . We have the following theorem (see [16] for a proof):

**THEOREM 10 (Mosco).** *Let  $f$  and  $f_n, n \in \mathbb{N}$ , be functions from  $\mathbb{R}^d$  to  $[-\infty, +\infty]$  which are convex and lower semi-continuous. Then  $(f_n)_{n \in \mathbb{N}}$  Mosco converges to  $f$  if and only if  $(f_n^*)_{n \in \mathbb{N}}$  Mosco converges to  $f^*$ .*

**PROPOSITION 11.** *Let  $\nu$  be a probability measure on  $\mathbb{R}^d$ . We denote by  $L$  its log-Laplace. Let  $(K_n)_{n \in \mathbb{N}}$  be a nondecreasing sequence of compact sets whose union is  $\mathbb{R}^d$ . For all  $n \in \mathbb{N}$ , we set  $\nu_n = \nu(\cdot | K_n)$  the probability  $\nu$  conditioned by  $K_n$ , and we denote by  $L_n$  its log-Laplace. Then  $(L_n)_{n \in \mathbb{N}}$  Mosco converges to  $L$ .*

**PROOF.** For  $n$  large enough, the compact set  $K_n$  meets the support of  $\nu$ . Thus, for  $n$  large enough and  $\lambda \in \mathbb{R}^d$ , we have

$$L_n(\lambda) = \ln \int_{\mathbb{R}^d} e^{\langle \lambda, z \rangle} d\nu_n(z) = \ln \int_{K_n} e^{\langle \lambda, z \rangle} d\nu(z) - \ln \nu(K_n).$$

By the monotone convergence theorem,

$$\lim_{n \rightarrow +\infty} L_n(\lambda) = \ln \int_{\mathbb{R}^d} \lim_{n \rightarrow +\infty} (\mathbb{1}_{K_n}(z) e^{\langle \lambda, z \rangle}) d\nu(z) - \lim_{n \rightarrow +\infty} \ln \nu(K_n) = L(\lambda).$$

Hence the second condition of Mosco convergence (with the  $\limsup$ ) is satisfied with the sequence  $(\lambda_n)_{n \in \mathbb{N}}$  constant equal to  $\lambda$ .

Let  $\lambda \in \mathbb{R}^d$  and  $(\lambda_n)_{n \in \mathbb{N}}$  be any sequence converging to  $\lambda$ . Fatou’s lemma implies that

$$\exp L(\lambda) = \int_{\mathbb{R}^d} \liminf_{n \rightarrow +\infty} \mathbb{1}_{K_n}(z) e^{\langle \lambda_n, z \rangle} d\nu(z) \leq \liminf_{n \rightarrow +\infty} \int_{\mathbb{R}^d} \mathbb{1}_{K_n}(z) e^{\langle \lambda_n, z \rangle} d\nu(z).$$

Therefore

$$L(\lambda) \leq \liminf_{n \rightarrow +\infty} (L_n(\lambda_n) + \ln \nu(K_n)) = \liminf_{n \rightarrow +\infty} L_n(\lambda_n).$$

Thus the first condition of Mosco convergence (with the  $\liminf$ ) is verified, and the proposition is proved. □

**COROLLARY 12.** *If  $\rho$  is a symmetric and nondegenerate probability measure on  $\mathbb{R}$ , then*

$$\forall(x, y) \in \Delta^*, \forall \varepsilon \in [0, |x|[ \quad I(x, y) - \frac{x^2}{2y} \geq I(\varepsilon, y) - \frac{\varepsilon^2}{2y}.$$

PROOF. For any  $n \in \mathbb{N}$ , we put  $K_n = [-n, n]^2$ . For  $n$  large enough so that  $K_n$  meets the support of  $\nu_\rho$ , we define  $\nu_n = \nu_\rho(\cdot|K_n)$ ,  $\Lambda_n$  its log-Laplace and  $I_n$  its Fenchel–Legendre transform. For all  $(u, v) \in \mathbb{R}^2$ ,

$$\Lambda_n(u, v) = \ln \int_{K_n} e^{us+vt} d\nu_\rho(s, t) - \ln \nu_\rho(K_n) \leq \Lambda(u, v) - \ln \nu_\rho(K_n).$$

Applying the Fenchel–Legendre transformation, we get

$$\forall (\varepsilon, y) \in \mathbb{R}^2 \quad I(\varepsilon, y) \leq I_n(\varepsilon, y) - \ln \nu_\rho(K_n).$$

Moreover the measure  $\nu_n$  has a bounded support, so Proposition 7 and the previous inequality imply that for any  $(x, \varepsilon, y) \in \mathbb{R} \times \mathbb{R} \times ]0, +\infty[$  such that  $0 \leq \varepsilon < x$ ,

$$I(\varepsilon, y) - \frac{\varepsilon^2}{2y} + \frac{x^2}{2y} \leq I_n(x, y) - \ln \nu_\rho(K_n).$$

It follows from Proposition 11 that  $(\Lambda_n)_{n \in \mathbb{N}}$  Mosco converges to  $\Lambda$ . Hence, by Mosco’s theorem,  $(I_n)_{n \in \mathbb{N}}$  Mosco converges to  $I$ . In particular, for  $(x, y) \in \mathbb{R}^2$  such that  $y > 0$  and  $x > \varepsilon$ , there exists a sequence  $(x_n, y_n) \in \mathbb{R}^2$  converging to  $(x, y)$  and such that

$$\limsup_{n \rightarrow +\infty} I_n(x_n, y_n) \leq I(x, y).$$

Since  $y > 0$  and  $x > \varepsilon$ , there exists  $n_0 \geq 1$  such that  $y_n > 0$  and  $x_n > \varepsilon$  for all  $n \geq n_0$ . Therefore

$$\forall n \geq n_0 \quad I(\varepsilon, y_n) - \frac{\varepsilon^2}{2y_n} + \frac{x_n^2}{2y_n} \leq I_n(x_n, y_n) - \ln \nu_\rho(K_n).$$

Moreover  $\nu_\rho(K_n)$  goes to 1 when  $n$  goes to  $+\infty$ . Hence

$$\limsup_{n \rightarrow +\infty} I(\varepsilon, y_n) - \frac{\varepsilon^2}{2y} + \frac{x^2}{2y} \leq I(x, y).$$

Finally  $I$  is lower semi-continuous, thus

$$\liminf_{n \rightarrow +\infty} I(\varepsilon, y_n) \geq I(\varepsilon, y).$$

This implies the announced inequality.  $\square$

We can now show that  $I - F$  has a unique minimum on  $\Delta^*$  at  $(0, \sigma^2)$ :

PROPOSITION 13. *If  $\rho$  is a symmetric probability measure on  $\mathbb{R}$  with variance  $\sigma^2 > 0$  and such that  $\Lambda$  is finite in a neighborhood of  $(0, 0)$ , then*

$$(x, y) \in \Delta^* \longmapsto I(x, y) - \frac{x^2}{2y}$$

*has a unique minimum at  $(0, \sigma^2)$  where it is equal to 0.*

PROOF. Corollary 12 implies that

$$\forall (x, y) \in \Delta^* \quad I(x, y) - \frac{x^2}{2y} \geq I(0, y).$$

Therefore  $I - F$  is a nonnegative function. Since  $(0, 0) \in \overset{\circ}{D}_\Lambda$ , the function  $I(0, \cdot)$  has a unique minimum at  $\sigma^2$ ; see Theorems 25.1 and 27.1 of [19]. As a consequence, if  $I - F$  has a minimum on  $\Delta^*$  at  $(x_0, y_0)$ , then  $y_0 = \sigma^2$  and  $I(x_0, \sigma^2) = x_0^2/(2\sigma^2)$ .

Moreover  $(0, \sigma^2) \in A_I$ , so there exists  $\varepsilon > 0$  such that  $B_\varepsilon$ , the open ball of radius  $\varepsilon$  centered at  $(0, \sigma^2)$ , is included in  $A_I$ . If  $(x, y)$  realizes a minimum of  $I - F$  on  $B_\varepsilon$ , then

$$(u(x, y), v(x, y)) = \nabla I(x, y) = \nabla F(x, y) = (x/y, -x^2/(2y^2)).$$

It follows from Lemma 6 that  $x = 0$  and thus  $u(x, y) = v(x, y) = 0$ . Therefore  $(x, y) = (0, \sigma^2)$ . Hence

$$\forall x \in ]-\varepsilon, 0[ \cap ]0, \varepsilon[ \quad I(x, \sigma^2) - \frac{x^2}{2\sigma^2} > 0.$$

Applying Corollary 12 with  $\varepsilon/2$ , we see that the above inequality holds for any  $x \neq 0$ . It follows that  $x_0 = 0$ .  $\square$

**5. Proof of Theorem 1 with a variant of Varadhan’s lemma.** Let  $\rho$  be a symmetric probability measure on  $\mathbb{R}$  with positive variance  $\sigma^2$  and such that  $(0, 0) \in \overset{\circ}{D}_\Lambda$ . The heuristics at the end of Section 3 and Proposition 13 suggest that, as  $n$  goes to  $+\infty$ , the law of  $(S_n/n, T_n/n)$  under  $\tilde{\mu}_{n,\rho}$  concentrates exponentially fast on  $(0, \sigma^2)$ , the minimum of  $I - F$ . Yet, in spite of the expression given in Proposition 4, we cannot apply Varadhan’s lemma (Theorem II.7.2 of [11]) directly since  $\Delta^*$  is not a closed set, and  $F$  is not continuous on  $\Delta$ .

In Section 5.5.1, we prove a variant of Varadhan’s lemma. We give the proof of Theorem 1 in Section 5.5.2.

5.1. *Around Varadhan’s lemma.*

PROPOSITION 14. *Let  $\rho$  be a probability measure on  $\mathbb{R}$ . We denote by  $\tilde{v}_{n,\rho}$  the distribution of  $(S_n/n, T_n/n)$  under  $\rho^{\otimes n}$ . We have*

$$\liminf_{n \rightarrow +\infty} \frac{1}{n} \ln \int_{\Delta^*} \exp\left(\frac{nx^2}{2y}\right) d\tilde{v}_{n,\rho}(x, y) \geq 0.$$

*Suppose that  $\rho$  is nondegenerate, symmetric and that  $(0, 0) \in \overset{\circ}{D}_\Lambda$ . We assume that there exists  $r > 0$  such that  $M_r + \ln \rho(\{0\}) < 0$  with*

$$M_r = \sup \left\{ \frac{x^2}{2y} : (x, y) \in \mathcal{C} \cap B_r \setminus \{(0, 0)\} \right\},$$

where  $B_r$  is the open ball of radius  $r$  centered at  $(0, 0)$ , and  $\mathcal{C}$  is the closed convex hull of  $\{(x, x^2) : x \text{ is in the support of } \rho\}$ . If  $A$  is a closed subset of  $\mathbb{R}^2$  which does not contain  $(0, \sigma^2)$ , then

$$\limsup_{n \rightarrow +\infty} \frac{1}{n} \ln \int_{\Delta^* \cap A} \exp\left(\frac{nx^2}{2y}\right) d\tilde{\nu}_{n,\rho}(x, y) < 0.$$

Let us give first some sufficient conditions to fulfill the hypothesis of the proposition. To ensure that there exists  $r > 0$  such that  $M_r + \ln \rho(\{0\}) < 0$ , it is enough that one of the following conditions is satisfied:

- (a)  $\rho$  has a density.
- (b)  $\rho(\{0\}) < 1/\sqrt{e}$ .
- (c) There exists  $c > 0$  such that  $\rho(]0, c[) = 0$ .
- (d)  $\rho$  is the sum of a finite number of Dirac masses.

Indeed, the function  $F$  is bounded by  $1/2$  on  $\mathcal{C} \setminus \{(0, 0)\} \subset \Delta^*$ . Thus for any  $r > 0$ ,  $M_r \leq 1/2$ . Therefore, if  $\rho$  has a density, or more generally if  $\rho(\{0\}) < e^{-1/2}$ , then for all  $r > 0$ ,  $M_r + \ln \rho(\{0\}) < 0$ .

On the other hand, if there exists  $c > 0$  such that  $]0, c[$  does not intersect the support of  $\rho$  (especially if  $\rho$  is the sum of a finite number of Dirac masses), then

$$\mathcal{C} \subset \{(x, y) \in \mathbb{R}^2 : c|x| \leq y\}.$$

Therefore

$$\forall (x, y) \in \mathcal{C} \cap B_r \setminus \{(0, 0)\} \quad \frac{x^2}{2y} = \frac{c|x|^2}{2cy} \leq \frac{|x|}{2c} \leq \frac{r}{2c}.$$

Hence for any  $r > 0$ ,  $M_r < r/2c$ . Since  $\rho$  is nondegenerate,  $\rho(\{0\}) < 1$ . Thus there exists  $r > 0$  such that  $\ln \rho(\{0\}) + r/2c < 0$ . Therefore conditions (c) and (d) imply that  $M_r + \ln \rho(\{0\}) < 0$ .

**PROOF OF PROPOSITION 14.** The large deviations principle satisfied by  $(\tilde{\nu}_{n,\rho})_{n \geq 1}$  implies that

$$\begin{aligned} & \liminf_{n \rightarrow +\infty} \frac{1}{n} \ln \int_{\Delta^*} \exp\left(\frac{nx^2}{2y}\right) d\tilde{\nu}_{n,\rho}(x, y) \\ & \geq \liminf_{n \rightarrow +\infty} \frac{1}{n} \ln \tilde{\nu}_{n,\rho}(\Delta^*) \geq -\inf\{I(x, y) : (x, y) \in \overset{\circ}{\Delta}\} = 0. \end{aligned}$$

We prove now the second inequality. Let  $\alpha > 0$ . The function  $I$  is lower semi-continuous on  $\mathbb{R}^2$ . Thus there exists a neighborhood  $\mathcal{U}$  of  $(0, 0)$  such that

$$\forall (x, y) \in \overline{\mathcal{U}} \quad I(x, y) \geq (I(0, 0) - \alpha) \wedge \frac{1}{\alpha} = (-\ln \rho(\{0\}) - \alpha) \wedge \frac{1}{\alpha}.$$

The above equality follows from Lemma 8. By hypothesis, there exists  $r > 0$  such that  $M_r + \ln \rho(\{0\}) < 0$ . Thus by choosing  $\alpha$  sufficiently small, we can assume that

$$M_r + \ln \rho(\{0\}) + \alpha < 0 \quad \text{and} \quad M_r - \frac{1}{\alpha} < 0.$$

Since  $M_r$  decreases with  $r$ , we can take  $r$  small enough so that  $B_r \subset \mathcal{U}$ . Notice next that  $(S_n/n, T_n/n) \in \mathcal{C}$  almost surely. Therefore, setting  $\mathcal{C}^* = \mathcal{C} \setminus \{(0, 0)\}$ ,

$$\int_{\Delta^* \cap A} \exp\left(\frac{nx^2}{2y}\right) d\tilde{v}_{n,\rho}(x, y) = \int_{\mathcal{C}^* \cap A} \exp\left(\frac{nx^2}{2y}\right) d\tilde{v}_{n,\rho}(x, y).$$

Let us decompose

$$\mathcal{C}^* \cap A \subset (\mathcal{C}^* \cap B_r) \cup (\mathcal{C} \cap B_r^c \cap A).$$

We have

$$\int_{\mathcal{C}^* \cap B_r} \exp\left(\frac{nx^2}{2y}\right) d\tilde{v}_{n,\rho}(x, y) \leq \exp(nM_r) \tilde{v}_{n,\rho}(\mathcal{U}).$$

The large deviation principle satisfied by  $(\tilde{v}_{n,\rho})_{n \geq 1}$  implies that

$$\begin{aligned} \limsup_{n \rightarrow +\infty} \frac{1}{n} \ln \int_{\mathcal{C}^* \cap B_r} \exp\left(\frac{nx^2}{2y}\right) d\tilde{v}_{n,\rho}(x, y) \\ \leq M_r - \inf_{\bar{u}} I \leq (M_r + \ln \rho(\{0\}) + \alpha) \vee \left(M_r - \frac{1}{\alpha}\right). \end{aligned}$$

Next, the set  $\mathcal{C} \cap B_r^c \cap A$  is closed and does not contain  $(0, 0)$ . Thus the function  $F$  is continuous on this set. Moreover  $F$  is bounded on  $\mathcal{C}^*$ . Hence Lemma B.3 in the Appendix and Lemma 1.2.15 of [7] imply that

$$\begin{aligned} \limsup_{n \rightarrow +\infty} \frac{1}{n} \ln \int_{\mathcal{C}^* \cap A} \exp\left(\frac{nx^2}{2y}\right) d\tilde{v}_{n,\rho}(x, y) \\ \leq \max\left(M_r + \ln \rho(\{0\}) + \alpha, M_r - \frac{1}{\alpha}, \sup_{\mathcal{C} \cap B_r^c \cap A} (F - I)\right). \end{aligned}$$

Since  $\rho$  is symmetric and  $(0, 0) \in \overset{\circ}{D}_\Delta$ , Proposition 13 implies that  $G = I - F$  has a unique minimum at  $(0, \sigma^2)$  on  $\Delta^*$ . Suppose that the infimum of  $G$  over  $\mathcal{C} \cap B_r^c \cap A$  is null. Then there exists a sequence  $(x_k, y_k)_{k \in \mathbb{N}}$  in  $\mathcal{C} \cap B_r^c \cap A \subset \Delta^*$  such that

$$\lim_{k \rightarrow +\infty} G(x_k, y_k) = \inf_{\mathcal{C} \cap B_r^c \cap A} G = 0.$$

For  $k$  large enough,  $G(x_k, y_k) \leq 1/2$ . Thus  $I(x_k, y_k) \leq 1$ ; that is,  $(x_k, y_k)$  belongs to the compact set  $\{(u, v) \in \mathbb{R}^2 : I(u, v) \leq 1\}$ . Up to the extraction of a subsequence, we suppose that  $(x_k, y_k)_{k \in \mathbb{N}}$  converges to some  $(x_0, y_0)$ , which belongs to the closed subset  $\mathcal{C} \cap B_r^c \cap A$ . Moreover  $G$  is lower semi-continuous, and hence

$$0 = \limsup_{k \rightarrow +\infty} G(x_k, y_k) \geq G(x_0, y_0) \geq 0.$$

Therefore  $G(x_0, y_0) = 0$ , and thus  $(x_0, y_0) = (0, \sigma^2) \in \mathcal{C} \cap B_r^c \cap A$ , which is absurd since  $A$  does not contain  $(0, \sigma^2)$ . Thus the infimum of  $G$  over  $\mathcal{C} \cap B_r^c \cap A$  is positive. Therefore

$$\max\left(M_r + \ln \rho(\{0\}) + \alpha, M_r - \frac{1}{\alpha}, \sup_{\mathcal{C} \cap B_r^c \cap A} (F - I)\right) < 0.$$

This proves the second inequality.  $\square$

**5.2. Proof of Theorem 1.** Let  $\rho$  be a symmetric probability measure on  $\mathbb{R}$  with positive variance  $\sigma^2$  and such that

$$\exists v_0 > 0 \quad \int_{\mathbb{R}} e^{v_0 z^2} d\rho(z) < +\infty.$$

This implies that  $\mathbb{R} \times ]-\infty, v_0[ \subset D_\Lambda$  and thus  $(0, 0) \in \mathring{D}_\Lambda$ . We assume that one of the four conditions given in the paragraph below Proposition 14 is satisfied.

We denote by  $\theta_{n,\rho}$  the distribution of  $(S_n/n, T_n/n)$  under  $\tilde{\mu}_{n,\rho}$ . Let  $U$  be an open neighborhood of  $(0, \sigma^2)$  in  $\mathbb{R}^2$ . Propositions 4 and 14 imply that

$$\begin{aligned} \limsup_{n \rightarrow +\infty} \frac{1}{n} \ln \theta_{n,\rho}(U^c) &= \limsup_{n \rightarrow +\infty} \frac{1}{n} \ln \int_{\Delta^* \cap U^c} \exp\left(\frac{nx^2}{2y}\right) d\tilde{\nu}_{n,\rho}(x, y) \\ &\quad - \liminf_{n \rightarrow +\infty} \frac{1}{n} \ln \int_{\Delta^*} \exp\left(\frac{nx^2}{2y}\right) d\tilde{\nu}_{n,\rho}(x, y) < 0. \end{aligned}$$

Hence there exist  $\varepsilon > 0$  and  $n_0 \in \mathbb{N}$  such that for any  $n > n_0$ ,

$$\theta_{n,\rho}(U^c) \leq e^{-n\varepsilon} \xrightarrow[n \rightarrow \infty]{} 0.$$

Thus, for each open neighborhood  $U$  of  $(0, \sigma^2)$ ,

$$\lim_{n \rightarrow +\infty} \tilde{\mu}_{n,\rho}\left(\left(\frac{S_n}{n}, \frac{T_n}{n}\right) \in U^c\right) = 0.$$

This means that, under  $\tilde{\mu}_{n,\rho}$ ,  $(S_n/n, T_n/n)$  converges in probability to  $(0, \sigma^2)$ . This completes the proof of Theorem 1.

**6. Expansion of  $I - F$  around its minimum.** In this section, which may be omitted on a first reading, we compute the expansion of the function  $I - F$  around  $(0, \sigma^2)$ , its minimum over  $\Delta^*$ . These computations are crucial because they explain why the fluctuations in Theorem 2 are of order  $n^{3/4}$ , and they give us the term in the exponential in the limiting law.

If  $\rho$  is a symmetric probability measure whose support contains at least three points and if  $(0, 0) \in \mathring{D}_L$ , then  $(0, \sigma^2) = \nabla \Lambda(0, 0) \in \nabla \Lambda(\mathring{D}_\Lambda) = A_I$ , the admissible domain of  $I$ . Proposition A.4 in the Appendix implies that  $I$  is  $C^\infty$  in the

neighborhood of  $(0, \sigma^2)$  and that

$$\begin{aligned} \nabla I(0, \sigma^2) &= (u(0, \sigma^2), v(0, \sigma^2)) = (\nabla \Lambda)^{-1}(0, \sigma^2) = (0, 0), \\ D_{(0, \sigma^2)}^2 I &= (D_{(0,0)}^2 \Lambda)^{-1} = \begin{pmatrix} \sigma^2 & 0 \\ 0 & \mu_4 - \sigma^4 \end{pmatrix}^{-1} = \begin{pmatrix} 1/\sigma^2 & 0 \\ 0 & 1/(\mu_4 - \sigma^4) \end{pmatrix}, \end{aligned}$$

since  $D_{(0,0)}^2 \Lambda$  is the covariance matrix of  $\nu_\rho$ . Hence, up to the second order, the expansion of  $I - F$  in the neighborhood of  $(0, \sigma^2)$  is

$$I(x, y) - F(x, y) = \frac{(y - \sigma^2)^2}{2(\mu_4 - \sigma^4)} + o(\|x, y - \sigma^2\|^2).$$

We need to push further the expansion of  $I - F$ .

Consider the case of the Gaussian  $\mathcal{N}(0, \sigma^2)$ . We can explicitly compute  $I$  in the following way:

$$\forall (x, y) \in \Delta^* \quad I(x, y) = \frac{1}{2} \left( \frac{y}{\sigma^2} - 1 - \ln \left( \frac{y - x^2}{\sigma^2} \right) \right).$$

In the neighborhood of  $(0, \sigma^2)$ , we have

$$I(x, y) - F(x, y) \sim \frac{x^4}{4\sigma^4} + \frac{(y - \sigma^2)^2}{4\sigma^2}.$$

In fact, we have a similar expansion in a more general case:

**PROPOSITION 15.** *If  $\rho$  is a symmetric probability measure on  $\mathbb{R}$  whose support contains at least three points and such that  $(0, 0) \in \mathring{D}_\Lambda$ , then  $I$  is  $C^\infty$  in the neighborhood of  $(0, \sigma^2)$ . If  $\mu_4$  denotes the fourth moment of  $\rho$ , then when  $(x, y)$  goes to  $(0, \sigma^2)$ ,*

$$I(x, y) - \frac{x^2}{2y} \sim \frac{(y - \sigma^2)^2}{2(\mu_4 - \sigma^4)} + \frac{\mu_4 x^4}{12\sigma^8}.$$

**PROOF.** If  $(0, 0) \in \mathring{D}_\Lambda$ , then  $(0, \sigma^2) = \nabla \Lambda(0, 0) \in \nabla \Lambda(\mathring{D}_\Lambda) = A_I$ , and Proposition A.4 in the Appendix implies that the function  $I$  is  $C^\infty$  on  $A_I$ . Moreover, if we denote the inverse function of  $\nabla \Lambda$  by  $(x, y) \mapsto (u(x, y), v(x, y))$ , then, for all  $(x, y) \in A_I$ ,

$$\nabla I(x, y) = (u(x, y), v(x, y)) \quad \text{and} \quad D_{(x,y)}^2 I = (D_{(u(x,y), v(x,y))}^2 \Lambda)^{-1}.$$

The hypothesis  $(0, 0) \in \mathring{D}_\Lambda$  also implies that  $\rho$  has finite moments of all orders. The expansion of  $F$  to the fourth order in the neighborhood of  $(0, \sigma^2)$  is

$$F(x, y) = \frac{x^2}{2\sigma^2} - \frac{x^2(y - \sigma^2)}{2\sigma^4} + \frac{x^2(y - \sigma^2)^2}{2\sigma^6} + o(\|x, y - \sigma^2\|^4).$$

Therefore, in the neighborhood of  $(0, 0)$ ,

$$\begin{aligned}
 I(x, h + \sigma^2) - F(x, h + \sigma^2) &= \frac{h^2}{2(\mu_4 - \sigma^4)} + a_{3,0}x^3 + a_{2,1}x^2h + a_{1,2}xh^2 + a_{0,3}h^3 \\
 &\quad + a_{4,0}x^4 + a_{3,1}x^3h + a_{2,2}x^2h^2 + a_{1,3}xh^3 + a_{0,4}h^4 + o(\|x, h\|^4),
 \end{aligned}$$

with, for any  $(i, j) \in \mathbb{N}$  such that  $i + j \in \{3, 4\}$ ,

$$a_{i,j} = \frac{1}{i!j!} \frac{\partial^{i+j} I}{\partial x^i \partial y^j}(0, \sigma^2),$$

except for

$$a_{2,1} = \frac{1}{2} \left( \frac{\partial^3 I}{\partial x^2 \partial y}(0, \sigma^2) + \frac{1}{\sigma^4} \right) \quad \text{and} \quad a_{2,2} = \frac{1}{4} \frac{\partial^4 I}{\partial x^2 \partial y^2}(0, \sigma^2) - \frac{1}{2\sigma^6}.$$

If we prove that  $a_{4,0} > 0$ , then the terms  $xh^2, h^3, x^3h, x^2h^2, xh^3$  and  $h^4$  are negligible compared to  $a_{4,0}x^4 + a_{0,2}h^2$  when  $(x, h)$  goes to  $(0, 0)$ . Next, the symmetry of  $I - F$  in the first variable implies that  $a_{3,0} = 0$ . If we show that  $a_{2,1} = 0$ , then when  $(x, y)$  goes to  $(0, \sigma^2)$ ,

$$I(x, y) - F(x, y) = \left( \frac{(y - \sigma^2)^2}{2(\mu_4 - \sigma^4)} + a_{4,0}x^4 \right) (1 + o(1)).$$

To conclude it is enough to show that  $a_{2,1} = 0$  and  $a_{4,0} = \mu_4 / (12\sigma^8)$ , that is,

$$\frac{\partial^3 I}{\partial x^2 \partial y}(0, \sigma^2) = -\frac{1}{\sigma^4} \quad \text{and} \quad \frac{\partial^4 I}{\partial x^4}(0, \sigma^8) = \frac{2\mu_4}{\sigma^2}.$$

For any  $j \in \mathbb{N}$ , we introduce the function  $f_j$  defined on  $\mathring{D}_\Lambda$  by

$$\forall (u, v) \in \mathring{D}_\Lambda \quad f_j(u, v) = \int_{\mathbb{R}} x^j e^{ux+vx^2} d\rho(x) \left( \int_{\mathbb{R}} e^{ux+vx^2} d\rho(x) \right)^{-1}.$$

These functions are  $C^\infty$  on  $\mathring{D}_\Lambda$ , and they verify the following properties:

★  $f_0$  is the identity function on  $\mathbb{R}^2$  and

$$f_1 = \frac{\partial \Lambda}{\partial u} \quad \text{and} \quad f_2 = \frac{\partial \Lambda}{\partial v}.$$

★ For all  $j \in \mathbb{N}$ ,  $f_j(0, 0) = \mu_j$  is the  $j$ th moment of  $\rho$ . It is null if  $j$  is odd, since  $\rho$  is symmetric. Moreover, for any  $j \in \mathbb{N}$ ,

$$\frac{\partial f_j}{\partial u} = f_{j+1} - f_j f_1 \quad \text{and} \quad \frac{\partial f_j}{\partial v} = f_{j+2} - f_j f_2.$$

Therefore, for all  $(x, y) \in A_I$ ,

$$D_{(x,y)}^2 I = (D_{(u(x,y),v(x,y))}^2 \Lambda)^{-1} \\ = \begin{pmatrix} f_2 - f_1^2 & f_3 - f_1 f_2 \\ f_3 - f_1 f_2 & f_4 - f_2^2 \end{pmatrix}^{-1} (u(x, y), v(x, y)).$$

Denoting by  $g = (f_2 - f_1^2)(f_4 - f_2^2) - (f_3 - f_1 f_2)^2$ , the determinant of the positive definite symmetric matrix  $D^2 \Lambda$ , we get that for any  $(x, y) \in A_I$ ,

$$D_{(x,y)}^2 I = \frac{1}{g(u(x, y), v(x, y))} \begin{pmatrix} f_4 - f_2^2 & f_1 f_2 - f_3 \\ f_1 f_2 - f_3 & f_2 - f_1^2 \end{pmatrix} (u(x, y), v(x, y)).$$

Moreover  $(u(0, \sigma^2), v(0, \sigma^2)) = (0, 0)$  thus

$$\frac{\partial u}{\partial x}(0, \sigma^2) = \frac{\partial^2 I}{\partial x^2}(0, \sigma^2) = \frac{f_4 - f_2^2}{g}(0, 0) = \frac{\mu_4 - \sigma^4}{\sigma^2(\mu_4 - \sigma^4)} = \frac{1}{\sigma^2},$$

$$\frac{\partial v}{\partial y}(0, \sigma^2) = \frac{\partial^2 I}{\partial y^2}(0, \sigma^2) = \frac{f_2 - f_1^2}{g}(0, 0) = \frac{\sigma^2}{\sigma^2(\mu_4 - \sigma^4)} = \frac{1}{\mu_4 - \sigma^4},$$

$$\frac{\partial u}{\partial y}(0, \sigma^2) = \frac{\partial v}{\partial x}(0, \sigma^2) = \frac{\partial^2 I}{\partial x \partial y}(0, \sigma^2) = \frac{f_1 f_2 - f_3}{g}(0, 0) = 0.$$

Differentiating with respect to  $y$ , we get

$$\frac{\partial^3 I}{\partial y \partial x^2} = \frac{\partial u}{\partial y} \times \frac{\partial}{\partial u} \left( \frac{f_4 - f_2^2}{g} \right) (u, v) + \frac{\partial v}{\partial y} \times \frac{\partial}{\partial v} \left( \frac{f_4 - f_2^2}{g} \right) (u, v).$$

The first term of the addition, taken at  $(0, \sigma^2)$ , is null. For the second term, we need to compute the partial derivative of  $(f_4 - f_2^2)/g$  with respect to  $v$ ,

$$\frac{\partial}{\partial v} \left( \frac{f_4 - f_2^2}{g} \right) = \frac{1}{g} \times \frac{\partial}{\partial v} (f_4 - f_2^2) - \frac{f_4 - f_2^2}{g^2} \times \frac{\partial g}{\partial v} \\ = \frac{f_6 - 3f_2 f_4 + 2f_2^3}{g} - \frac{f_4 - f_2^2}{g^2} \times \frac{\partial g}{\partial v}.$$

Let us differentiate with respect to  $v$ ,

$$\frac{\partial g}{\partial v} = f_2(f_6 - f_4 f_2) + f_4(f_4 - f_2^2) - f_1^2(f_6 - f_4 f_2) \\ - 2f_4 f_1(f_3 - f_1 f_2) - 3f_2^2(f_4 - f_2^2) - 2f_3(f_5 - f_3 f_2) \\ + 2f_1 f_2(f_5 - f_3 f_2) + 2f_2 f_3(f_3 - f_1 f_2) + 2f_1 f_3(f_4 - f_2^2).$$

Taken at  $(0, 0)$ , each term with even subscript vanishes and we have

$$\frac{\partial g}{\partial v}(0, 0) = \sigma^2(\mu_6 - \mu_4 \sigma^2) + \mu_4(\mu_4 - \sigma^4) - 3\sigma^4(\mu_4 - \sigma^4) \\ = \sigma^2 \mu_6 - 3\mu_4 \sigma^4 + 2\sigma^8 + (\mu_4 - \sigma^4)^2.$$

Finally

$$\begin{aligned} & \frac{\partial}{\partial v} \left( \frac{f_4 - f_2^2}{g} \right) (0, 0) \\ &= \frac{\mu_6 - 3\sigma^2\mu_4 + 2\sigma^6}{\sigma^2(\mu_4 - \sigma^4)} - \frac{\sigma^2\mu_6 - 3\mu_4\sigma^4 + 2\sigma^8 + (\mu_4 - \sigma^4)^2}{\sigma^4(\mu_4 - \sigma^4)}, \end{aligned}$$

which is equal to  $(\sigma^4 - \mu_4)/\sigma^4$  after simplification. Therefore

$$\begin{aligned} \frac{\partial^3 I}{\partial y \partial x^2} (0, \sigma^2) &= 0 + \frac{\partial v}{\partial y} (0, \sigma^2) \frac{\partial}{\partial v} \left( \frac{f_4 - f_2^2}{g} \right) (0, 0) \\ &= \frac{1}{\mu_4 - \sigma^4} \times \frac{\sigma^4 - \mu_4}{\sigma^4} = -\frac{1}{\sigma^4}. \end{aligned}$$

This is what we wanted to prove. Let us compute now the fourth partial derivative of  $I$  with respect to  $x$ . We have to obtain first an expression of the third partial derivative of  $I$  with respect to  $x$ ,

$$\frac{\partial^3 I}{\partial x^3} = \frac{\partial u}{\partial x} \times \frac{\partial}{\partial u} \left( \frac{f_4 - f_2^2}{g} \right) (u, v) + \frac{\partial v}{\partial x} \times \frac{\partial}{\partial v} \left( \frac{f_4 - f_2^2}{g} \right) (u, v).$$

The only term we do not know is the partial derivative with respect to  $u$  of  $(f_4 - f_2^2)/g$ . We have

$$\begin{aligned} \frac{\partial}{\partial u} \left( \frac{f_4 - f_2^2}{g} \right) &= \frac{1}{g} \times \frac{\partial}{\partial u} (f_4 - f_2^2) - \frac{f_4 - f_2^2}{g^2} \times \frac{\partial g}{\partial u} \\ &= \frac{f_5 - f_4 f_1 - 2f_2 f_3 + 2f_2^2 f_1}{g} - \frac{f_4 - f_2^2}{g^2} \times \frac{\partial g}{\partial u}, \end{aligned}$$

with

$$\begin{aligned} \frac{\partial g}{\partial u} &= f_2(f_5 - f_4 f_1) + f_4(f_3 - f_2 f_1) - f_1^2(f_5 - f_4 f_1) \\ &\quad - 2f_4 f_1(f_2 - f_1^2) - 3f_2^2(f_3 - f_2 f_1) - 2f_3(f_4 - f_3 f_1) \\ &\quad + 2f_1 f_2(f_4 - f_3 f_1) + 2f_2 f_3(f_2 - f_1^2) + 2f_1 f_3(f_3 - f_2 f_1). \end{aligned}$$

Notice that this quantity vanishes at  $(0, 0)$ . Therefore the partial derivative of  $(f_4 - f_2^2)/g$  with respect to  $u$ , taken at  $(0, 0)$ , is null as well, and we get back that the third partial derivative of  $I$  with respect to  $x$ , taken at  $(0, \sigma^2)$ , is null. Differentiating once more, we obtain

$$\begin{aligned} \frac{\partial^4 I}{\partial x^4} &= \frac{\partial^2 u}{\partial x^2} \times \frac{\partial}{\partial u} \left( \frac{f_4 - f_2^2}{g} \right) (u, v) + \frac{\partial^2 v}{\partial x^2} \times \frac{\partial}{\partial v} \left( \frac{f_4 - f_2^2}{g} \right) (u, v) \\ &\quad + \frac{\partial u}{\partial x} \times \left( \frac{\partial u}{\partial x} \times \frac{\partial^2}{\partial u^2} \left( \frac{f_4 - f_2^2}{g} \right) (u, v) + \frac{\partial v}{\partial x} \times \frac{\partial^2}{\partial v \partial u} \left( \frac{f_4 - f_2^2}{g} \right) (u, v) \right) \\ &\quad + \frac{\partial v}{\partial x} \times \left( \frac{\partial u}{\partial x} \times \frac{\partial^2}{\partial u \partial v} \left( \frac{f_4 - f_2^2}{g} \right) (u, v) + \frac{\partial v}{\partial x} \times \frac{\partial^2}{\partial v^2} \left( \frac{f_4 - f_2^2}{g} \right) (u, v) \right). \end{aligned}$$

Let us compute it at  $(0, \sigma^2)$ ,

$$\begin{aligned} \frac{\partial^4 I}{\partial x^4}(0, \sigma^2) &= \frac{1}{\sigma^2} \left( \frac{1}{\sigma^2} \frac{\partial^2}{\partial u^2} \left( \frac{f_4 - f_2^2}{g} \right) (0, 0) + 0 \right) \\ &\quad + 0 + \frac{\sigma^4 - \mu_4}{\sigma^4} \frac{\partial^2 v}{\partial x^2}(0, \sigma^2) + 0, \end{aligned}$$

with

$$\frac{\partial^2 v}{\partial x^2}(0, \sigma^2) = \frac{\partial}{\partial x} \left( \frac{\partial^2 I}{\partial x \partial y} \right) (0, \sigma^2) = \frac{\partial^3 I}{\partial x^2 \partial y}(0, \sigma^2) = -\frac{1}{\sigma^4}$$

and

$$\begin{aligned} \frac{\partial^2}{\partial u^2} \left( \frac{f_4 - f_2^2}{g} \right) &= \frac{1}{g} \frac{\partial^2}{\partial u^2} (f_4 - f_2^2) - \frac{2}{g^2} \frac{\partial g}{\partial u} \frac{\partial}{\partial u} (f_4 - f_2^2) \\ &\quad - \frac{f_4 - f_2^2}{g^2} \frac{\partial^2 g}{\partial u^2} + \frac{2}{g^3} \left( \frac{\partial g}{\partial u} \right)^2 (f_4 - f_2^2). \end{aligned}$$

Hence

$$\frac{\partial^2}{\partial u^2} \left( \frac{f_4 - f_2^2}{g} \right) (0, 0) = \frac{1}{\sigma^4(\mu_4 - \sigma^4)} \left( \sigma^2 \frac{\partial^2}{\partial u^2} (f_4 - f_2^2) (0, 0) - \frac{\partial^2 g}{\partial u^2} (0, 0) \right).$$

The two remaining terms are the derivatives of quantities which we have already computed. In the following, we evaluate them directly at  $(0, 0)$ , which is straightforward since  $f_j(0, 0) = 0$  when  $j$  is odd:

$$\frac{\partial^2}{\partial u^2} (f_4 - f_2^2) (0, 0) = \frac{\partial}{\partial u} (f_5 - f_4 f_1 - 2 f_2 f_3 + 2 f_2^2 f_1) (0, 0) = \mu_6 - 3\sigma^2 \mu_4 + 2\sigma^6$$

and

$$\begin{aligned} \frac{\partial^2 g}{\partial u^2} (0, 0) &= \frac{\partial}{\partial u} \left( \frac{\partial g}{\partial u} \right) (0, 0) = \sigma^2(\mu_6 - \mu_4 \sigma^2) + \mu_4(\mu_4 - \sigma^4) - 0 - 2\mu_4 \sigma^4 \\ &\quad - 3\sigma^4(\mu_4 - \sigma^4) - 2\mu_4^2 + 2\sigma^4 \mu_4 + 2\sigma^4 \mu_4 + 0. \end{aligned}$$

This is equal to  $\sigma^2 \mu_6 - \mu_4^2 + 3\sigma^8 - 3\mu_4 \sigma^4$  after simplification. Thus we have

$$\begin{aligned} \frac{\partial^2}{\partial u^2} \left( \frac{f_4 - f_2^2}{g} \right) (0, 0) &= \frac{\sigma^2 \mu_6 - 3\sigma^4 \mu_4 + 2\sigma^8 - \sigma^2 \mu_6 + \mu_4^2 - 3\sigma^8 + 3\mu_4 \sigma^4}{\sigma^4(\mu_4 - \sigma^4)} \\ &= \frac{\mu_4^2 - \sigma^8}{\sigma^4(\mu_4 - \sigma^4)} = \frac{\mu_4 + \sigma^4}{\sigma^4}. \end{aligned}$$

Finally

$$\frac{\partial^2 I}{\partial x^4}(0, \sigma^2) = \frac{\mu_4 + \sigma^4}{\sigma^8} - \frac{\sigma^4 - \mu_4}{\sigma^8} = \frac{2\mu_4}{\sigma^8}.$$

We obtain the announced term and the proof is complete.  $\square$

**7. Proof of Theorem 2.** We first give conditions on the probability measure  $\rho$  in order to apply Theorem A.5 (see Appendix A) to the distribution  $\nu_\rho$ . We will use then Laplace’s method, as we announced in the heuristics of Section 3, to obtain the fluctuations Theorem 2. The proof will rely on the expansion of  $I - F$  around  $(0, \sigma^2)$  given in Proposition 15. We will also use the variant of Varadhan’s lemma, stated in Proposition 14. We start with the following lemma:

LEMMA 16. *If  $\rho$  has a probability density  $f$  with respect to the Lebesgue measure on  $\mathbb{R}$ , then  $\nu_\rho^{*2}$  has the density*

$$f_2 : (x, y) \mapsto \frac{1}{\sqrt{2y - x^2}} f\left(\frac{x + \sqrt{2y - x^2}}{2}\right) f\left(\frac{x - \sqrt{2y - x^2}}{2}\right) \mathbb{1}_{x^2 < 2y}$$

with respect to the Lebesgue measure on  $\mathbb{R}^2$ .

PROOF. Let  $h$  be a bounded continuous function from  $\mathbb{R}^2$  to  $\mathbb{R}$ . We have

$$\begin{aligned} \int_{\mathbb{R}^2} h(x, y) d\nu_\rho^{*2}(x, y) &= \int_{\mathbb{R}^2} h((z, z^2) + (t, t^2)) d\rho(z) d\rho(t) \\ &= \int_{D^+} h(z + t, z^2 + t^2) f(z) f(t) dz dt \\ &\quad + \int_{D^-} h(z + t, z^2 + t^2) f(z) f(t) dz dt, \end{aligned}$$

with  $D^+ = \{(z, t) \in \mathbb{R}^2 : z > t\}$  and  $D^- = \{(z, t) \in \mathbb{R}^2 : z < t\}$ . Indeed, the Lebesgue measure of the set  $\{(z, t) \in \mathbb{R}^2 : z = t\}$  is null. Let us denote, respectively, by  $I_+$  and  $I_-$  the two previous integrals.

We define  $\phi : (z, t) \in \mathbb{R}^2 \mapsto (u, v) = (z + t, z^2 + t^2)$ . It is a one to one map from  $D^+$  (resp., from  $D^-$ ) onto  $\Delta_2 = \{(u, v) \in \mathbb{R}^2 : u^2 < 2v\}$ . Moreover  $\phi$  is  $C^1$  on  $D^+ \cup D^-$ , and its Jacobian in  $(z, t)$  is  $2|z - t| = 2\sqrt{2v - u^2} \neq 0$ . The change of variables given by  $\phi$  yields

$$I_+ = \int_{\Delta_2} h(u, v) \frac{1}{2\sqrt{2v - u^2}} f\left(\frac{u + \sqrt{2v - u^2}}{2}\right) f\left(\frac{u - \sqrt{2v - u^2}}{2}\right) du dv,$$

and  $I_- = I_+$ . By adding these two terms, we get the lemma.  $\square$

By Theorem A.5 in the Appendix, the expansion of  $g_n$  holds as soon as there exists  $q \in [1, +\infty[$  such that  $\widehat{f}_2 \in L^q(\mathbb{R}^d)$ . However the computation of  $\widehat{f}_2$  is not feasible in general. Proposition A.6 says that the previous condition is satisfied if there exists  $p \in ]1, 2]$  such that  $f_2 \in L^p(\mathbb{R}^d)$  so that the expansion is true. Let us

take a look at this:

$$\begin{aligned} & \int_{\mathbb{R}^2} |f_2(u, v)|^p \, du \, dv \\ &= \int_{\mathbb{R}^2} \frac{f^p((u + \sqrt{2v - u^2})/2) f^p((u - \sqrt{2v - u^2})/2)}{(2v - u^2)^{p/2}} \mathbb{1}_{u^2 < 2v} \, du \, dv. \end{aligned}$$

Let us make the change of variables given by

$$(u, v) \mapsto (x, y) = \frac{1}{2}(u + \sqrt{2v - u^2}, u + \sqrt{2v - u^2}),$$

which is a  $C^1$ -diffeomorphism from  $\Delta_2$  to  $D^+$  (see the proof of the previous lemma) with Jacobian in  $(u, v)$ ,  $2\sqrt{2v - u^2} = 2(y - x) > 0$ :

$$\int_{\mathbb{R}^2} |f_2(u, v)|^p \, du \, dv = \int_{\mathbb{R}^2} \frac{f^p(x) f^p(y)}{(y - x)^p} 2(y - x) \mathbb{1}_{y > x} \, dx \, dy.$$

By symmetry in  $x$  and  $y$ , we get

$$\int_{\mathbb{R}^2} |f_2(u, v)|^p \, du \, dv = \int_{\mathbb{R}^2} f^p(x) f^p(y) |y - x|^{1-p} \, dx \, dy.$$

Then we get the following proposition:

**PROPOSITION 17.** *Suppose that  $\rho$  has a density  $f$  with respect to the Lebesgue measure on  $\mathbb{R}$  such that, for some  $p \in ]1, 2]$ ,*

$$\int_{\mathbb{R}^2} f^p(x + y) f^p(y) |x|^{1-p} \, dx \, dy < +\infty.$$

*Then, for  $n$  large enough,  $\tilde{\nu}_{n,\rho}$  has a density  $g_n$  with respect to the Lebesgue measure on  $\mathbb{R}^2$  such that, for any compact subset  $K_I$  of  $A_I$ , when  $n$  goes to  $+\infty$ , uniformly over  $(x, y) \in K_I$ .*

$$g_n(x, y) \sim \frac{n}{2\pi} (\det D_{(x,y)}^2 I)^{1/2} e^{-nI(x,y)}.$$

Let us prove now Theorem 2. Suppose that  $\rho$  is a probability measure on  $\mathbb{R}$  with an even density  $f$  such that there exist  $v_0 > 0$  and  $p \in ]1, 2]$  such that

$$\int_{\mathbb{R}} e^{v_0 z^2} f(z) \, dz < +\infty \quad \text{and} \quad \int_{\mathbb{R}^2} f^p(x + y) f^p(y) |x|^{1-p} \, dx \, dy < +\infty.$$

The first inequality implies that  $\mathbb{R} \times ]-\infty, v_0[ \subset D_\Lambda$  and thus  $(0, 0) \in \overset{\circ}{D}_\Lambda$ . Moreover  $\rho$  is symmetric (since  $f$  is even), and its support contains at least three points (since  $\rho$  has a density). Proposition 15 implies that there exists  $\delta > 0$  such that

$$(*) \quad \forall (x, y) \in B_\delta \quad G(x, y) = I(x, y) - \frac{x^2}{2y} \geq \frac{(y - \sigma^2)^2}{4(\mu_4 - \sigma^4)} + \frac{\mu_4 x^4}{24\sigma^8},$$

where  $\mu_4$  denotes the fourth moment of  $\rho$  and  $B_\delta$  the open ball of radius  $\delta$  centered at  $(0, \sigma^2)$ . We can reduce  $\delta$ , in order to have  $B_\delta \subset K_I$  where  $K_I$  is a compact subset of  $A_I$ . Moreover  $A_I \subset \overset{\circ}{D}_I \subset \Delta^*$  thus  $B_\delta \cap \Delta^* = B_\delta$ .

Let  $n \in \mathbb{N}$ , and let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a bounded continuous function. We have

$$\mathbb{E}_{\tilde{\mu}_{n,\rho}} \left( f \left( \frac{S_n}{n^{3/4}} \right) \right) = \frac{1}{Z_n} \int_{\Delta^*} f(xn^{1/4}) \exp\left(\frac{nx^2}{2y}\right) d\tilde{\nu}_{n,\rho}(x, y) = \frac{A_n + B_n}{Z_n},$$

with

$$A_n = \int_{B_\delta} f(xn^{1/4}) \exp\left(\frac{nx^2}{2y}\right) d\tilde{\nu}_{n,\rho}(x, y),$$

$$B_n = \int_{\Delta^* \cap B_\delta^c} f(xn^{1/4}) \exp\left(\frac{nx^2}{2y}\right) d\tilde{\nu}_{n,\rho}(x, y).$$

In what follows, we introduce  $e^{-nI(x,y)}$  in the expression of  $A_n$ , in order to use Proposition 17:

$$A_n = n \int_{B_\delta} f(xn^{1/4}) e^{-nG(x,y)} H_n(x, y) dx dy,$$

where we set  $H_n(x, y) = e^{nI(x,y)} g_n(x, y)/n$ . We define

$$B_{\delta,n} = \{(x, y) \in \mathbb{R}^2 : x^2/\sqrt{n} + y^2/n \leq \delta^2\}.$$

Let us make the change of variables given by  $(x, y) \mapsto (xn^{-1/4}, yn^{-1/2} + \sigma^2)$ , with Jacobian  $n^{-3/4}$ ,

$$A_n = n^{1/4} \int_{B_{\delta,n}} f(x) \exp\left(-nG\left(\frac{x}{n^{1/4}}, \frac{y}{\sqrt{n}} + \sigma^2\right)\right) \times H_n\left(\frac{x}{n^{1/4}}, \frac{y}{\sqrt{n}} + \sigma^2\right) dx dy.$$

We check now that we can apply the dominated convergence theorem to this integral. The uniform expansion of  $g_n$  (see Proposition 17) means that for any  $\alpha > 0$ , there exists  $n_0 \in \mathbb{N}$  such that

$$\forall (x, y) \in K_I, \forall n \geq n_0 \quad |H_n(x, y) 2\pi (\det D_{(x,y)}^2 I)^{-1/2} - 1| \leq \alpha.$$

If  $(x, y) \in B_{\delta,n}$ , then  $(x_n, y_n) = (xn^{-1/4}, yn^{-1/2} + \sigma^2) \in B_\delta \subset K_I$ . Thus for all  $n \geq n_0$  and  $(x, y) \in B_{\delta,n}$ ,

$$\left| H_n\left(\frac{x}{n^{1/4}}, \frac{y}{\sqrt{n}} + \sigma^2\right) 2\pi (\det D_{(x_n,y_n)}^2 I)^{-1/2} - 1 \right| \leq \alpha.$$

Moreover  $(x_n, y_n)$  goes to  $(0, \sigma^2)$ . Thus by continuity,

$$(D_{(x_n,y_n)}^2 I)^{-1/2} \xrightarrow{n \rightarrow +\infty} (D_{(0,\sigma^2)}^2 I)^{-1/2} = (D_{(0,0)}^2 \Lambda)^{1/2},$$

whose determinant is equal to  $\sqrt{\sigma^2(\mu_4 - \sigma^4)}$ . Therefore

$$\mathbb{1}_{B_{\delta,n}}(x, y)H_n\left(\frac{x}{n^{1/4}}, \frac{y}{\sqrt{n}} + \sigma^2\right) \xrightarrow{n \rightarrow +\infty} (4\pi^2\sigma^2(\mu_4 - \sigma^4))^{-1/2}.$$

The expansion of  $G$  in the neighborhood of  $(0, \sigma^2)$  implies that

$$\exp\left(-nG\left(\frac{x}{n^{1/4}}, \frac{y}{\sqrt{n}} + \sigma^2\right)\right) \xrightarrow{n \rightarrow +\infty} \exp\left(-\frac{y^2}{2(\mu_4 - \sigma^4)} - \frac{\mu_4 x^4}{12\sigma^8}\right).$$

Let us check that the integrand is dominated by an integrable function, which is independent of  $n$ . The function

$$(x, y) \mapsto (D_{(x,y)}^2 I)^{-1/2}$$

is bounded on  $B_\delta$  by some  $M_\delta > 0$ . The uniform expansion of  $g_n$  implies that for all  $(x, y) \in B_\delta$ ,  $H_n(x, y) \leq C_\delta$  for some constant  $C_\delta > 0$ . Finally, the inequality (\*) above yields

$$\begin{aligned} \mathbb{1}_{B_{\delta,n}}(x, y)f(x) \exp\left(-nG\left(\frac{x}{n^{1/4}}, \frac{y}{\sqrt{n}} + \sigma^2\right)\right)H_n\left(\frac{x}{n^{1/4}}, \frac{y}{\sqrt{n}} + \sigma^2\right) \\ \leq \|f\|_\infty C_\delta \exp\left(-\frac{y^2}{4(\mu_4 - \sigma^4)} - \frac{\mu_4 x^4}{24\sigma^8}\right). \end{aligned}$$

The right term is an integrable function on  $\mathbb{R}^2$ ; thus it follows from the dominated convergence theorem that

$$A_n \underset{+\infty}{\sim} n^{1/4} \int_{\mathbb{R}^2} \frac{f(x)}{\sqrt{2\pi\sigma^2}\sqrt{2\pi(\mu_4 - \sigma^4)}} \exp\left(-\frac{y^2}{2(\mu_4 - \sigma^4)} - \frac{\mu_4 x^4}{12\sigma^8}\right) dx dy.$$

By Fubini’s theorem, we get

$$A_n \underset{+\infty}{\sim} \frac{n^{1/4}}{\sqrt{2\pi\sigma^2}} \int_{\mathbb{R}} f(x) \exp\left(-\frac{\mu_4 x^4}{12\sigma^8}\right) dx.$$

Let us focus now on  $B_n$ . The distribution  $\rho$  is symmetric, it has a density and  $(0, 0)$  belongs to the interior of  $D_\Lambda$ ; thus Proposition 14 implies that there exist  $\varepsilon > 0$  and  $n_0 \geq 1$  such that for any  $n \geq n_0$ ,

$$\int_{\Delta^* \cap B_\delta^c} \exp\left(\frac{nx^2}{2y}\right) d\tilde{\nu}_{n,\rho}(x, y) \leq e^{-n\varepsilon},$$

and thus  $B_n \leq \|f\|_\infty e^{-n\varepsilon}$ , so that  $B_n = o(n^{1/4})$ . Therefore

$$A_n + B_n \underset{+\infty}{\sim} \frac{n^{1/4}}{\sqrt{2\pi\sigma^2}} \int_{\mathbb{R}} f(x) \exp\left(-\frac{\mu_4 x^4}{12\sigma^8}\right) dx.$$

Applying this to  $f = 1$ , we get

$$Z_n \underset{+\infty}{\sim} \frac{2n^{1/4}}{\sqrt{2\pi\sigma^2}} \int_0^{+\infty} \exp\left(-\frac{\mu_4 x^4}{12\sigma^8}\right) dx = \frac{n^{1/4}}{\sqrt{2\pi\sigma^2}} \frac{1}{2} \left(\frac{12\sigma^8}{\mu_4}\right)^{1/4} \Gamma\left(\frac{1}{4}\right),$$

where we made the change of variables  $y = \mu_4 x^4 / (12\sigma^8)$ . Finally

$$\mathbb{E}_{\tilde{\mu}_{n,\rho}}\left(f\left(\frac{S_n}{n^{3/4}}\right)\right) \underset{+\infty}{\sim} \left(\frac{4\mu_4}{3\sigma^8}\right)^{1/4} \Gamma\left(\frac{1}{4}\right)^{-1} \int_{\mathbb{R}} f(x) \exp\left(-\frac{\mu_4 x^4}{12\sigma^8}\right) dx.$$

The ultimate change of variables  $s = \mu_4^{1/4} x / \sigma^2$  gives us Theorem 2.

### APPENDIX A: GENERAL RESULTS ON THE CRAMÉR TRANSFORM

We present here some general results on the Cramér transform of distributions on  $\mathbb{R}^d$ .

A probability measure  $\mathbb{R}$  is said to be degenerate if it is a Dirac mass. The following definition generalizes this notion for measures on  $\mathbb{R}^d$ :

**DEFINITION A.1.** A probability measure  $\nu$  on  $\mathbb{R}^d$ ,  $d \geq 2$ , is said to be degenerate if its support is included in a hyperplane of  $\mathbb{R}^d$ ; that is, there exists a hyperplane  $\mathcal{H}$  of  $\mathbb{R}^d$  such that  $\nu(\mathcal{H}) = 1$ .

A first consequence of the nondegeneracy of  $\nu$  is that its covariance matrix is a symmetric positive definite matrix; see Section III.5 of [13] for a proof.

From this point forward, we consider  $\nu$  a nondegenerate probability measure on  $\mathbb{R}^d$ . The log-Laplace  $L$  of  $\nu$  is defined in  $\mathbb{R}^d$  by

$$\forall \lambda \in \mathbb{R}^d \quad L(\lambda) = \ln \int_{\mathbb{R}^d} e^{\langle \lambda, z \rangle} d\nu(z),$$

where  $\langle \cdot, \cdot \rangle$  denotes the inner product in  $\mathbb{R}^d$ . It is a convex function on  $\mathbb{R}^d$  which takes its values in  $]-\infty, +\infty]$ . The Fenchel–Legendre transform of  $L$  is called the Cramér transform of  $\nu$  and is defined on  $\mathbb{R}^d$  by

$$\forall x \in \mathbb{R}^d \quad J(x) = \sup_{\lambda \in \mathbb{R}^d} (\langle \lambda, x \rangle - L(\lambda)).$$

It is a nonnegative, convex and lower semi-continuous function. We denote by  $D_L$  and  $D_J$  the convex sets where  $L$  and  $J$  are finite. Noticethat if  $\overset{\circ}{D}_L$  is nonempty, then  $L$  is  $C^\infty$  on  $\overset{\circ}{D}_L$ . We refer to Section 2.2 of [7], Section VII.5 of [11] and Sections 25 and 26 of [19] for the main results on  $L$  and  $J$ . Cramér’s theorem (Theorem B.4 in the Appendix) links  $J$  and the large deviations of  $S_n/n$  where  $S_n$  is the sum of  $n$  independent random variables with common distribution  $\nu$ .

We are interested in the points  $\lambda$  realizing the supremum defining  $J(x)$ , for  $x \in D_J$ . We denote by  $\mathcal{C}$  the closed convex hull of the support of  $\nu$ .

LEMMA A.2. *Let  $\nu$  be a nondegenerate probability measure on  $\mathbb{R}^d$ . The interior of  $\mathcal{C}$  is not empty and  $\overset{\circ}{\mathcal{C}} \subset D_J \subset \mathcal{C}$ . Moreover for any  $x \in \overset{\circ}{\mathcal{C}}$ , the supremum defining  $J(x)$  is realized for some value  $\lambda(x) \in D_L$ .*

PROOF. The nondegeneracy of  $\nu$  means that its support is not included in a hyperplane of  $\mathbb{R}^d$ . Therefore the support of  $\nu$  contains  $d$  linearly independent vectors, and the interior of the convex hull of these vectors is nonempty. Thus  $\overset{\circ}{\mathcal{C}}$  is nonempty.

Suppose that  $\mathcal{C} \neq \mathbb{R}^d$  (otherwise it is immediate that  $D_J \subset \mathcal{C}$ ). Let  $x \notin \mathcal{C}$ . By the Hahn–Banach theorem, there exists  $\lambda \in \mathbb{R}^d$  and  $a \in \mathbb{R}$  such that

$$\forall y \in \mathcal{C} \quad \langle \lambda, y \rangle \leq a < \langle \lambda, x \rangle.$$

Since  $\nu(\mathcal{C}) = 1$ , Jensen’s inequality implies that

$$\forall t > 0 \quad J(x) \geq -\ln \int_{\mathcal{C}} \exp(t \langle \lambda, y \rangle - t \langle \lambda, x \rangle) d\nu(y) \geq t(\langle \lambda, x \rangle - a).$$

Sending  $t$  to  $+\infty$ , we conclude that  $J(x) = +\infty$ . Thus  $D_J \subset \mathcal{C}$ .

Let  $x \in \overset{\circ}{\mathcal{C}}$ , and let  $(\lambda_n)_{n \in \mathbb{N}}$  be a sequence in  $\mathbb{R}^d$  such that

$$\begin{aligned} J(x) &= \lim_{n \rightarrow +\infty} \left( \langle \lambda_n, x \rangle - \ln \int_{\mathbb{R}^d} \exp(\langle \lambda_n, z \rangle) d\nu(z) \right) \\ &= -\ln \lim_{n \rightarrow +\infty} \int_{\mathbb{R}^d} \exp(\langle \lambda_n, z - x \rangle) d\nu(z). \end{aligned}$$

We suppose that  $|\lambda_n|$  goes to  $+\infty$ , and we show that it leads to a contradiction. For all  $n \in \mathbb{N}$ , we set  $u_n = \lambda_n |\lambda_n|^{-1}$ . Then  $(u_n)_{n \in \mathbb{N}}$  is a bounded sequence. Thus, up to the extraction of a subsequence, we might assume that it converges to some vector  $u \in \mathbb{R}^d$  whose norm is 1. Let  $v$  belong to the support of  $\nu$ , and let  $U$  be an open subset of  $\mathbb{R}^d$  containing  $v$ . We have then  $\nu(U) > 0$ . Suppose that for any  $z \in U$ ,  $\langle u, z - x \rangle > 0$ . Then, by Fatou’s lemma,

$$\begin{aligned} +\infty &= \int_U \liminf_{n \rightarrow +\infty} \exp(|\lambda_n| \langle u_n, z - x \rangle) d\nu(z) \\ &\leq \liminf_{n \rightarrow +\infty} \int_U \exp(|\lambda_n| \langle u_n, z - x \rangle) d\nu(z). \end{aligned}$$

Hence

$$\exp(-J(x)) = \lim_{n \rightarrow +\infty} \int_{\mathbb{R}^d} \exp(|\lambda_n| \langle u_n, z - x \rangle) d\nu(z) = +\infty.$$

Thus  $J(x) = -\infty$ , which is absurd since  $J$  is a nonnegative function. We conclude that for all  $v$  in the support of  $\nu$  and for any open subset  $U$  of  $\mathbb{R}^d$  containing  $v$ , there exists  $z \in U$  such that  $\langle u, z - x \rangle \leq 0$ . It follows that, for any  $v$  in the support of  $\nu$ ,  $\langle u, v \rangle \leq \langle u, x \rangle$ . This inequality is stable by convex combinations, thus

$$\forall y \in \mathcal{C} \quad \langle u, y \rangle \leq \langle u, x \rangle.$$

Since  $x \in \overset{\circ}{\mathcal{C}}$ , there exists a ball  $B_x$  centered at  $x$  and contained in  $\mathcal{C}$ . Thus there exists  $y_0 \in B_x$  such that  $\langle u, y_0 \rangle > \langle u, x \rangle$ , which is absurd. Therefore  $(\lambda_n)_{n \in \mathbb{N}}$  is a bounded sequence. Hence there exists a subsequence  $(\lambda_{\phi(n)})_{n \in \mathbb{N}}$  and  $\lambda(x) \in \mathbb{R}^d$  such that  $\lambda_{\phi(n)}$  goes to  $\lambda(x)$ . By Fatou's lemma,

$$\begin{aligned} J(x) &= \langle \lambda(x), x \rangle - \ln \lim_{n \rightarrow +\infty} \int_{\mathbb{R}^d} \exp(\langle \lambda_n, z \rangle) d\nu(z) \\ &\leq \langle \lambda(x), x \rangle - \ln \int_{\mathbb{R}^d} \liminf_{n \rightarrow +\infty} \exp(\langle \lambda_n, z \rangle) d\nu(z) \\ &= \langle \lambda(x), x \rangle - \ln \int_{\mathbb{R}^d} \exp(\langle \lambda(x), z \rangle) d\nu(z) \leq J(x). \end{aligned}$$

Thus  $J(x) = \langle \lambda(x), x \rangle - L(\lambda(x))$ . Since  $L(\lambda(x)) \neq -\infty$ , this formula implies that  $J(x) < +\infty$  and thus that  $\overset{\circ}{\mathcal{C}} \subset D_J$ . Moreover if  $L(\lambda(x)) = +\infty$ , then  $J(x) = -\infty$ , which is absurd. Therefore  $L(\lambda(x)) < \infty$ . This shows that the supremum defining  $J(x)$  is realized at a point  $\lambda(x)$  with  $\Lambda(\lambda(x)) < +\infty$ .  $\square$

If  $D_L$  is an open subset of  $\mathbb{R}^d$ , then for all  $(x, y) \in \overset{\circ}{D}_J = \overset{\circ}{\mathcal{C}}$ , the supremum defining  $J(x)$  is realized at some  $\lambda(x) \in \overset{\circ}{D}_L$ . This is the case when the support of  $\nu$  is bounded, and also for the distribution  $\nu_\rho$  when  $\rho$  is the Gaussian  $\mathcal{N}(0, \sigma^2)$ , where we have then  $D_L = \mathbb{R} \times ]-\infty, 1/(2\sigma^2)[$ .

Now we study the smoothness of  $J$ .

NOTATION. If  $f$  is a differentiable function on an open subset  $U$  of  $\mathbb{R}^d$ , we denote by  $D_x f$  the differential of  $f$  at  $x \in U$ . If  $f$  is real-valued, we denote:

- ★  $D_x^2 f$  its second differential at  $x \in U$  (considered as a matrix of size  $d \times d$ ).
- ★  $\nabla f$  the function  $U \rightarrow \mathbb{R}^d$  such that

$$\forall x \in U, \forall y \in \mathbb{R}^d \quad \langle \nabla f(x), y \rangle = D_x f(y).$$

We define the admissible domain of  $J$ :

DEFINITION A.3. Let  $\nu$  be a nondegenerate probability measure on  $\mathbb{R}^d$  such that the interior of  $D_L$  is nonempty. The admissible domain of  $J$  is the set  $A_J = \nabla L(\overset{\circ}{D}_L)$ .

The following proposition states that  $A_J$ , the admissible domain of  $J$ , is an open subset of  $\mathbb{R}^d$ , and that  $J$  is  $C^\infty$  on  $A_J$ .

PROPOSITION A.4. Let  $\nu$  be a nondegenerate probability measure on  $\mathbb{R}^d$  such that the interior of  $D_L$  is nonempty. Let  $A_J$  be the admissible domain of  $J$ . We have:

(a) The function  $\nabla L$  is a  $C^\infty$ -diffeomorphism from  $\mathring{D}_L$  to  $A_J$ . Moreover

$$A_J \subset D_J = \{x \in \mathbb{R}^d : J(x) < +\infty\}.$$

(b) Denote by  $\lambda$  the inverse  $C^\infty$ -diffeomorphism of  $\nabla L$ . Then the function  $J$  is  $C^\infty$  on  $A_J$  and for any  $x \in A_J$ ,

$$J(x) = \langle x, \lambda(x) \rangle - L(\lambda(x)),$$

$$\nabla J(x) = (\nabla L)^{-1}(x) = \lambda(x) \quad \text{and} \quad D_x^2 J = (D_{\lambda(x)}^2 L)^{-1}.$$

(c) If  $D_L$  is an open subset of  $\mathbb{R}^d$ , then  $A_J = \mathring{D}_J = \mathring{C}$  where  $C$  denotes the convex hull of the support of  $\nu$ .

PROOF. The points (a) and (b) are proved in Section 2 of [1], Section 1.5 of [6] and Section 26 of [19]; see also Section VIII.4 of [11] in the case where  $D_L = \mathbb{R}^d$ . Let us prove point (c). If  $D_L$  is an open subset of  $\mathbb{R}^d$ , then Lemma A.2 implies that for  $x \in \mathring{C} = \mathring{D}_J$ , the supremum defining  $J(x)$  is realized at some point  $\lambda(x) \in D_L = \mathring{D}_L$ . The function  $L$  is differentiable at  $\lambda(x)$ , and point (b) yields that

$$x = \nabla L(\lambda(x)) \in \Lambda(\mathring{D}_L) = A_J.$$

Thus  $\mathring{D}_J \subset A_J$ . Finally  $A_J \subset D_J$ , and  $A_J$  is open; thus  $A_J = \mathring{D}_J = \mathring{C}$ . This proves (c).  $\square$

Let  $\nu$  be a probability distribution on  $\mathbb{R}^d$  having a density with respect to the Lebesgue measure, and let  $S_n$  be the sum of  $n$  independent and identically distributed random variables with distribution  $\nu$ . The following theorem states that, under some hypothesis allowing the Fourier inversion, the density of the distribution of  $S_n/n$  is asymptotically a function of  $J$ , the Cramér transform of  $\nu$ . We refer to Section 3 of the article of Andriani and Baldi [1] for a proof.

**THEOREM A.5.** *Let  $\nu$  be a nondegenerate probability measure on  $\mathbb{R}^d$ . We denote by  $L$  its log-Laplace and by  $J$  its Cramér transform. Suppose that  $\mathring{D}_L \neq \emptyset$  and that there exists  $n_0 \geq 1$  such that*

$$\widehat{\nu^{*n_0}} \in L^1(\mathbb{R}^d).$$

*We denote by  $A_J$  the admissible domain of  $J$ . Let  $(X_n)_{n \geq 1}$  be a sequence of independent and identically distributed random variables with distribution  $\nu$ . For any  $n \geq n_0$ , the random variable  $\bar{X}_n = (X_1 + \dots + X_n)/n$  has a density  $g_n$  with respect to the Lebesgue measure on  $\mathbb{R}^d$ . If  $K_J$  is a compact subset of  $A_J$ , then uniformly over  $x \in K_J$ , when  $n$  goes to  $+\infty$ ,*

$$g_n(x) \sim \left(\frac{n}{2\pi}\right)^{d/2} (\det D_x^2 J)^{1/2} e^{-nJ(x)}.$$

PROPOSITION A.6. *Let  $\nu$  be a nondegenerate probability measure on  $\mathbb{R}^d$  such that  $\mathring{D}_L \neq \emptyset$ . If there exists  $m \in \mathbb{N}$  and  $p \in ]1, 2]$  such that  $\nu^{*m}$  has a density  $f_m \in L^p(\mathbb{R}^d)$ , then the hypotheses of Theorem A.5 are verified.*

PROOF. The Hausdorff–Young inequality (see Theorem 1.2.1 of [5]) implies that  $\widehat{f}_m \in L^r(\mathbb{R}^d)$ , with  $r = p/(p - 1)$ . Moreover  $\widehat{f}_m$  is bounded, so  $\widehat{f}_m \in L^q(\mathbb{R}^d)$ , where  $q$  is a positive integer larger than  $r$ . Therefore

$$\widehat{\nu^{*mq}} = (\widehat{\nu^{*m}})^q = (\widehat{f}_m)^q \in L^1(\mathbb{R}^d).$$

Hence the hypotheses of the theorem are verified with  $n_0 = mq$ .  $\square$

### APPENDIX B: SOME RESULTS ON LARGE DEVIATIONS

Let  $(\mathcal{X}, \mathcal{B})$  be a topological space. We refer to the Section 1.2 of [7] for the two following definitions:

DEFINITION B.1. A rate function on  $\mathcal{X}$  is a nonnegative map  $J$  defined on  $\mathcal{X}$  and which is lower semi-continuous; that is, for any  $\alpha > 0$ , the level set  $\{x \in \mathcal{X} : J(x) \leq \alpha\}$  is a closed subset of  $\mathcal{X}$ . A good rate function is a rate function for which all these level sets are compact sets of  $\mathcal{X}$ .

DEFINITION B.2. A sequence  $(\mu_n)_{n \geq 1}$  of probability measures on  $\mathcal{X}$  satisfies a large deviation principle with speed  $n$  and which is governed by the rate function  $J$  if, for any  $A \in \mathcal{B}$ ,

$$\begin{aligned} -\inf\{J(x) : x \in \overset{\circ}{A}\} &\leq \liminf_{n \rightarrow +\infty} \frac{1}{n} \ln \mu_n(A) \\ &\leq \limsup_{n \rightarrow +\infty} \frac{1}{n} \ln \mu_n(A) \leq -\inf\{J(x) : x \in \overline{A}\}. \end{aligned}$$

The following lemma is a variant of the upper bound of Varadhan’s lemma; see Lemma 4.3.6 of [7].

LEMMA B.3. *Let  $\mathcal{X}$  be a regular topological space endowed with its Borel  $\sigma$ -field  $\mathcal{B}$ . Let  $(\nu_n)_{n \geq 1}$  be a sequence of probability measures defined on  $(\mathcal{X}, \mathcal{B})$  which satisfies a large deviation principle with speed  $n$ , governed by the good rate function  $J$ . For any bounded continuous function  $f : \mathcal{X} \rightarrow \mathbb{R}$ , we have for any closed subset  $A$  of  $\mathcal{X}$ ,*

$$\limsup_{n \rightarrow +\infty} \frac{1}{n} \ln \int_A e^{nf(x)} d\nu_n(x) \leq \sup_{x \in A} (f(x) - J(x)).$$

We end this [Appendix](#) with the Cramér theorem in  $\mathbb{R}^d$  (see Theorem 2.2.30 of [7]):

**THEOREM B.4 (Cramér).** *Let  $\nu$  be a probability measure on  $\mathbb{R}^d$ ,  $d \geq 1$ . We denote by  $L$  its log-Laplace and by  $J$  its Cramér transform. Let  $(X_n)_{n \geq 1}$  be a sequence of independent random variables with common law  $\nu$ . We define*

$$\forall n \geq 1 \quad S_n = X_1 + \cdots + X_n.$$

*If  $L$  is finite in the neighborhood of 0, then  $J$  is a good rate function, and the sequence of the laws of  $S_n/n$ ,  $n \geq 1$  satisfies the large deviation principle with speed  $n$  and governed by  $J$ .*

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## REFERENCES

- [1] ANDRIANI, C. and BALDI, P. (1997). Sharp estimates of deviations of the sample mean in many dimensions. *Ann. Inst. Henri Poincaré Probab. Stat.* **33** 371–385. [MR1457057](#)
- [2] ASCHWANDEN, M. J., ed. (2013). *Self-Organized Criticality Systems*. Open Academic Press, Berlin. Available at <http://ojs.antek666.website.pl/SOC1.pdf>.
- [3] BAK, P. (1996). *How Nature Works: The Science of Self-Organized Criticality*. Copernicus, New York. [MR1417042](#)
- [4] BAK, P., TANG, C. and WIESENFELD, K. (1988). Self-organized criticality. *Phys. Rev. A* (3) **38** 364–374. [MR0949160](#)
- [5] BERGH, J. and LÖFSTRÖM, J. (1976). *Interpolation Spaces. An Introduction. Grundlehren der Mathematischen Wissenschaften* **223**. Springer, Berlin. [MR0482275](#)
- [6] BOROVKOV, A. A. and MOGUL'SKIĬ, A. A. (1992). Large deviations and testing statistical hypotheses. I. Large deviations of sums of random vectors. *Siberian Adv. Math.* **2** 52–120. [MR1181029](#)
- [7] DEMBO, A. and ZEITOUNI, O. (2010). *Large Deviations Techniques and Applications. Stochastic Modelling and Applied Probability* **38**. Springer, Berlin. [MR2571413](#)
- [8] DE BOER, J., DERRIDA, B., FLYVBJERG, H., JACKSON, A. D. and WETTIG, T. (1994). A simple model of self-organized biological evolution. *Phys. Rev. Lett.* **73** 906–909.
- [9] DHAR, D. (2006). Theoretical studies of self-organized criticality. *Phys. A* **369** 29–70. [MR2246566](#)
- [10] EISELE, T. and ELLIS, R. S. (1988). Multiple phase transitions in the generalized Curie–Weiss model. *J. Stat. Phys.* **52** 161–202. [MR0968583](#)
- [11] ELLIS, R. S. (2006). *Entropy, Large Deviations, and Statistical Mechanics*. Springer, Berlin. [MR2189669](#)
- [12] ELLIS, R. S. and NEWMAN, C. M. (1978). Limit theorems for sums of dependent random variables occurring in statistical mechanics. *Z. Wahrsch. Verw. Gebiete* **44** 117–139. [MR0503333](#)
- [13] FELLER, W. (1971). *An Introduction to Probability Theory and Its Applications. Vol. II*, 2nd ed. Wiley, New York. [MR0270403](#)
- [14] GORNY, M. (2012). Le modèle d'Ising Curie–Weiss et ses généralisations. Masters thesis, Université Paris-Sud.

- [15] GORNY, M. (2014). A Curie–Weiss model of self-organized criticality: The Gaussian case. *Markov Processes Related Fields* **20** 563–576.
- [16] MOSCO, U. (1971). On the continuity of the Young–Fenchel transform. *J. Math. Anal. Appl.* **35** 518–535. [MR0283586](#)
- [17] PRUESSNER, G. (2012). *Self-Organised Criticality: Theory, Models and Characterisation*. Cambridge Univ. Press, Cambridge.
- [18] RÁTH, B. and TÓTH, B. (2009). Erdős–Rényi random graphs + forest fires = self-organized criticality. *Electron. J. Probab.* **14** 1290–1327. [MR2511285](#)
- [19] ROCKAFELLAR, R. T. (1970). *Convex Analysis*. *Princeton Mathematical Series* **28**. Princeton Univ. Press, Princeton, NJ. [MR0274683](#)
- [20] SORNETTE, D. (2006). *Critical Phenomena in Natural Sciences: Chaos, Fractals, Selforganization and Disorder: Concepts and Tools*, 2nd ed. Springer, Berlin. [MR2220576](#)
- [21] TURCOTTE, D. L. (1999). Self-organized criticality. *Rep. Progr. Phys.* **62** 1377.

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